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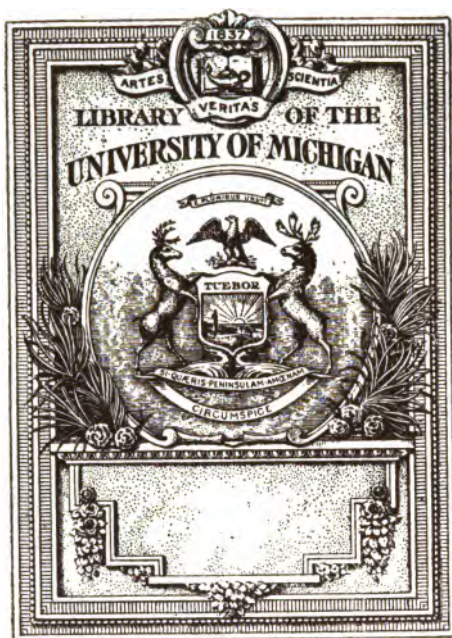
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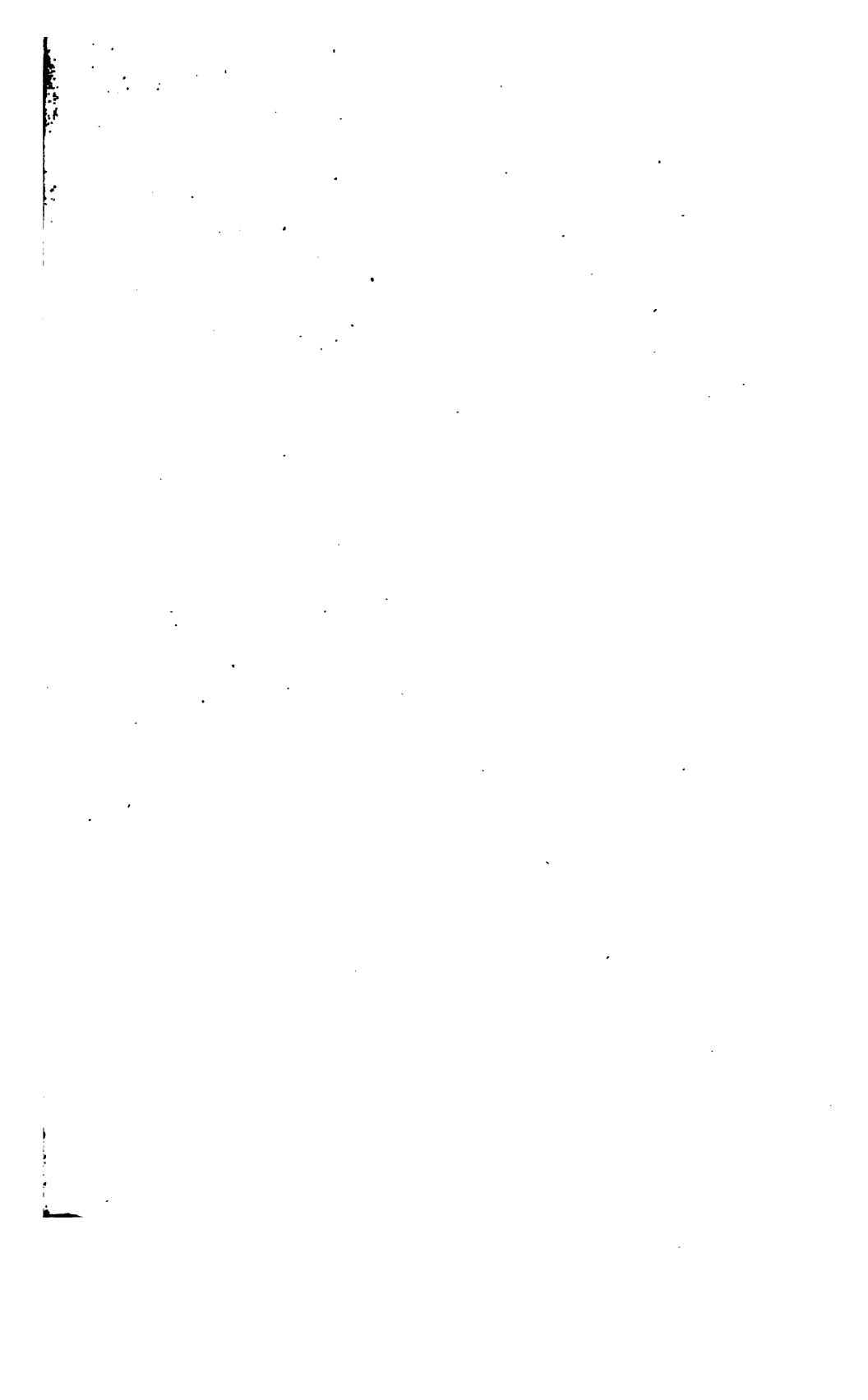


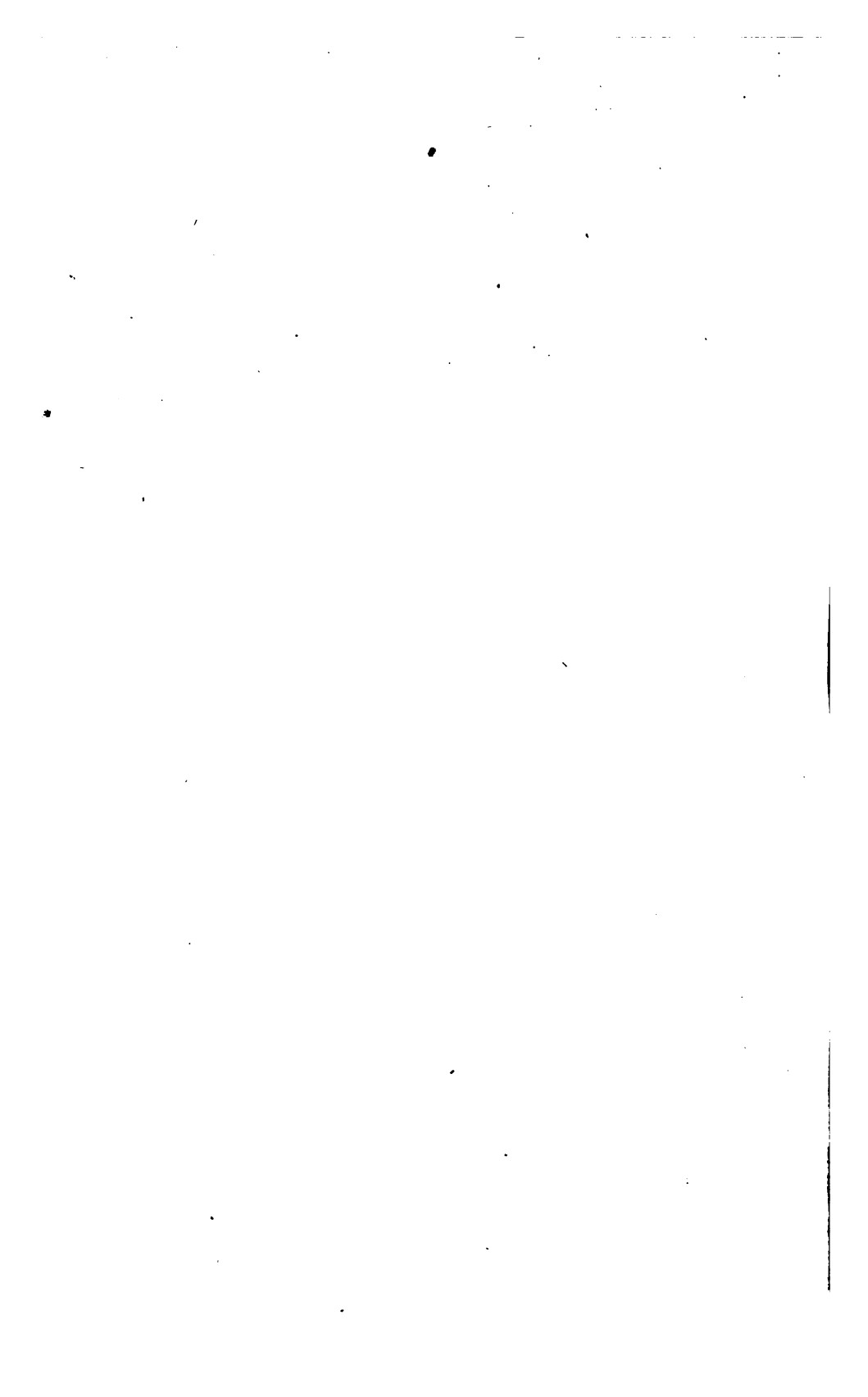
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THE
MATHEMATICAL QUESTIONS,
PROPOSED IN THE
LADIES' DIARY,
AND THEIR ORIGINAL ANSWERS,
TOGETHER WITH SOME NEW SOLUTIONS,

FROM ITS COMMENCEMENT IN THE YEAR

1704 to 1816. 

IN FOUR VOLUMES.

BY THOMAS LEYBOURN,

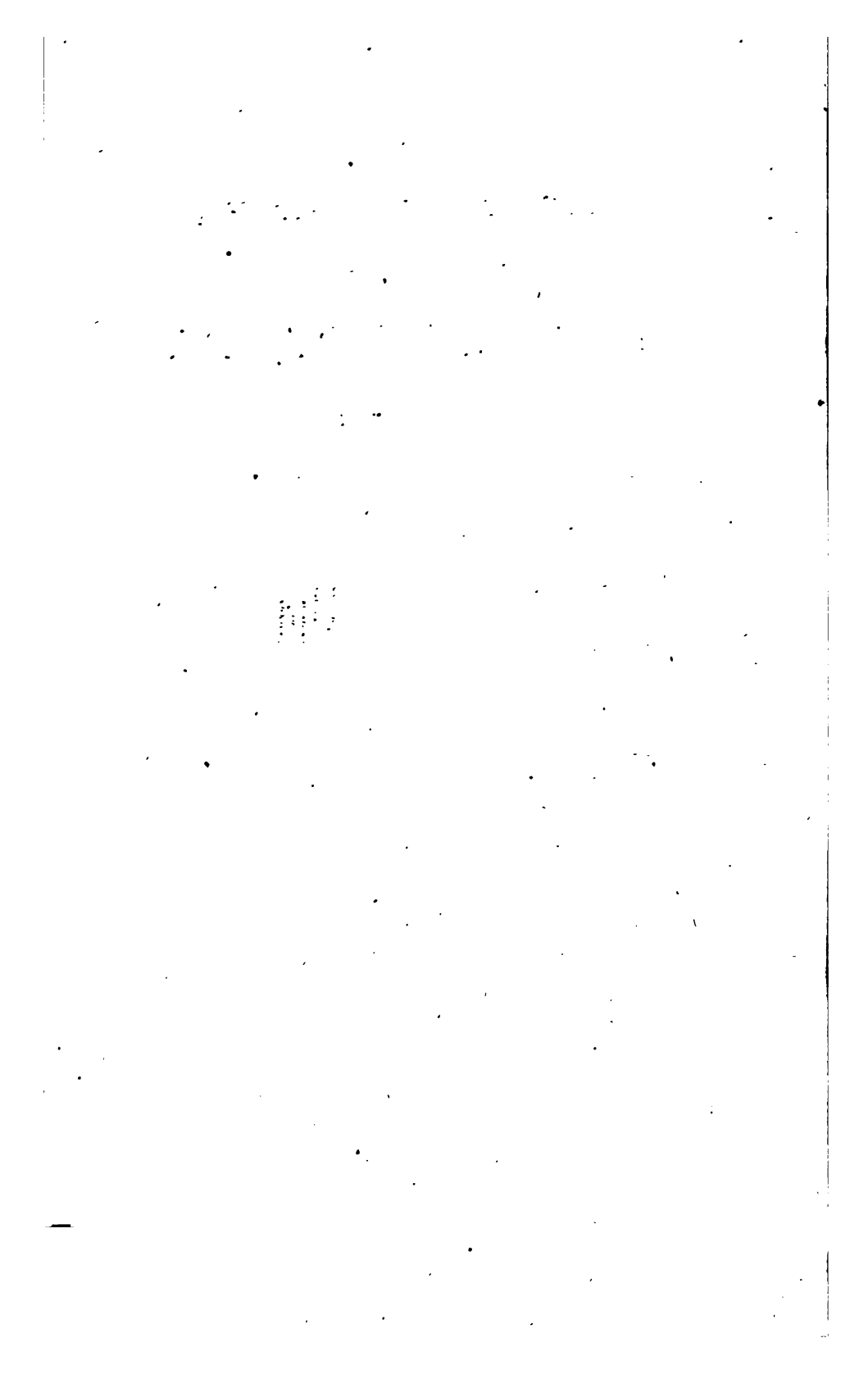
OF THE ROYAL MILITARY COLLEGE.

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1817.



MATHEMATICAL QUESTIONS

PROPOSED IN THE

LADIES' DIARY,

AND THEIR ANSWERS.

Questions proposed in 1802 and answered in 1803.

I. QUESTION 1089, by P. Panglos.

A pendulum vibrates as often in a minute as it is inches in length.
Query, what that length is ?

*Answered by Master John Golding, at Mr. Gregory's Academy,
Cambridge.*

Put x for the pendulum's length in inches, and also its number of vibrations in a minute; the length of the second's pendulum being $39\frac{1}{2}$ inches. Then, the number of vibrations being inversely proportional to the square of the lengths, as $\sqrt{x} : \sqrt{39\frac{1}{2}} :: 60 : x$; hence, multiplying extremes and means, $x\sqrt{x} = 60\sqrt{39\frac{1}{2}}$; and, by squaring, $x^3 = 3600 \times 39\frac{1}{2} = 140850$; therefore $x = \sqrt[3]{140850} = 52.02982$ inches, the length required.

The same by Mr. J. H. Hearing, Alderbury School.

The lengths of pendulums are to one another reciprocally as the squares of their vibrations made in the same time: therefore, putting x for the length required, $x : 39\frac{1}{2} :: 60^2 : x^2$; hence, $x^2 = 60^2 \times 39\frac{1}{2} = 140850$, and $x = \sqrt{140850} = 52.02982$ inches.

The same by Mr. Jos. Kaye, Aldmonbury.

It is demonstrated by mathematician's (Hutton's Course of Mathematics, vol. 2, pa. 174, &c.) that the times of vibrations of pendulums are as the square root of their lengths; or the number of vibrations in a given time reciprocally as the square root of their lengths. Let x = the number of vibrations in a minute, and also the length; then

$x^3 : 60^3 :: 39\frac{1}{8} : x$; hence $x^3 = 60^3 \times 39\frac{1}{8} = 140850$, and $x = \sqrt[3]{140850} = 52.03$, the required length and number of vibrations in a minute.

The same by Mr. Alexander Rowe, Reginnis.

Let x denote the pendulum's length. Then, by Dr. Hutton's Course, vol. 2, page 176, as $\sqrt{x} : \sqrt{39\frac{1}{8}} :: 60 : x$; mult. extremes and means, $x\sqrt{x} = 60\sqrt{39\frac{1}{8}}$; squaring, $x^3 = 60^3 \times 39\frac{1}{8}$; hence, $x = \sqrt[3]{60^3 \times 39\frac{1}{8}} = 52.02982$ inches $= 4.335818$ feet as required.

The same by Mr. John Selkirk, Newcastle.

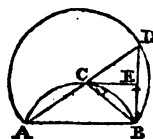
Put $x =$ length of the pendulum. Then, by Dr. Hutton's Course, vol. 2, p. 176, as $\sqrt{x} : \sqrt{39\frac{1}{8}} :: 60 : x$; hence $x = 52.02982$ inches; the length of the pendulum required.

II. QUESTION 1090, by Mr. Wm. Bewley, Hawkshead.

Having given the base, the sum of the sides, and the vertical angle; to construct the triangle.

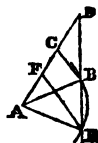
Answered by Mr. Wm. Bardon, Acaster Malbis.

On the given base AB describe (by prob. 14, Dr. Hutton's Geom.) a segment of a circle ADB capable of containing an angle equal to half the given vertical angle; from A apply $AD =$ the given sum of the sides: bisect AD with the perpendicular CE , meeting AD in c ; join BC , then ABC will be the required triangle. For, the triangles BEC , DEC being equal in all respects, $BC = CD$; therefore $AC + CB = AC + CD = AD$, and $\angle ACB = \angle CDB + \angle CBD = 2\angle CDB =$ the given vertical angle by construction.



The same by the Rev. Mr. Ewbank, Vicar of Thornton Steward.

Draw $AD =$ the given sum of the sides, and DBE making the $\angle D =$ half the given vertical angle: with centre A , and radius the given base, describe an arc cutting DE in B and E : draw BC or EF to make with DE an angle $= \angle D$, and the thing is done. For either ABC or AEF may be the triangle required, as is evident.



The same by Mr. Henry Hann, Schoolmaster, Saltwick.

On the given base AB (see the first fig.) describe the circular seg-

ment ACB to contain the given angle, and also ADB to contain half the same angle; take AD equal to the sum of the given sides, and join BC , so shall ABC be the triangle required. For join BD . Then $\angle C = \angle B + \angle D = 2\angle D$ by construction; therefore $\angle B = \angle D$, and $CB = CD$; consequently $AC + CB = AC + CD$ the sum of the sides.

Note. This prob. is constructed in Simpson's Algebra, p. 315, and it is also solved algebraically in Sir I. Newton's Arith. p. 110, 2d edition.

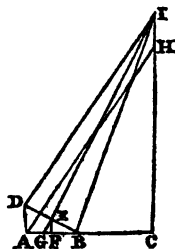
III. QUESTION 1001, by Mr. Wm. Cole, Colchester.

The front of a house is of such a height, that if a ladder, of a certain length, be placed at the distance of 12 feet from it, the top of the ladder will just reach to the top of the house; but if the ladder be placed twenty feet from the front, its top will fall 4 feet below the top of the house. Required the height of the house and the length of the ladder?

Answered by Mr. W. Buttermann, Dronfield.

Construction. On the given base $AC = 20$, erect the perpendiculars CI and $AD (= 4)$; take $AB = 8$, and join BD , which bisect in E with the perpendicular CEI ; then will CI be the height of the building, BI the length and first position of the ladder, and AH the second position of the same, if HI be taken $= AD = 4$. For, joining DI , then $AH =$ and parallel to DI , because $AD =$ and parallel to HI ; also $BI = DI$ because EI bisects BD perpendicularly; therefore $AH = BI$.

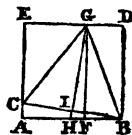
Calculation. Let fall the perpendicular EF , which will be $= \frac{1}{2}AD$; and, E being a right angle, the two triangles BFE , GFE are equiangular, therefore $FB : FE :: FE : FG = 1$; and, again by similar triangles, as $GF : FE :: GC : CI = 34$ the height of the building: then $BI = \sqrt{(BC^2 + CI^2)} = \sqrt{(12^2 + 34^2)} = \sqrt{1300} = 36.0555$ the length of the ladder.



The same by Mr. John Cavill, Beighton.

Construction. In the line AB , take $AF = 20$, and $BF = 12$, also AC perpendicular to AB and $= 4$, join BC , which bisect by the perpendicular HIG , meeting the perpendicular FG in G ; draw BG , GC ; then will FG be the height of the house, and GB or GC the length of the ladder. For, completing the rectangle $ABDE$, because $CI = BI$, and I is a right angle, therefore $GB = GC =$ the length of the ladder, because $AC =$ the given difference made in the height by moving the ladder; also $FG = BD$ the building's height, and CE the less height, AC being the given difference.

Calculation. By similar triangles as $AB : BC :: BI : BH = 16\frac{1}{4}$; then



$FH = BH - BF = 16\frac{1}{2} - 12 = 4\frac{1}{2}$; and again by similar triangles as $AC : AB :: FH : FG = 34$ the height of the building, also $CE = 30$, lastly, $CG = \sqrt{(CE^2 + GE^2)} = \sqrt{(30^2 + 20^2)} = \sqrt{1300} = 36.055513$ the length of the ladder.

The same by Mr. E. Birch, Schoolmaster, Castle Donnington.

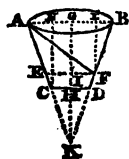
In the first figure $AC = 20$, $CB = 12$, $IH = 4$, AH or BI the ladder, and CI the height of the house. Put $x = AH$ or BI the ladder, and $y = CI$; then $x^2 = 12^2 + y^2$, and $x^2 = 20^2 + (y - 4)^2$; therefore $12^2 + y^2 = 20^2 + y^2 - 8y + 16$; hence $8y = 20^2 - 12^2 + 4^2 = 272$; consequently $y = 34$, and $x = \sqrt{1300} = 36.0555$ as required.

IV. QUESTION 1092, by Mr. S. Hefford, of Albrighton.

Two Bacchanalians being allowed a large copper of ale, its form a conical frustum, standing on the less end, the dimensions of which were, top diameter 5, bottom 3, and depth 6 inches, they could not agree about the division of it; they therefore determined to have a mark made on the side, so that the first, by taking half the liquor, should drink till the bottom edge of the liquor just touched the mark. It is required to find the place for the mark on the side?

Answered by Mr. W. Buttermann, Dronfield.

Draw the lines as in the annexed fig. where $ABDC$ denotes the vessel and ABK the cone completed, also AF the surface of the liquor when it is just one half exhausted, F being the supposed mark on the side, so that the content of the ungula ABF may be just equal to that of $ACDF$. Now, by Hutton's Mensuration, page 174, as $AFK^3 : AFB^3 :: AB^3 : EF^3$; hence $EF = 3.5884$; then by similar triangles as $BC : GK :: BL : LF = 4.2348$; hence $BF = \sqrt{(BL^2 + LF^2)} = 4.2932$ inches, the distance required.



The same by Mr. Joseph Gittins, Whitechapel.

Let $ABDC$ represent the copper, where AB is the greater given diameter, CD the less, and CH the given depth; also let AF denote the surface of the liquor when its lowest edge just touched the required mark at F . Draw EF parallel to CD ; continue the sides AC and BD till they meet at K , completing the cone ABK , of which $ABDC$ is a frustum, the whole axis being GK . Now, by similar triangles $AN : AG :: NC : GK$. Find the solidity of ABF , the half of the frustum $ABDC$, by the 6th prob. p. 188, in the Mensuration, 2d edition, for which put s ; as also the content of the whole cone ABK , for which put s ; then the diff,

$s - s'$ will be the content of the oblique cone AFK . Then, by Cor. 4, page 288, Mensuration, $s^3 : (s - s')^3 :: AB^3 : EF^3 :: BK^3 : FK^3$; hence $BF = BK - FK = 4.2932$ inches, the distance of the mark along the side below the top of the vessel, as required.

V. QUESTION 1093, by Mr. John Blackwell, *Hungerford*.

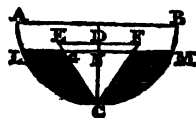
I have a conical piece of dry oak, the base diameter and altitude being 24 and 29 inches, which I connected by its vertex at the bottom of a semispherical vessel of 60 inches diameter. Now I desire to know what is the least quantity of water which must be poured into the said vessel to raise or set the axis of the cone in a vertical position?

Answered by Mr. Tho. Cook, Broughton.

When the axis of the cone is raised by the water to an upright position, the weight of the water displaced will be equal to the whole weight of the cone; and, by hydrostatics, the content of the cone is to the content of the part immersed, as the specific gravities of water and oak. Therefore, 16704*n* being the content of the whole cone, (*n* being = .2618), we have $1000 : 925 :: 16704n : 15451\frac{1}{2}n$ the content of the part under water; also the altitudes of similar cones are as the cube root of their solidities, therefore as $\sqrt[3]{1000} : \sqrt[3]{925} :: 29 : 28.45607$ the depth of the water, or the versed sine of the spherical segment: then, $28.45607^3 \times .5236 \times (60 \times 3 - 28.25607 \times 2) = 51623.496$ the content of the whole segment, from which taking 4045.124 the content of the cone under water, leaves 47578.373 the number of cubic inches, which being divided by 282, gives 168.717 the number of gallons required.

The same by Mr. P. Leeming, Horsforth.

Let ACB be the vessel, ECF the cone, and LM the surface of the water. Then $24^3 \times .2618 \times 29 = 4373.107$ inches the solidity of the cone; and $1000 : 925 :: 4373.107 : 4045.124$ the cubic inches of a cone of water equal in weight to the oaken cone, or the solid content of the cone GCH ; then, by similar cones, as cone ECF ; cone $GCH :: DC^3 : NC^3 = 22559.825$, hence $NC = 28.256$ the depth of the water; then, by the property of the circle, $LN = 29.9492$, and by Mensuration, $(3LN^3 + CN^3) \times CN \times .5236 = 51623.29944$ the solidity of the segment LCM , from which taking the cone GCH , the remainder will be 47578 inches = 168.7169 gallons, to be poured in.



VI. QUESTION 1094, by Mr. T. Swanwick, *Teacher of Mathematics.*

Required a theorem for finding the time of descent of heavy bodies from vast heights, supposing no resistance: as for example, the moon to the earth, if her projectile force should cease?

Answered by Mr. O. G. Gregory, Cambridge.

In a former letter I spoke pretty much at large on the subject of this question, and the various solutions which have been given of it. It does not appear necessary to repeat all those particulars again. I shall just observe here, that Mr. Thorp, in his Commentary on Newton's Principia, pa. 193, makes the time in the example of the moon's falling to the earth, to be 4da. 19h. 55½m. reckoning the earth as a point. In Mr. Whiston's Mathematical Philosophy, page 72, the time is stated at 4da. 20h. In Dr. Hutton's Conics and Select Exercises, page 183, and in his Course, vol. 2, pa. 337, the time is 4da. 19h. 46m. 46a. accounting the earth's radius 3965 miles, and the moon's distance 60 times that radius. The problem is discussed very clearly, though popularly, in Baxter's Matho, vol. 2, page 306, 308; but he has not calculated for this particular example.

The same by Mr. Tho. Towan, Redruth.

Put $r = cs$ the earth's radius, $a = sA$ the height above the earth's centre, $t =$ the time of falling through As , and $g = 16\frac{1}{2}$ feet; then, in Dr. Hutton's Mathematics, vol. 2, p. 337, we have the following theorem, $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{AD + DS}{sc}$ for the time of descent to the earth's surface at s . In the case of a body's falling from the height of the moon; $r = 3965$ miles, $a = 60r$, and $t = 416806$ sec. = 4da. 19h. 46m. 46 seconds the time of falling from the moon to the earth.

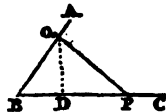


VII. QUESTION 1095, by Mr. Wm. White, *Chaddesden.*

If an indefinite right line BA form a given $\angle ABC$ of 44° with a given right line $AC = 400$ feet; and if a body q begin to move from B along BA, at the same instant that another body r begins to move from C along CB; the former body moving with an accelerated velocity of one foot in the first second, 5 feet the next, 9 the third, &c. and the latter body r with a uniform velocity of 6 feet in a second. To find the time of their nearest approach towards each other, as also their position and distance at that time,

Answered by Mr. John Campbell.

The space moved over by the point a , in the first, second, third, fourth, &c. seconds, form the arithmetical progression 1, 5, 9, 13, &c. feet, the common difference being 4. If x denote the number of its terms, or the time in motion; then $4(x-1) + 1 = 4x - 3$ will be the last term, to which add the first term 1, gives $4x - 2$ the sum of the extremes; then $\frac{1}{2}x(4x - 2) = 2x^2 - x$ is the whole sum of the series = the line or distance BQ passed over. In the same time x the distance CP passed over by the uniform motion is $6x$, consequently $BP = 400 - 6x = a - 6x$, putting $a = 400$. Let QD be perpendicular to BP , and put s and $c = \text{sine and cosine } \angle B (44^\circ)$; then, by trigonometry, $1 : BQ :: s : DQ = s(2x^2 - x) :: c : BD = c(2x^2 - x)$; hence, $DP = BP - BD = a - 6x - c(2x^2 - x) = a - 2cx^2 - x(6 - c) = a - 2cx^2 - dx$, putting $d = 6 - c$; then $PQ^2 = PD^2 + DQ^2 = (a - 2cx^2 - dx)^2 + s^2(2x^2 - x)^2$; the fluxion of this put = 0, the resulting equation in numbers will be $16x^3 + 39.7925x^2 - 2245.1515x = 4224.528$; hence, by Dr. Hutton's easy approximation in his Course of Mathematics, vol. 1, find $x = 11.588$ seconds, the time required. Then $BQ = 2x^2 - x = 256.93$; $DQ = 178.479$; $BD = 184.82$; $DP = 145.658$; $QP = 230.3712$ feet.



VIII. QUESTION 1096, by Mr. Alex. Rowe, *Reginnis*.

Suppose the greatest possible square pyramid to be built, having its slant side = 120 feet. It is required to find the side of its base and its perpendicular altitude, as also the number of bricks requisite to build it, the dimensions of a brick, including the joints of mortar, to be 10 inches long, 5 inches broad, and 3 inches thick?

Answered by Mr. Joseph Brewer, Preston.

It is shewn by the writers on Fluxions, that a pyramid is the greatest, when its altitude is one-third the given slant side drawn into $\sqrt{3}$, and the side of its square base two-thirds the slant side drawn into $\sqrt{6}$; therefore $49\sqrt{3} = 69.2829323$ is the altitude, and $80\sqrt{6} = 195.95941794$ is the side of the base, consequently $80^3 \times 6 \times 40\sqrt{3} \div 3 = 80^3 \sqrt{3}$ feet = $960^3 \sqrt{3}$ inches the solid content, this divided by 150, the cubic inches in a brick, gives $64 \times 96 \times 960\sqrt{3} = 10216051\frac{1}{2}$ nearly, the number of bricks required.

The same by Mr. Alex. Rowe, Reginnis, near Pensance.

Put the slant side 120 feet = a , and the altitude = $3x$, then $2\sqrt{(a^2 - 9x^2)}$ = the length of a side of the base; hence $4a^2 - 36x^2$ is the area of the base, and $4a^2x - 36x^3$ is the solidity, a maximum. The fluxion of which being made = 0, and reduced, gives $x = \frac{1}{3}a\sqrt{3}$.

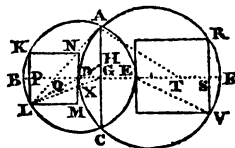
Hence, $3x = \frac{1}{2}a\sqrt{3} = 69.282$ &c. is the altitude, and $\frac{1}{2}a\sqrt{6} = 195.959$, &c. the side of the base. Also the solidity in feet is $886810 = 1532407680$ cubic inches, which divided by $150 (= 10 \times 5 \times 3)$ gives $10216051\frac{1}{3}$, the number of bricks required to build the pyramid.

IX. QUESTION 1097, by Mr. O. G. Gregory, of Cambridge.

ABCE, in the annexed figure, is a circle whose radius is 10; and ED is the radius of another circle ADCF which cuts off half the former circle: It is required to find the distance between the centres of the greatest squares which can be inscribed in the two lunes ABCD and AFCE; also the distance between each of these centres and either horn A or c of the lunes?

Answered by Mr. William Burdon, Acaster Malbis.

Construction. Join AC, cutting DF passing through the centres of the circles in G; make $GH = \frac{1}{2}DG$, and produce HD till it meets the circumference in I; draw LK perpendicular to BD; then will LMNK evidently be the square inscribed in the lune ABCDA. In the same manner may the greatest square be inscribed within the other lune. N. B. The method of inscribing a square within ADCEA is given at quest.394.



Calculation. From the answer to question 964, Ladies' Diary for 1794, the radius ED, to cut off half the given area of the other circle, will be found = 11.587167 . And, by a well known property of the circle, $(ED + DF) \times (BD^2 - DF) = PK^2 = \frac{1}{4}DF^2$, which gives $DF = 7.851978$, the side of the square LMNK. Again, $BE : ED :: ED : EG = 5.713122$, and by 47 Euc. 1, $AQ = \sqrt{(GQ^2 + ED^2 - EG^2)} = 12.840834$, the distance of the centre of the square from the horn A or c. And nearly in the same manner the side of the square inscribed within the lune AECFA is found = $\frac{2}{3}ED\sqrt{5} = 10.363935$, and the distance of its centre from A or c = 15.188468 ; whence the distance between the centres of the squares is 20.595123 .

The same by Mr. Gregory, the Proposer.

It is now pretty well known, or at least may be learned from Martin's Magazine for July 1762, page 898, and the Ladies' Diary 1794, page 42, that $BE : DF :: 1 : 1.15873$, under the conditions expressed in the question; hence EF or ED = 11.5873 , and $BD = 20 - ED = 8.4127$.

Let PK or PQ be denoted by x ; then, by the nature of the circle, as $EP : PK :: PK : PB$, or as $11.5873 + 2x : x :: x : 8.4127 - 2x$, hence, multiplying extremes and means, &c. $x^2 + 1.26984x =$

19.496095742, and $x = 3.825934 = PK = QN$. After a method nearly similar might RS be found by a *simple* equation. But it may be otherwise found by considering that as $2RS = SE$, and E is the centre, the angle VER is such that its cosine is double its sine, and therefore its sine is $\frac{1}{2}\sqrt{5}$, and answers to about $26^\circ 33\frac{1}{4}'$; hence, as radius $1 : \frac{1}{2}\sqrt{5} :: EV = EF : ES = 11.5873 \times \frac{1}{2}\sqrt{5} = 5.1819988$ or 5.182 very nearly $= ST$ or TE . We therefore readily find $GT = GD + DE + ET = 20.595234$, the distance between the centres of the two inscribed squares.

Now it may be easily demonstrated that $BF - DE : DE :: FG - GE$ (or FE) : GE , that is $20 : 11.5873 :: 11.5873 : 6.71325 = GE$; then $BE - GE = 13.28675 = BG$, and GA or $GC = \sqrt{(BG \times GE)} = 9.44443$. Again, $DG = DE - GE = 4.87405$; therefore $GQ = QD + DG = 8.099984$, also $GT = GE + ET = 11.89525$. Lastly, $\sqrt{(GQ^2 + GA^2)} = 12.840735 = AQ$, and $\sqrt{(GT^2 + GA^2)} = 15.188522 = AT$, the distances sought.

X. QUESTION 1098, by Mr. J. Macdonald, of Spindleston.

If, in air of the uniform density of that at the earth's surface, a ball of cork, falling from an infinite height, acquire a uniform velocity of 10 feet per second; it is required to find what is its diameter, the specific gravity of cork and air being 240 and $1\frac{1}{2}$?

Answered by Mr. P. Barlow, Shipdham.

Let $N = 240$, $n = 1\frac{1}{2}$, $v = 10$, $g = 16\frac{1}{12}$ feet, $p = 3.1416$, and x the diameter of the ball in feet. Then, by Dr. Hutton's Course of Mathematics, vol. 2, page 354, $\frac{1}{2}px^3(N - n)$ is the force by which the ball is urged, and $pnv^2x^3 \div 32g$ the resistance it meets with; which are equal when the velocity becomes uniform; therefore $\frac{1}{2}px^3(N - n) = pnv^2x^3 \div 32g$, hence $x = 3nv^2 \div 16g(N - n) = .0059673$ of a foot, or $\frac{1}{14}$ of an inch very nearly; the diameter of the ball.

Answered by Mr John Barron, Spilsby.

Put $N = 240$, $n = 1\frac{1}{2}$, $v = 10$ feet, $g = 16\frac{1}{12}$ feet, $p = 3.1416$, and x the ball's diameter. Then, (by Dr. Hutton's Course, vol. 2, page 354) $\frac{1}{2}px^3(N - n)$ is the force or weight by which the ball is urged, and $pnv^2x^3 \div 32g$ is the resistance it meets with; but these being equal by the quest. reduction gives $x = 3nv^2 \div 16g(N - n) = .0059673$ of a foot, the diameter of the ball required.

The same by Mr. Wm. Francis, jun. Academy, Maidenhead.

Put x for the cork's diameter, $g = 16\frac{1}{12}$, $N = 240$, $n = 1\frac{1}{2}$. Then, by Dr. Hutton's Course, vol. 2, pa. 354, $\sqrt{(49 \times \frac{1}{2}x \times (N - n) \div n)} = 10$. Hence, $64\frac{1}{2} \times \frac{1}{2}x \times 238\frac{1}{2} \div 1\frac{1}{2} = 10^2$, or $x = .00596735$ foot, the cork's diameter.

The same by the Rev. Mr. J. Furness, Ponteland.

Let x = the globe's diameter, $n = 240$, $n = 1\frac{1}{2}$, $v = 10$ feet, $g = 16\frac{1}{2}$, and $p = 3.1416$; but $n - n$ is the relative gravity of the globe in the fluid. Then, by the Select Exercises or the Course, $\frac{1}{2}px^3(n - n)$ is the weight by which it is urged, and $pnv^2x^3 \div 32g$ is the resistance, consequently $\frac{1}{2}px^3(n - n) = pnv^2x^3 \div 32g$ when the velocity becomes uniform; hence, $x = 3nv^2 \div 16g(n - n) = .00596734$ of a foot, is the diameter of the cork.

XI. QUESTION 1099, by Leonardus Pisanus.

To find n , when $n^2 + 13$, and $n^2 - 13$, and n^2 , are all three rational squares.

N. B. This is the same as Question 4 in the Diary for 1800, which is repropoed here, as the solutions that were given to it did not find n rational, as it ought to be.

Answered by the Proposer.

Theorem 1. If d be any number, then $(d^2 + (d + 1)^2) \times ((d + 1)^2 + (d + 2)^2) \pm (d + 1)^2 \times 4$ will be two squares, whose roots are $2d^2 + 4d + 3$, and $2d^2 + 4d + 1$.

Theorem 2. If a and b be any two numbers; then $(a^2 + b^2) \pm (a + b) \times (a - b) \times 4ab$ will be two squares, the roots being $a^2 \pm 2ab - b^2$.

The number 13 is composed of two squares, $4 + 9$, whose roots are 2 and 3 + 1; if therefore we make $d = 2$, the first theorem gives the two squares $13 \times 25 + 36$ and $13 \times 25 - 36$.

Let $c = 13$, $t^2 = 25$, $s^2 = 36$; and put ct^2 for a , and s^2 for b in the second theorem; then $(c^2t^4 + s^4) \pm (c^2t^4 + s^4) \times (c^2t^4 - s^4) \times 4c^2t^2s^2$ will be two squares. But $(c^2t^4 - s^4) \times 4s^2t^2$ will be a square number, because $c^2t^4 - s^4$ (the product of two square numbers) is a square; and consequently

$$\frac{(c^2t^4 + s^4)^2}{(c^2t^4 - s^4) \times 4c^2t^2s^2} = 30 \frac{164568241}{375584400}$$
 is a square num-

ber, and is that required (n^2) ; for $\frac{(c^2t^4 + s^4)^2}{(c^2t^4 - s^4) \times 4c^2t^2s^2} \pm c =$

$$\frac{(c^2t^4 + s^4)^2 \pm (c^2t^4 - s^4) \times 4c^2t^2s^2}{(c^2t^4 - s^4) \times 4c^2t^2s^2}$$
 where the numerator is the two squares as before.

Hence it appears that the question will always admit of an answer when the given number *plus* and *minus* a square, are both squares. If the given number is the difference of two squares, whose sum is a

square, the answer may be found by means of the second theorem only.

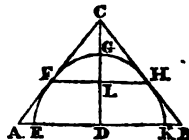
This question is in Lucas de Burgo's *Summa de Arithmetica, &c. an.* 1494. The answer is given without the process. We find two or three other questions of the same kind, but the author has omitted the investigations, except in a particular and very obvious case. His answers however were probably obtained by means of a table of *congruous numbers and squares* (as he calls them): these numbers are such that the latter *plus* and *minus* the former, are both squares: they are readily found by the second theorem above; thus, if $a = 2$, $b = 1$; then 24 and 25 are his first pair; for 25 ± 24 are both squares. If $a = 3$, $b = 2$; then 120 and 169 are two others; &c.

XII. QUESTION 1100, by Mr. T. Hewitt, London.

Given the base and height of a cone, it is required to find the height of the greatest parabolic conoid which can be inscribed in the cone?

Answered by Mr. Wm. Burdon, Acaster Malbis.

The parabolic conoid $EFCHK$, inscribed in the cone ABC , is the greatest when the subtangent CL is $= \frac{1}{2}CD$, by the Scholium to prop. 19, Simpson on the *Maxima et Minima*. Now, by cor. 2, prop. 5, page 105, Dr. Hutton's *Conic Sections*, $LG = GC = \frac{1}{2}CD$, therefore $GD = \frac{1}{2}CD$ the height of the paraboloid. By similar triangles $CD : AD :: CL : LF = \frac{1}{2}AD$; and, by prop. 1, page 98, *Conics*, $\sqrt{GL} : \sqrt{GD} :: FL : ED = \frac{1}{2}AD \sqrt{2}$ the radius of the conoid's base.



Scholium. The solidity of the cone, is to that of its greatest inscribed paraboloid as 9 is to 8.

The same by Mr. J. Chantry, Boston.

Let $AB = b$, $CD = a$, $CG = x$, $GD = a - x$, $FL = y$; then, by the properties of the paraboloid $GL = x$, and $x : y^2 :: a - x : (ay - xy^2) \div x = DE^2$; and, by similar triangles $a : \frac{1}{2}b :: 2x : bx \div a = y$; this substituted for y , gives $(ab^2x - b^2x^2) \div a^2 = DE^2$. Then, the solidity of the conoid $= (ab^2x - b^2x^2) \times 3.1416 \times (a - x) \div 2a^2$ a maximum, or $a^2x - 2ax^2 + x^3$ a maximum, this in fluxions and reduced, gives $x = \frac{1}{2}a$, consequently $GD = \frac{1}{2}a = \frac{1}{2}CD$, or the height of the conoid is $\frac{1}{2}$ of that of the cone.

The same by Mr. Wm. Eaton, jun. Sutton-on-the-Hill.

Put $CD = a$, $AD = b$, and $x = CG = GL$ by the nature of the parabola; by similar triangles $a : b :: 2x : 2bx \div a = FL = cx$ put-

ting $c = 2b \div a$; and by the parab. $x : c^2x^2 :: a - x : ac^2x - c^2x^2 = DE^2$; hence the content is $3.1416 \times (ac^2x - c^2x^2) \times \frac{1}{2}(a - x)$, a maximum, this in fluxions, &c. gives $x = \frac{1}{3}a$; therefore $DE = \frac{2}{3}CD$, when the conoid is the greatest.

The same by Mr. Gregory, Cambridge.

Let ABC represent a section through the axis CD of the cone, and EFB of the paraboloid; we have given the diameter $AB = 2c$ and $CD = a$. Put $DE = x$; then $GL = a - x = CG$ (by theor. 5, on the parabola, Hutton's Conics), theref. $CL = 2a - 2x$. By similar triangles $CD : AD :: CL : FL$, that is $a : c :: 2a - 2x : c(2a - 2x) \div a$, and by the nature of the parab. (Hutton's Theor. 1) $GL : GD :: FL^2 : DE^2$, that is, $a - x : x :: c^2(2a - 2x)^2 \div a^2 : c^2(4ax - 4x^2) \div a^2 = DE^2$. When the paraboloid is the greatest, $DE^2 \times DG$ is a maximum, that is $c^2(4ax - 4x^2) \times x \div a^2$; or (dropping $4c^2 \div a^2$ which is constant) $ax^2 - x^3$ is a maximum, this in fluxions, &c. gives $x = \frac{1}{3}a$, or $GD = \frac{2}{3}CD$ whatever the diameter AB may be.

Remark. Since a paraboloid is half a cylinder of equal base and altitude; the content of the greatest inscribed paraboloid becomes $\frac{1}{2} \frac{4}{3} apc^2$, putting $p = 3.1416$. And it has been proved by various writers, that the greatest cylinder in a cone, is that whose altitude is $\frac{1}{3}$, and diameter of the base $\frac{2}{3}$, those of the cone, and consequently its solidity $(\frac{2}{3}c)^2 \times \frac{1}{3}a \times p = \frac{4}{3} apc^2$. Hence it appears that the greatest paraboloid is just double the greatest cylinder, in the same cone.

XIII. QUESTION 1101, by Mr. John Carr, Giggleswick School.

Mr. Tho. Simpson, in his Miscellaneous Tracts, where he treats of the lunar orbit, assumes the following series, viz. $e \times (1 - b \cos. \beta z - c \cos. \gamma z - d \cos. \delta z \&c.)$ for the reciprocal of the moon's distance from the centre of the earth. Quere, is this assumption admissible as a constant expression for the reciprocal of the distance?

Answered by Mr. John Carr, the Proposer.

To enter fully into the nature of the series, $e(1 - b \cos. \beta z - c \cos. \gamma z - d \cos. \delta z \&c.)$, which Mr. Simpson assumes for the reciprocal of the moon's distance from the earth's centre, would require more room than can be allotted to it in the Diary. I shall therefore briefly answer it in the following manner.

In the assumed series, the quantities $e, b, c, d, \beta, \gamma, \delta$, &c. are all supposed to be constant quantities; it follows therefore, that when z returns to any given magnitude, the reciprocal of the distance, and consequently the distance of itself, will *always* return to a corresponding given magnitude: but this is impossible, on account of the motion of the apoge, if z be meant to express the angular distance of the moon

from a fixed point, which she is supposed to depart from; and if z be supposed to express the angular distance of the moon from the apoge, then it will have a different value from the z used in finding the fluxionary equation of the orbit in the preceding part of the work. These two therefore having different values cannot be properly incorporated as if they had one and the same value, as is actually done in the subsequent part of the work.

On the former supposition therefore the assumed series can express the reciprocal of the distance only for one periodical revolution; and on the latter supposition the whole of the subsequent work must be erroneous.

There is something similar to this at page 162 of that performance; where, in order to express the angular distance of the moon from the sun in terms of z , (that the assumed forces may be compared with the time) the sun and moon are supposed to depart from a given line, *at the same time*: and, in consequence of that supposition, double the angular distance is supposed to be pz ; where $p = 2(1 - m)$, and m the periodical time of the moon in parts of a year. But here again it is plain that pz can express the double angular distance of the moon from the sun only for one revolution; it cannot properly therefore be made the *constant* representative of that distance. It plainly appears then, from these considerations, that the whole of that ingenious, intricate, and elaborate performance, is limited to the ascertaining of the lunar orbit for one revolution of the moon only, from the time of her first departure from the fixed point she is supposed to be set off from, under the circumstances and restrictions mentioned, or understood, in the performance alluded to, and consequently the inference made by the ingenious author, at the close of that work, concerning the return of the equations to the same magnitude cannot be fairly deduced from any thing he has there done.

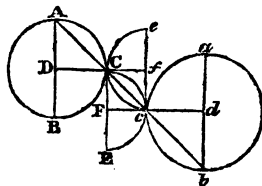
XIV. QUESTION 1102, by Mr. J. G.

It is required geometrically to draw the line of quickest descent between the circumferences of two given circles?

Answered by Mr. J. G. the Proposer.

Let AB and ab be the vertical diameters of the two given circles ACB , acb ; join Ab , cutting the two circles in c and c ; then shall the line cc be the line of quickest descent between the two circumferences.

First, from whatever point c , in the circle ACB , the body begins to descend, the required line cc from that point when produced, must pass through the lowest



point b of the other circle. For through the point c draw the radius dc , produced to meet the vertical line ce in r : then because of the parallels, the two triangles bdc , crc are similar; and because $db = dc$, therefore $rc = rc$; consequently a circle crc , described with the centre r and radius rc , will pass through the point c , and be a tangent to the circle acb at that point, and it is therefore the least circle, having its centre in the vertical ce and any where meeting the given circle acb . But, by mechanics, the time of descent down the chord cc is equal to the time in the diameter ce , which diameter being the least, the time in it and consequently in cc is the least also.

In like manner, to whatever point c , in the lower circle, the body must descend, the line dc produced upwards, must pass through the highest point a ; this is proved in the same way, by drawing dcf from the centre c to meet the vertical line cfe in f , and from the centre f describing the least or tangential circle cce ; then, again, the time in the chord cc is = the time in the diameter ec , which is the least as before. Consequently the line must be drawn from the highest point a to the lowest point b , to have the intercepted part cc passed over in the shortest time.

PRIZE QUESTION, by Amicus.

Suppose a solid, generated by the revolution of a curve about an axis, to be of such a nature, that the solidity of any one cone inscribed in it, so as to stand on the solid's base, and touch its surface, may be the greatest possible; and that when the slant height of such cone is the shortest possible, its length is given. From hence the species, the quadrature, and the rectification of the generating curve, with the cubature and surface of the generated solid, may be found by the help of the conic sections only, and are here required, with the investigation?

Answered by Amicus.

Since every inscribed cone is the greatest possible, its greatest inscribed cylinder must also touch the solid in the same circle parallel to the base. Let y = the radius of this circle; and s = the length of the corresponding cylinder, whose solidity, being constant, make = $4pc^3$, which is therefore = py^3s , or $y^3s = 4c^3$; hence the curve is the second hyperbolium of the parabola, or Newton's 65th species.

Also $y^3 = 4c^3 \div s$, and $y = 2 \sqrt{(c^3 \div s)}$, and $ys = 2s \sqrt{(c^3 \div s)}$, the fluent = $4c^{\frac{3}{2}} s^{\frac{1}{2}}$, or $4c^{\frac{3}{2}} \times (s^{\frac{1}{2}} - d^{\frac{1}{2}}) =$ the quadrature, where d is the value of s at the point where the curve is supposed to begin.

And $py^2s = \frac{4pc^3}{s} \cdot s$, and $4pc^3 \times \text{hyp. log. } \frac{s}{d}$ the cubature.

Again, $y = -\frac{c^{\frac{3}{2}}}{s^{\frac{3}{2}}}$; and $\sqrt{(s^2 + y^2)} = s \sqrt{(1 + \frac{c^2}{s^2})}$ the fluxion of the rectification, its fluent is $s \sqrt{(1 + \frac{c^2}{s^2})} +$ fluent of

$$\frac{3c^{\frac{3}{2}}}{2s^{\frac{3}{2}} \sqrt{(1 + \frac{c^2}{s^2})}}. \text{ Let } 1 + \frac{c}{s} = w; \frac{c}{s} = w - 1; s = \frac{c}{w - 1}; \dot{s} = -\frac{c\dot{w}}{(w - 1)^2}; \frac{3c^{\frac{3}{2}}}{2s^{\frac{3}{2}} \sqrt{(1 + \frac{c^2}{s^2})}} = \frac{3(w - 1)^{\frac{3}{2}} \times c\dot{w}}{(w - 1)^2 \times 2\sqrt{(1 + (w - 1)^2)}} \\ = -\frac{3c}{2} \cdot \frac{(w - 1)\dot{w}}{w^{\frac{1}{2}} \sqrt{(3 - 3w + w^2)}} = -\frac{3c}{2} \times \frac{(w^{\frac{1}{2}} - w^{-\frac{1}{2}})\dot{w}}{\sqrt{(3 - 3w + w^2)}}.$$

This being compared with theorems 21 and 22, table 12 of Landen's

Memoirs, $y = w$; $f = \frac{3}{2}$; $g = \sqrt{3}$; $x = \frac{3}{2} - w + \sqrt{(3 - 3w + w^2)}$; $\frac{a^2 - b^2}{2a}$

$$= \frac{3}{2}; \frac{a^2 + b^2}{2a} = \sqrt{3}; a = \frac{3}{2} + \sqrt{3}; \frac{b^2}{a} = \sqrt{3} - \frac{3}{2}; \text{DP} =$$

$\sqrt{2aw} = \sqrt{(w \times (3 + 2\sqrt{3}))}$: ae = area of the ellipsis adjoining to the conjugate axis, whose transverse axis = $2\sqrt{(a^2 + b^2)}$, and conjugate = $2b$, or whose semitransverse = $\sqrt{(a^2 + b^2)} = \sqrt{3} \times (\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}})$, and semiconjugate = $\frac{1}{2}\sqrt{3} = b$, abscissa $\sqrt{(2\sqrt{3})} \sqrt{(a - x)} = \sqrt{(2\sqrt{3})} \times \sqrt{(\sqrt{3} + w - \sqrt{(3 - 3w + w^2)})}$, ordinate $b \sqrt{(x \div a)} = \sqrt{(x \div (2 - \frac{1}{2}\sqrt{3}))}$; AD = the arc of an hyperbola whose semitransverse axis = a , semiconjugate = b , tangent $DP = \sqrt{(w(3$

$+ 2\sqrt{3}))$, perpendicular $CP = \sqrt{ax}$; fluent of $\frac{w^{-\frac{1}{2}}\dot{w}}{\sqrt{(3 - 3w + w^2)}}$

$$= \frac{2\sqrt{2a}}{b^2} \times (ae + AD - DP), \text{ and of } \frac{w^{\frac{1}{2}}\dot{w}}{\sqrt{(3 - 3w + w^2)}} =$$

$$\frac{2\sqrt{2a}}{b^2} \times (\frac{1}{2}ae + \sqrt{3} \times (AD - DP) + \frac{b^2}{2a} DP); \text{ their difference} =$$

$$\frac{2\sqrt{2a}}{b^2} \times (-\frac{1}{2}ae + (1 - \sqrt{3}) \times (AD - DP) - \frac{2a}{b} \times DP) \text{ is the fluent of}$$

$$\frac{(w^{-\frac{1}{2}} - w^{\frac{1}{2}})\dot{w}}{\sqrt{(3 - 3w + w^2)}}. \text{ But at the beginning of the curve, when } s = d$$

$$\text{and } w = 1 + c \div d, DP = \sqrt{((1 + \frac{c}{d}))} \times (3 + 2\sqrt{3}) = \sqrt{\frac{a}{x}} \times$$

$$\sqrt{(\frac{3}{2} + 3x - x^2)}, \text{ or } (2 + \frac{2c}{d})x = \frac{3}{2} + 3x - x^2, \text{ and } x = \frac{1}{2} -$$

$$\frac{c}{d} + \sqrt{(1 - \frac{c}{d} + \frac{c^2}{d^2})}. \text{ Let the value of } ae \text{ to this value of } x \text{ be} =$$

A, that of $AD = B$, and of $DP = \sqrt{(2a \times \frac{d+c}{d})} = c$; then the corrected fluent of $\frac{(w^{\frac{1}{2}} - w^{\frac{1}{2}}) \dot{w}}{\sqrt{(3 - 3w + w^2)}}$ is $\frac{2\sqrt{2a}}{b^2} \times \left(\frac{A - ac}{2} - (\sqrt{3} - 1) \times (AD - DP - B + c) - \frac{b^2}{2u} (DP - c) \right)$; and the required rectification $= s\sqrt{(1 + \frac{c^2}{s^2})} - d\sqrt{(1 + \frac{c^2}{d^2})} + \frac{3c\sqrt{2a}}{b^2} \times (\frac{1}{2} \times (A - ac) + (1 - \sqrt{3}) \times (AD - B - DP + c) - \frac{b^2}{2a} \times (DP - c))$.

Again, the flux. of the superficies of the solid is $2py\sqrt{(y^2 + s^2)} = \frac{4pc\sqrt{c}}{\sqrt{s}} s\sqrt{(1 + \frac{c^2}{s^2})}$. Let $\frac{c}{s} = u$; then $\frac{c}{u} = s$; $\dot{s} = -\frac{c}{u^2} \dot{u}$;

and $2py\sqrt{(y^2 + s^2)} = 4pc^2 \times -\dot{u}\sqrt{(1 + \frac{1}{u^2})}$, which fluxion being of the very same form as that for the rectification above, its fluent will be found in the very same manner.

Moreover, since $s = \frac{1}{2}$ of the altitude of the cone, and $y = \frac{2}{3}$ of the radius of the cone's base, its slant height $= \sqrt{(9s^2 + \frac{4}{9}y^2)} = \frac{2}{3}\sqrt{(4s^2 + y^2)} = 3\sqrt{(s^2 + \frac{c^2}{s^2})}$, which is a minimum when $2s = \frac{c}{s}$ or $s = \frac{c}{\sqrt{2}}$, the slant height being then $= \frac{3c}{\sqrt{2}}\sqrt{(1 + 2)} = \frac{3\sqrt{3}}{\sqrt{2}} \times c$, which being given by the question, c is given, and consequently all the rest.



Questions proposed in 1803 and answered in 1804.

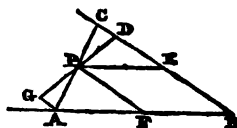
I. QUESTION 1104, by Mr. Henry Rocktree, *Olive Park*.

Required a solution to question 47 of Dr. Hutton's translation of Montucla's *Recreations*, vol. 1, page 426; viz. An angle, and a point within it, being given; required the position of the line passing through the given point, when the rectangle of the sides of the angle cut off by it, towards the vertex, is the least possible?

Answered by Mr. Tho. Hopper, Excise-Officer.

Let abc be the given angle, and p the given point, also are the re-

quired line. Draw FE parallel to AB , and FF parallel to BC , putting $FE = a$, $FF = b$, and $AF = x$; then, by reason of the parallels, $AF : FF :: FE : CE$, or $x : b :: a : ab \div x = CE$, then $EC = b + ab \div x$, and $AB = a + x$, hence $AB \times BC = a$ minimum, this in fluxions and reduced gives $x = a$ that is $AF = FF = FB$.



The same Geometrically, by Mr. Henry Lightbown.

It is well known that when the vertical angle is given and the rectangle of the sides viz. $AB \times BC$, a minimum, that the triangle ABC cut off by the said lines is a minimum also, and this will be the case when the given point F bisects the line AC , by Simpson on the Maxima and Minima, theorem 8. Hence this construction. Make ABC the given angle, and F the given point, draw FF parallel to BC , and set off $FA = FB$, join AF , which produce to C , and APC is the line required. For, by parallel lines, $AF : FB :: AP : FC$, a ratio of equality.

The same by Mr. John Smith, of Alton Park.

Let ABC be the given angle, and F the given point. Draw FF parallel to CB , make $AF = FB$, and from A through F draw APC , which will be the line required. For AC is bisected by the point F (Hutton's Geom. theorem 82), therefore the triangle ABC will be the least that can be formed by the angle ABC and a line passing through F (by Simpson on the Maxima and Minima, theorem 8), consequently the rectangle of its sides, $AB \times BC$ will be the least also (Hutton's Geom. cor. 4, theor. 81).

The same by Mr. Tho. Smith, Hethersett.

Let ABC be the angle, and F the point. Draw FD to any point in BC , and produce DF to G making $FG = FD$; draw GA parallel to BC , and through A and F draw APC which will be the line required. For the two triangles APG , CPD are mutually equal, and the line $AF = FC$, or AC is bisected in F . And, by Simpson on the Maxima and Minima, the area of the triangle ABC is least, when the base AC is bisected by the given point; and it is well known that $AB \times BC \times \frac{1}{2} \sin. \angle B$ is the area of the triangle $ABC = a$ min. but the $\angle B$ is constant; therefore $AB \times BC$ is a minimum also.

II. QUESTION 1105, by Mr. Wm. Maddocks, Gravesend.

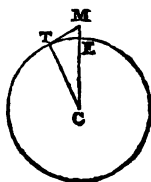
A gentleman has in his garden a stone, in the form of an oblong spheroid, the transverse diameter of which is 100 inches, and the conjugate 70, which he is desirous of having made into the largest rolling stone that it will admit of; but the stone mason, being unskilled in

III. QUESTION 1106, by Mr. Tho. Squire, Baldock.

The pike of Teneriffe lying in north lat. $28^{\circ} 29'$; now on Midsummer Day 1800, the sun was observed to rise 11 minutes $28\frac{1}{2}$ seconds sooner on the summit of the mountain than on the plain below; on this observation it is required to compute the height of the mountain, allowing for the sun's semidiameter, refraction, and parallax; and taking the earth's diameter at 7964 miles?

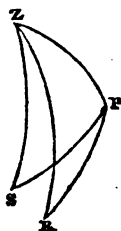
Answered by Mr. H. Codling, South Walsham.

The near time of the sun's rising at Teneriffe, on June 24, is nearly at 5 o'clock; at which time the declination is $23^{\circ} 26' 45''$, the semidiameter $15' 49''$, and refraction $33' 45''$; from the sum of these two take $10''$ the horizontal parallax, there remains $49' 24''$ the depression of the sun's centre at the time of observation on the plane. Then, in the opposite hemisphere are given the sun's altitude $49' 24''$, the polar distance $113^{\circ} 26' 45''$, and lat. $28^{\circ} 29'$, hence the time of the day is found $5^h 1' 22\frac{1}{2}''$ from which take $11' 28\frac{1}{2}''$ and the diff. is $4^h 49' 54\frac{1}{2}''$ the time of observation on the summit of the mountain. Then, by another common prob. the depression at this time is found $= 3^{\circ} 2' 29''$; from this taking the former depression, the remainder $2^{\circ} 13' 5''$ is the angle c at the centre of the earth, m denoting the top of the mountain, and mt the tangent or visual line. Hence, in the right-angled triangle ctm , besides the angle c , there is given the leg $ct = cz = 3982$ miles, then cm is found $= 3985$ nearly; consequently $mx = 3$ miles, the height of the mountain.



The same by Mr. John Cullyer, Wicklemood.

In the spherical triangles zps , zpr , we have $zp = 61^{\circ} 31'$ the co-lat. and $ps = pr = 66^{\circ} 34'$ the co-declination and $zs = 90^{\circ} 33'$ the zenith distance, allowing $33'$ for the difference between the sun's horizontal refraction and parallax; hence is found the $\angle zps = 104^{\circ} 18' 16''$ the angle of time at sun-rise; to which add $2^{\circ} 52' 6'' = 11$ min. $28\frac{1}{2}$ sec. gives $107^{\circ} 10' 22'' = \angle zpr$, the angle of time at sun-rise on the mountain. Then, in the triangle zpr , are given zp and pr as before, and the $\angle zpr$, to find $zr = 92^{\circ} 46' 40''$, the sun's zenith distance at the time he rises on the mountain; from which subtracting $90^{\circ} 33'$, leaves $2^{\circ} 13' 40''$ the angle at the earth's centre subtended by the tangent line, or visual ray; then by plane trigon. is found 3.013 miles nearly, the height of the mountain.

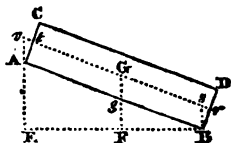


IV. QUESTION 1107, by the Rev. J. Furnass, Ponteland.

Two person's, A and B, found a square ingot of pure silver, being 40 inches in length, and 8 inches on the side, which they carried home between them, each extremity resting on their shoulders; and then agreed to share it in proportion to the weight or pressure sustained by each, the height of A's shoulder being 5 feet, and that of B's 4 feet; required the value each man's share, at the rate of five shillings the ounce?

Answered by Mr. O. G. Gregory, Woolwich.

Let ABCD represent the ingot of silver; resting on A's shoulder at the point A, and on B's shoulder at the end B, the difference of height's being $AE = 12$ inches, and the length or hypotenuse $AB = 40$ inches; hence $EB = \sqrt{(AB^2 - AE^2)} = 4\sqrt{91} = 38.157568$. Now the weights borne at B and A, are as EF and FB, (GF being the perpendicular let fall from the centre of gravity) or by reason of the similar triangles EGF , BAE , and the parallels AB , vs , they are as Ag , gB or as vg , gs . The equal triangles Avt , BSr , are both similar to AEB , hence $BE : AE :: BR : rs :: At : tv$, that is $4\sqrt{91} : 12 :: 4 : 1.257942$. which added to, and taken from 20 or $\frac{1}{2}tr$, gives $21.257942 = vg = Ag$, and $18.742058 = gs = gB$. These are the respective lengths of the two pieces into which the ingot is divided, each of which multiplied by 8^2 or 64, the area of the base, gives 1360.508288 inches for the greater part, and 119.9491712 for the less. Then, taking 11091 for the specific gravity of pure silver, we find 8732.290175 and 7704.607973 for the weights in avoirdupois ounces, and the ounce avoirdupois being to the ounce troy, as 175 to 192, those numbers being reduced in this proportion give 7959.118649 and 7022.429142 troy ounces; which at 5s. the ounce, produce £1989.779662 and £1755.607285, or 1989*l.* 15*s.* 7*d.* and 11765*l.* 12*s.* 1*d.* for the respective values of the shares of B and A.



V. QUESTION 1108, by Mr. Wm. Eaton, jun. Sutton-on-the-Hill.

D, aged 66 years, is heir to an estate of £120 per annum, at the decease of C, aged 73, if it should happen that C survives A and B, who are now in possession for their lives, and of the ages of 68 and 50. Now D wishes to part with his interest in the estate, for an annuity for his life; what ought this to be, allowing interest at 5 per cent. also supposing the decrements of life to be equal, and the extremity of old age 86?

Answered by Mr. Wm. Eaton, jun.

First, to find the present value of D's interest according to the proposed order of survivorship, let $t = 36$, $m = 20$, $n = 18$, and $v = 13$, be the several complements of life, also $r = 1.05$, $p = 20$ the perpetuity, $mv = 6.6614$, and $nv = 6.355$ the value of an annuity certain for v years. if the lives (whose comps. are m and n) shall continue so long: Then, by what is done at question 56, vol. 3, Dodson's Mathematical Repository, the present value will be found

$$(mv - \frac{v^2}{6rm}) \times 2m - (nv - \frac{v^2}{6rn}) \times (2m - v) + \frac{v^2}{6rn} \times (n - v) \times 2 \times$$

$\frac{p}{4mt} = .66912$, which multiplied by 120 gives 80.4*l.* the present worth.

Next, to find what annuity 80.4*l.* will purchase for his life: the present value of a life of 66, is found to be 6.9 years purchase; therefore $80.4 \div 6.9 = 11.65 = 11*l.* 13*s.*$ is what D ought to receive yearly.

VI. QUESTION 1109, *by Mr. Rd. Smithson, Upleatham.*

There is a field CBD, limited by a curve CB, which is the involute of a circle, whose radii DC, DE form an angle CDE of 145 degrees; DC is equal 12 chains, and BE is a tangent to the generating circle, at right angles with DE: hence the length of the curve and area of the field are required?

Answered by Mr. J. Chantry, Boston.

By the common rules, the whole circumference of the circle, to the radius 12, is 75.3984; therefore as $360^\circ : 145^\circ :: 75.3984 : 30.368 = BE = CE$. Then, by *Simp. Flux.* p. 163, $BE^2 \div 2CD = 38.425$ links, the length of the curve CB; and, by page 131 *ib.* $BE^3 \div 6CD = 38$ acres 3 r. 23½ p. the area of the field CBD.



The same by Mr. Henry Hann, Saltwick.

As $360^\circ : 145^\circ :: 24 \times 3.1416 : 30.3688$ the length of the arch $CE = EB$. Again, by article 139, *Simp. Flux.* the length of the curve $CB = BE^2 \div 2CD = 38.46929$; and, by article 121 of the same, the area of CBD is $= BE^3 \div 6CD = 389.42208$ square chains = 38 acres 3 r. 30 p.

The same by Mr. Tho. Hornby, of Wombledon.

By proportion, as $1 : 3.1416 :: 24 : 75.3984$ chains, the whole circumference of the circle and as $360^\circ : 145^\circ :: 75.3984 : 30.3688$

the length of the arch $cs =$ the tangent es ; then, by Simp. Flux. articles 139 and 121, $es' \div 2cd = 38.4276$ chains, the length of the arch cs and $es' \div 6cd = 38$ ac. 3 r. 24 p. the area of the field cds .

VII. QUESTION 1110, *by Mr. Wm. Francis, jun. Maidenhead.*

A clock maker having put a new pendulum to an old time-piece, found it gained 5 minutes in 12 hours; but after he had lengthened it two inches, it lost five seconds per hour: what ought to be the true length of the pendulum, and how often will it vibrate per minute?

Answered by Mr. Peter Barlow, Shepdlham.

The numbers of vibrations made by the pendulum, before and after it was lengthened, are as 3625 to 3595; and, by the law of pendulums, the numbers of vibrations made in equal times, are inversely as the square roots of their lengths; therefore calling the first length x , as $\sqrt{x} : \sqrt{(x+2)} :: 3595 : 3625$; hence $x = 119.33541$ inches. Then $3600^2 : 3625^2 :: 119.33541 : 120.9986$ the true length of the pendulum; and $\sqrt{39\frac{1}{8}} : \sqrt{120.9986} :: 60 : 34.118$, the number of vibrations per minute.

The same by Mr. Francis, jun. the Proposer.

The pendulum vibrated $\frac{1}{144}$ too often at first, and $\frac{1}{720}$ too few when lengthened. Put $x^2 =$ its length in inches, and $y^2 = x^2 + 2$. Then as $\sqrt{x^2} : 60 :: \sqrt{39\frac{1}{8}} : \frac{375.3}{x}$. Also as $\sqrt{y^2} : 60 :: \sqrt{39\frac{1}{8}} : \frac{375.3}{y}$. But $\frac{375.3}{x} \times \frac{143}{144} = \frac{375.3}{y} \times \frac{719}{720}$. Hence $y = 1.00839x$, or $y^2 = 1.01685x^2 = x^2 + 2$; consequently $x^2 = 118.66887$ inches, the first length; and theref. it made 34.45167 vibrations, when it should have made only 34.2124 per minute. Lastly, as $34.2124^2 : 60^2 :: 39\frac{1}{8} : 120.33423$ inches, the true length required.

The same by Mr. J. H. Hearing, Adderbury.

If a pendulum gain 5 minutes in 12 hours, it will gain 25 seconds per hour; therefore putting x for the first length of that required; and $n = 3600$, by the nature of pendulums $x : x+2 :: (n-25)^2 : (n+5)^2$, hence $x = (n-25)^2 \div (30n-30) = 118.6687$ inches; put this $= p$, and let $y =$ the increment to be added, to correct its motion, then as $p : p+y :: (n-25)^2 : n^2$, consequently $p+y = pn^2 \div (n-25)^2 = 120.3346$ the true length of the pendulum, then as $\sqrt{120.33426} : \sqrt{39\frac{1}{8}} :: 60'' : 34.2124$ the number of vibrations per minute.

The same by Mr. Sam. Jones, of Irondish.

As the first length of pendulum gained 25 seconds and the latter lost 5'' per hour, the former gained of the latter $\frac{1}{144}$ every vibration; therefore as $120'' : 121'' :: x : x + 2$, putting x for the first length; hence $x = 119.5$ inches. Again, as the pendulum gained 25 seconds per hour, it gained $\frac{1}{144}$ every vibration, then as $145'' : 144'' :: 119.5 : 121.1177$ inches, the true length; and as $\sqrt{121.117} : \sqrt{391} :: 60 : 34.1016$ the number of vibrations per minute.

VIII. QUESTION 1111, *by Mr. M. H. Murck.*

To find two numbers with these properties, that if unity be added to each, they shall be squares; also if unity be added to their sum or difference, they will make two other squares?

Answered by Mr. John Cornthwaite, Garstang.

Put $x^2 + 2x$ and $x^2 - 2x$ = the two numbers, which answer the first two conditions; for $x^2 + 2x + 1$ and $x^2 - 2x + 1$ are both squares. It remains then to make $4x + 1$ (the difference + 1) and $2x^2 + 1$ (the sum + 1) squares. Let $4x + 1 = p^2$, this gives $x = (p^2 - 1) \div 4$, hence $2x^2 + 1 = (2p^4 - 4p^2 + 18) \div 16$ a square, or $2p^4 - 4p^2 + 18$ a square. Put $p = r + 1$, then $2p^4 - 4p^2 + 18$ becomes $2r^4 + 8r^3 + 8r^2 + 16$; assume it equal $(r^2 + 4)^2 = r^4 + 8r^2 + 16$, then $r^4 = -8r^3$, and $r = -8$. Hence $p = r + 1 = -7$, and $x = 12$; then the two numbers are 168 and 120.

The same by Mr. John Cullyer, Wicklowood.

Let $x^2 + 2x$ and $x^2 - 2x$ be the two numbers; then by the quest. $2x^2 + 1$ and $4x + 1$ must be two squares, which put $= y^2$ and z^2 , hence $x = \frac{1}{2}z^2 - \frac{1}{2}$, and $2x^2 - 4x = z^2 - x^2$. Now put $y = x + v$, and $z = x - v$, then we have $2x^2 - 4x = 4vx$, or $x = 2v + 2 = \frac{1}{2}z^2 - \frac{1}{2}$, and $y = 3v + 2$, and $z = 2 + v$ and $x = 2v + 2 = (z^2 - 1) \div 4 = (v^2 + 4v + 3) \div 4$, whence $v^2 - 4v = 5$, which gives $v = 5$, and $x = 12$, whence $x^2 + 2x = 168$, and $x^2 - 2x = 120$, the two numbers sought.

The same by Mr. J. Lockwood.

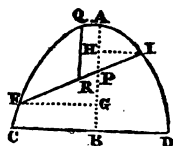
Let $x^2 + 2x$ and $x^2 - 2x$ represent the required numbers, which it is evident answer two of the conditions in the question; there now only remains to make $2x^2 + 1$ and $4x + 1$ each a square. Now put this last $= n^2$ then $x = (n^2 - 1) \div 4$, hence the first becomes $(n^4 - 2n^2 + 9) \div 8$ a square number, which it will be when $n = 7$; hence $x = 12$, consequently the two numbers are 120 and 168, which answer all the conditions.

IX. QUESTION 1112, by Mr. Wm. Burdon, of Acaster Malbis.

A gentleman has a garden in the form of a parabola, the base or double ordinate of which is 100 feet, and the abscissa or height 200 feet. A straight walk, which goes obliquely across the garden, terminating in the curve on both sides, bisects the abscissa, and the length of the walk on the upper part, is to that on the lower, as two is to three. Required the area of the two parts the garden is divided into by the said walk?

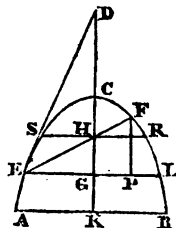
Answered by Mr. O. G. Gregory.

Since $FF' : PI :: 3 : 2$, it is plain by similar triangles, FG to HI , and FG to PI are also in the same ratio of 3 to 2, so that if $PH = 2x$, then will $FG = 3x$. Also if $AB = 200 = 4a$, then $AP = PB = CB = 2a$, and $CB = a$. Hence, by the nature of the parabola, $\sqrt{AB} : BC :: \sqrt{AH} : HI$, or $\sqrt{4a} : a :: \sqrt{(2a - 2x)} : \frac{1}{2}\sqrt{(2a^2 - 2ax)}$, hence $2\sqrt{(2ax + 3x)} = 3\sqrt{(2a - 2x)}$, which gives $x = \frac{1}{3}a = 16\frac{2}{3}$; hence $AH = 66\frac{2}{3}$, and $AG = 150$. Now $\sqrt{AB} : BC :: \sqrt{AG} : FG$, that is $\sqrt{200} : 50 :: \sqrt{150} : 25\sqrt{3} = FG$; hence $FP = \sqrt{(FG^2 + GP^2)} = 25\sqrt{7}$, and $PI = \frac{2}{3}FP = \frac{50}{3}\sqrt{7}$; therefore $FI = \frac{13}{3}\sqrt{7}$ the base of the parabolic segment FAI . By Dr. Hutton's Conics, pr. 19, Parab. $FP \times PI : (\frac{1}{2}FI)^2 :: AP : QR$ the absciss of the ordinate FI , that is $3 \times 2 : 2\frac{1}{3} \times 2\frac{1}{3} :: 100 : \frac{2500}{4} = QR$; and $FG \div FP = 25\sqrt{3} \div 25\sqrt{7} = \sqrt{\frac{3}{7}} = \frac{1}{7}\sqrt{21}$ the nat. sin. $\angle P$ or $\angle R$; therefore, by the rule at p. 277, Dr. Hutton's Mensuration, 3d edition, we have $\frac{2}{3} \times \frac{13}{3}\sqrt{7} \times \frac{2500}{4} \times \sqrt{\frac{3}{7}} = \frac{78125}{2}\sqrt{3} = 5011\frac{1}{2}72106$ the area of the segment FAI ; also $\frac{2}{3} \times 200 \times 100 = 13333\frac{1}{3}$ is the area of the whole CAD ; consequently their diff. $8321\cdot61227$ is the area of the frustum $CFID$.



The same by Mr. Tho. Hornby, of Wombledon.

Let ABC denote the garden, EF the walk, SR a double ordinate bisecting the absciss CK in H , and ED a tangent to the curve at the point E . Then, by prop. 15, Dr. Hutton's Conics, as $2 : 3 :: CH = 100 : CD = CG = 150$; again, by prop. 1, as $CK : CG :: AB^2 : EL^2$, or $\sqrt{200} : \sqrt{150} :: 100 : 50\sqrt{3} = EL$; and by similar triangles as $EH : EF :: 3 : 5 :: \frac{1}{5}EL$ or $25\sqrt{3} : EP = \frac{1}{3}\sqrt{3}$; but $\frac{2}{3}CG \times EL = \frac{2}{3} \times 150 \times 50\sqrt{3} = 5000\sqrt{3}$ = area of ECL , and as $EL^3 : EP^3 :: 6^3 : 5^3 :: 5000\sqrt{3} : \frac{1}{2}\sqrt{3} \times 5000\sqrt{3} = 5011\cdot72$ the area of ECF ; but $\frac{2}{3}CK \times$



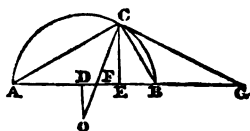
$AB = 13333.33 =$ the area of the whole field ABC ; their dif. 8321.61 is the lower oblique frustum $AEFB$.

X. QUESTION 1113, by Mr. Wm. Cole, Colchester.

In a plane triangle are given, the vertical angle, the base, and the sum of the perpendicular and one segment of the base a maximum; to construct the triangle?

Answered by Mr. Wm. Cole, Colchester.

On the given base AB describe a segment of a circle to contain the given vertical angle. From the centre o draw OD , perpendicular to AB ; draw OC , making the angle $DOC =$ half a right angle, and meeting the circle in C . Join A, C , and B, C , so shall ABC be the triangle required.



For, let fall the perpendicular CE , and in AB produced take $EG = EC$. Draw CG , OA , and OB . Then the base AB , and the vertical angle ACB , are equal to those given, by construction; and because OD is perpendicular to AB , $CE = EG$, and DOC half a right angle, the angles F, FCE, ECG, EGC are each half a right angle. Therefore OCG is a right angle, and $AG (=$ the sum of the segment and perpendicular by construction) is evidently a maximum when CG touches the circle.

The same by Mr. Tho. King, Basseldon.

On the given base AB describe a segment of a circle to contain the given vertical angle. From the centre o draw the perpendicular OD on the given base, and $DF = DO$, then join OF , which produce to the circle at c ; draw AC and BC , so shall ABC be the triangle required.

For, let fall the perpendicular CE , and in AB produced take $EG = EC$, and draw CG . Now the base AB and the vertical $\angle ACB$ are equal to those given by the construction. And because $DF = DC$, and $CE = EG$, the triangles ODF, FCE, ECG are all similar; and the angles FCE, GCE are each half a right angle, therefore OCG is a right angle; also $AE + EC = AG$, which is evidently a maximum when CG touches the circle.

XI. QUESTION 1114, by the Rev. Thomas Scurr, Hexham.

Required the dimensions of a cylindrical vessel, open at top, capable of containing 56400 cubic inches, so that it may be lined with lead of a given thickness at the least possible expence. Also the vessel being kept always full by a supplying cock at the top, and two circular holes, each half an inch in diameter, being open in its perpen-

The same by Mr. John Smith, Alton Park.

When a vessel is such that it may be lined with lead, of a given thickness, at the least possible expence, the difference of the solids made up of the external and internal dimensions of the lining, must be a minimum. (See question 62, vol. 1, pa. 166). But if we suppose the lining to be very thin, the internal superficies may be the minimum; and then, the vessel being cylindrical, it is well known that the diameter must be double the depth; whence, by similar solids, as $2^2 \times .7854 : 1^2 :: 56400 : 17952.6356$; the cube root of which is 26.1844 the depth, consequently 52.3688 is the diameter.

Now c and d being the two holes, at the respective distances of 24 and 12 inches from the bottom, or 2.1844 and 14.1844 from the top; then (by Dr. Hutton's Maths. vol. 2, schol. to pr. 61), $BF = 2\sqrt{(AC \times CB)} = 14.4811$ inches, and $BC = 2\sqrt{(AD \times DB)} = 26.0931$ inches, the distances at which the water will fall on the horizontal plane.

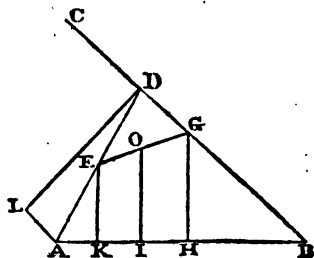
Lastly, let $h = 2.1844$, the height of the water above the upper hole, $a = .1309 = \frac{1}{3}$ of its area, and $g = 193$ inches; then (ibid. cor. 3,) $2a\sqrt{gh} = 5.375434$ cubic inches, is the quantity discharged per second; and this mult. by 5 min. or 300 sec. gives $1612.6302 = 5.7185$ gallons, the quantity discharged by the upper orifice in 5 min. And (ibid. cor. 1,) the quantities discharged at different depths below the surface, are as the square roots of the depths, therefore as $\sqrt{2.1844} : \sqrt{14.1844} :: 5.7185 : 14.572$ gallons, the quantity discharged by the lower orifice in the same time.

XII. QUESTION 1115, by Mr. Da. Henry, Preston Lancashire.

Let AD and BDC represent two beams of timber, of given dimensions and quality, standing on a horizontal plane, at a given distance AB , and to lean against each other at D , but moveable about the lower ends A and B : it is required to determine their positions when in equilibrio?

Answered by Mr. Gregory.

Let EG which joins the centres of gravity of the given beams be divided in O , so that EO may be to $OG :: w : w$, w being the weight of the beam BC , and w that of AD ; then O will be the common centre of gravity, and when the system is in equilibrio will be at the greatest distance possible from the horizontal line AB . Draw EK , OI and GH perpendicular to AB , and because $EO : OG :: w : w$, it is evident that



$$OI \text{ is } = \frac{w \cdot EK + w \cdot HG}{w + w};$$

therefore $w \cdot EK + w \cdot HG$ must be a maximum.

Put $a = AB$, $b = AD$, $c = AE$, $d = EG$, then

$$EK = c \sin A, GH = d \sin B;$$

therefore $wc \sin A + wd \sin B$ a maximum, and taking fluxions

$$\frac{\dot{A}}{B} = - \frac{wd \cos B}{wc \cos A}.$$

Again $b : a :: \sin B : \sin (B + A)$ or $\sin D$,

$$\text{or } a \sin B = b \sin (B + A),$$

and taking fluxions

$$\frac{\dot{A}}{B} = \frac{a \cos B}{b \cos (A + B)} - 1.$$

$$\text{Therefore } - \frac{wd \cos B}{wc \cos A} = \frac{a \cos B}{b \cos (A + B)} - 1,$$

$$\text{or } wdb \cos B \cos D = wc \cos A (a \cos B + b \cos D).$$

Now put $x = BD$, then

$$\cos A = \frac{a^2 + b^2 - x^2}{2ab}$$

$$\cos B = \frac{a^2 + x^2 - b^2}{2ax}$$

$$\cos D = \frac{b^2 + x^2 - a^2}{2bx}$$

$$a \cos B - b \cos D = x,$$

these values being substituted in the preceding equation, we have

$$wdb (a^2 + x^2 - b^2) (b^2 + x^2 - a^2) = 2acx^3 (a^2 + b^2 - x^2)$$

$$\text{or } wdb \{x^4 - (a^2 - b^2)x^2\} = 2acx^3 \{a^2 + b^2 - x^2\}.$$

From this equation x may be determined and the problem is then resolved.

If instead of finding the relation of the angles we put y for the perpendicular drawn from D upon the base AB , we have

$$w \cdot EK + w \cdot HG = \frac{wcy}{b} + \frac{wdy}{x}$$

$$\text{or } y \times \left\{ \frac{wc}{b} + \frac{wd}{x} \right\} \text{ a maximum.}$$

But, by a well known property of triangles,

$$y = \frac{\sqrt{\{4a^2b^2 - (a^2 + b^2 - x^2)^2\}}}{a}$$

$$\text{therefore } \sqrt{\{4a^2b^2 - (a^2 + b^2 - x^2)^2\}} \times \left\{ \frac{wc}{b} + \frac{wd}{x} \right\} \text{ a max.}$$

taking fluxions and reducing we shall obtain the same equation as before.

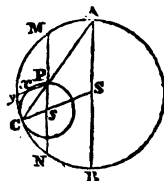
XII. QUESTION 1116, by Mr. J. G.

To find the position of the line of quickest descent from the circum-

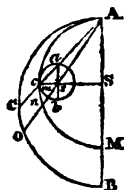
ference of one circle to that of another, the former circle lying wholly within the latter?

Answered by Mr. J. G. the Proposer.

Lemma. The line of quickest descent from any point within a circle to the circumference of that circle, passes through the superior extremity of the diameter perpendicular to the horizon. For, let P be the point, AB the vertical or perpendicular diameter. Join AP , producing it to c ; through P draw MPN parallel to AB ; join c, s , the centre of ABC , by a line cutting MPN in s . Now by similar triangles psc , asc , $As : sc :: ps : sc$; but $As = sc$, therefore $ps = sc$; consequently a circle described with ps radius will touch ACB in c , and therefore pc is the line of quickest descent: For if any other line pxy be drawn cutting the two circles in x and y ; then the time in $px =$ time in pc , therefore time in py is greater than the time in pc .



Solution. To find the line of quickest descent from the circle acb to the circle ACB . Let AB, ab be the vertical diameters. By the lemma, whatever point be taken in the circumference acb , the line of quickest descent from that point to the circumference ACB passes through A . Draw $Aacc$, cutting the circles in c and c , then cc is the line required. For through c and s draw css meeting AB in s . By similar triangles Asc , asc , $As : sc :: as : sc$, therefore $As = sc$; therefore a circle described with sc as radius will touch ACB in A , and acb in c , and cc is the line of quickest descent from acb to ACB . For draw from A , any other line Amn , cutting the three circles in m, n, o ; then, by a former prob. time in $cc =$ time in no , or less than the time in mo .



XIV. QUESTION 1117, by Mr. O. G. Gregory, Cambridge.

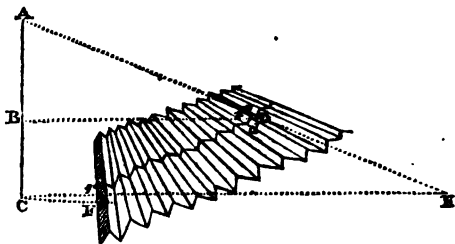
At Peterborough, in North latitude $52^{\circ} 36'$ there is a flight of 20 steps which front the South, the breadth of each step being 12 inches, and the height of each 6 inches; and at the distance of 6 feet southward of the lowest step there is a pillar, the altitude of which is 12 feet: it is required to determine the length from the foot of the pillar to the extremity of its shadow, on Feb. the 4th, 1803, at 11 o'clock A. M. (the sun then shining), and to shew what portion of that length will be found on the steps?

Answered by the Proposer.

In solving this question it will be necessary in the first place to de-

termine the sun's altitude and azimuth, at 11 o'clock A. M. on Feb. 4, 1803; when the sun's declination is $16^{\circ} 25\frac{1}{2}' S.$ that at noon being $16^{\circ} 24' 44'' S.$ Here we shall have, in an oblique spherical triangle, two sides and the included angle; namely, one side equal to $37^{\circ} 24'$, the colatitude, another equal to $106^{\circ} 25\frac{1}{2}'$ the sun's north polar distance, and the included angle equal to 15° , the hour angle from noon, to find another angle, namely $164^{\circ} 42' 17''$ the sun's azimuth from the north, and the third side $70^{\circ} 14' 15''$ the sun's co-altitude.

Then in the marginal figure, where AC represents the pillar standing, perpendicularly, we have in the triangle ACE right-angled at C , $AC = 12$ feet, and the angle $A =$ the sun's co-altitude, whence we obtain 33.3998519 for



CE the length of the shadow, supposing it to fall upon the horizontal line. By the question we have given cr , the nearest distance from the foot of the pillar to the bottom step, $= 6$ feet; and the shadow cr is manifestly the hypotenuse of a right-angled triangle crr of which the angle c is equal to $15^{\circ} 17' 43''$ the supplement of the azimuth, and as those portions of the shadow which fall upon the horizontal parts, or treads, of the steps, will all lie parallel to cr , we shall have

$$\text{rad.} : \sec. 15^{\circ} 17' 43'' :: \left\{ \begin{array}{l} 6 : 6.2203364 = cr, \\ 1 : 1.0367217 = id, \end{array} \right\} \text{ the length of}$$

the shadow on the face, or tread, of a step. From sim. tri. we have $AC : CE :: AB : BD$. Now, by supposing the line AE terminating the shadow to coincide with d , the bottom of the 12th step, we have $AB = 6\frac{1}{2}$, and using the above proportion we find BD comes to *more* than $cr + 11id$; and supposing AE to coincide with e the top of that step, then $AB = 6$, and BD comes out *less* than $cr + 11id$. Hence then it appears that the top of the shadow will fall on the vertical line, or riser, between the 11th and 12th step; therefore BD will be equal to $6.2203304 + 1.0367217 \times 11 = 17.6242683$. And we have $CE : AC :: BD : BA = 6.3321077$ feet; this taken from 12, leaves 5.6678923, for the sum of the portions of the shadow found upon the vertical parts, or risings, of the first 12 steps, these parts of the shadow being obviously parallel to ac : of this we have $.1678923 = dd$, the portion on the riser of the 12th step. Again $1.0397217 \times 11 = 11.4039387$ sum of the portions of the shadows found upon the horizontal parts, or treads, of the steps. Theref. $5.6678923 + 11.4039387 = 17.071831$ feet, length of shadow upon the steps; which added to $6.2203304 (= cr)$ gives 23.2921614 feet, distance, measuring along the steps, from the foot of the pillar to the extremity of the shadow.

N. B. In this solution no notice has been taken of refraction, or of the sun's semidiameter and parallax; had these been taken into the

account ce would have been shorter—and d might probably have fallen upon the face of the 11th step. They were omitted in order to shew the nature of the calculation when the extremity of the shadow was found on the rising part of the step.

PRIZE QUESTION, *by Amicus.*

The prime number 13 has the following twelve remarkable properties: there is a rational square number, whose root I call No. 1, and if 13 be both added to, and subtracted from this square, the sum and difference will be two other rational square numbers whose roots I call No. 2, and No. 3; there is a square integer, whose root I call No. 4, and if this square be multiplied by 13 the product will be the area of a right-angled plane triangle, whose sides are three integers, which I call Nos. 5, 6, and 7. The number 13 may be divided into two rational squares, whose roots I call No. 8, and No. 9, whose double rectangle is a square, whose root I call No. 10, and if this square be both added to, and subtracted from 13, there will thus be produced two other rational squares, whose roots I call No. 11, and No. 12: What are these twelve numbers?

Answered by Amicus, the Proposer.

Since the first square must necessarily be a fraction, let its root or No. 1, be denoted by $\frac{a^2+b^2}{q}$, and Nos. 2 and 3 by $\frac{2ab+a^2-b^2}{q}$ and $\frac{a^2-b^2-2ab}{q}$, so that $\frac{(a^2+b^2)^2}{q^2} + 13 = \frac{(2ab+a^2-b^2)^2}{q^2}$, and $\frac{(a^2+b^2)^2}{q^2} - 13 = \frac{(a^2-b^2-2ab)^2}{q^2}$. The diff. of these two equations

gives $13q^2 = 4ab(a^2 - b^2)$, and $\frac{1}{4}$ of $13q^2 = ab(a^2 - b^2)$ is the area of the right-angled triangle whose legs are $a^2 - b^2$ and $2ab$, and hypotenuse $a^2 + b^2$, which are Nos. 5, 6, 7, and $\frac{1}{2}q =$ No. 4. In the equation $13q^2 = 4ab(a^2 - b^2)$, for a and b write r^2 and s^2 , then

$\frac{13q^2}{r^2s^2} = 2(r^2 + s^2) \times 2(r^2 - s^2)$. Let $2(r^2 - s^2) = 16v^2$, then $r^2 - s^2 = 8v^2$, $r = 2v^2 + 1$, $s = 2v^2 - 1$, $r^2 = 4v^4 + 4v^2 + 1$, $s^2 = 4v^4 - 4v^2 + 1$, $r^2 + s^2 = 8v^4 + 2$, $r^2 - s^2 = 64v^2 + 16v^2 = 16v^2(4v^2 + 1) = 13t^2 = \frac{13q^2}{4r^2s^2}$, $\frac{13t^2}{16v^2} = 4v^2 + 1$, let $\frac{t^2}{16v^2} = w^2$, then $13u^2 = 4v^2 + 1$, therefore $13u^2 - 1$, must be a square,

which, by the known methods of reducing equations of this form, will be when $u = 5$; hence $13u^2 - 1 = 324 = 4v^4$, $2v^2 = 18$, $v^2 = 9$, $v = 3$, $t = 4vu = 60$, $r = 19$, and $s = 17$. Again, let $r' = 13t'$

$$+ s' = (s^2 + nt)^2, \quad r^2 = s^2 + nt, \quad n = \frac{r^2 - s^2}{t} = \frac{72}{60}, \quad r' = s' +$$

$$2nts^2 + n^2t^2 = s'^2 + 13t'^2, \quad 2ns^2 + n^2t = 13t', \quad \frac{2ns^2}{13 - n^2} = t', \quad r^2$$

$$= s^2 + \frac{2n^2s^2}{13 - n^2} = \frac{13 + n^2}{13 - n^2}s^2, \text{ that is, } 13 + n^2 \text{ and } 13 - n^2 \text{ are}$$

both square numbers. Now r being $= 19$, and $s = 17$, also $t = 60$, and $n = \frac{18}{5} = \frac{6}{5} = \frac{1}{5}$, $n^2 = \frac{1}{25}$, $x^2 + y^2 = 13$, $2xy = n^2$, $(x+y)^2 = 13 + n^2$, $(x-y)^2 = 13 - n^2$, $x + y = \frac{1}{5}$, $x - y = \frac{1}{5}$, $x = \frac{1}{5}$, $y = \frac{1}{5}$; where x is No. 8, y is No. 9, n is No. 10, $x+y$ No. 11, and $x-y$ No. 12. Also $a = 19^2 = 361$, $b = 17^2 = 289$, $a - b = 72$, $a + b = 650$, $13q^2 = 4ab(a^2 - b^2) = 4 \cdot 361 \cdot 289 \cdot 72 \cdot 650$, $q^2 = 4 \cdot 361 \cdot 289 \cdot 72 \cdot 50$, $q = 19 \cdot 17 \cdot 12 \cdot 10$, hence $\frac{1}{5}a = \text{No. } 4 = 19 \cdot 17 \cdot 60 = 19380$, $2ab = 208658$, $a^2 - b^2 = 46800$, and $a^2 + b^2 = 213842$, which are the three sides of the right-angled triangle, or

$$\text{Nos. } 5, 6, 7. \text{ Also } \frac{a^2 + b^2}{q} = \frac{321842}{38760} = \frac{106921}{19380}, \quad \frac{a^2 - b^2 + 2ab}{q} = \frac{127729}{19380}, \text{ and } \frac{a^2 - b^2 - 2ab}{q} = \frac{80929}{19380}, \text{ which are Nos. } 1, 2, 3.$$

The twelve numbers or roots required being $\frac{106921}{19380}$, $\frac{127729}{19380}$, $\frac{80929}{19380}$, 19380, 208658, 46800, 213842, $\frac{1}{5}$, $\frac{1}{5}$, $\frac{6}{5}$, $\frac{1}{5}$, $\frac{1}{5}$. And from them others ad libitum may be determined.

The same answered by Leonardus Pisanus.

The first part of this question is already answered in last year's Diary; the roots being $5\frac{106921}{19380}$, $6\frac{127729}{19380}$, and $4\frac{80929}{19380}$, or Nos. 1, 2, 3.

Retaining the notation in that solution, viz. $c = 13$, $t^2 = 25$, and $s^2 = 36$; then $c^2t^2 - s^4$ will be a square; therefore $(c^2t^2 - s^4) \times s^2t^2$ must also be a square. But $c^2t^2 - s^4$ and $2cs^2t^2$, are the legs of a right-angled triangle whose hypothenuse is $c^2t^2 + s^4$; and its area will be $(c^2t^2 - s^4) \times c^2t^2$, or c multiplied by the square $(c^2t^2 - s^4) \times s^2t^2$. The root of $(c^2t^2 - s^4) \times s^2t^2$ is 9690, or No. 4. And $c^2t^2 - s^4$, $2cs^2t^2$, and $c^2t^2 + s^4$, or 104329, 23400, and 106921, are Nos. 5, 6, 7.

In the same answer it is found by theorem 1, that $13 \times 25 - 36$ is a square; consequently $\frac{13 \times 25}{25} - \frac{36}{25}$ must be a square. But 13

or $\frac{13 \times 25}{25}$ or $\frac{325}{25}$ is composed of the squares $\frac{324}{25} + \frac{1}{25}$, whose roots are $\frac{18}{5}$ and $\frac{1}{5}$, being Nos. 8 and 9, for their double rectangle $\frac{18}{5} \times \frac{1}{5} \times 2$ or $\frac{36}{25}$ (a square) whose root is $\frac{6}{5}$, or No. 10. And since

the sum of two squares plus and minus twice the rectangle of their roots, are always squares, therefore $13 + \frac{1}{4}$ and $13 - \frac{1}{4}$ will be squares, the roots being $3\frac{1}{2}$ and $3\frac{3}{2}$ which are Nos. 11 and 12.

Questions proposed in 1804, and answered in 1805.

QUESTION 1119, by Mr. Henry Lightbown, near Blackburn.

Given $\sqrt[3]{x} - \sqrt[3]{y} = a$, and $\sqrt[3]{(x-y)} = b$; to find x and y .

Answered by Mr. J. Bennett, Barton-under-Needwood.

From the first of the given equations, we have by transposition $\sqrt[3]{x} = a + \sqrt[3]{y}$, and from the second $\sqrt[3]{x} = \sqrt[3]{(b^3 + y)}$: consequently $\sqrt[3]{(b^3 + y)} = a + \sqrt[3]{y}$. This involved to the third power produces $b^3 + y = a^3 + 3a^2\sqrt[3]{y} + 3a\sqrt[3]{y^2} + y$; whence by transposition $3a\sqrt[3]{y^2} + 3a^2\sqrt[3]{y} = b^3 - a^3$, and by division $y^{\frac{2}{3}} + ay^{\frac{1}{3}} = (b^3 - a^3) \div 3a$. Completing the square and reducing we have

$$y^{\frac{2}{3}} = \sqrt{\left(\frac{b^3 - a^3}{3a} + \frac{a^2}{4}\right) - \frac{a}{2}} = \sqrt{\frac{4b^3 - a^3}{12a} - \frac{a}{2}}.$$

$$\text{Therefore } y = \left(\sqrt{\frac{4b^3 - a^3}{12a} - \frac{a}{2}}\right)^3,$$

$$\text{and } x = b^3 + \left(\frac{4b^3 - a^3}{12a} - \frac{a}{2}\right)^3 = \left(\sqrt{\frac{4b^3 - a^3}{12a} + \frac{a}{2}}\right)^3.$$

The same by Mr. Joseph Giffins, Whitechapel.

Put $n^3 = x$, and $m^3 = y$, then the given equations become $\sqrt[3]{n^3} - \sqrt[3]{m^3} = a$, and $\sqrt[3]{(n^3 - m^3)} = b$. From the first of these $m = n - a$, and from the second $n^3 - m^3 = n^3 - (n - a)^3 = b^3$: which reduced,

&c. gives $n^3 - an = \frac{b^3}{3a} - \frac{a^3}{3}$; whence after comp. the square, &c.

we have $n = \frac{a}{2} + \sqrt{\left(\frac{b^3}{3a} - \frac{a^3}{12}\right)}$; and consequently $m =$

$$\sqrt{\left(\frac{b^3}{3a} - \frac{a^3}{12}\right) - \frac{a}{2}}. \text{ Hence } x = \left(\sqrt{\left(\frac{b^3}{3a} - \frac{a^3}{12}\right) + \frac{a}{2}}\right)^3,$$

$$\text{and } y = \left(\sqrt{\left(\frac{b^3}{3a} - \frac{a^3}{12}\right) - \frac{a}{2}}\right)^3.$$

The same by Mr. O. Gregory, Royal Military Academy.

From the second equation $\sqrt[3]{(x-y)} = b$, is found $x = b^3 + y$: substituting this value of x for it in the first equation $\sqrt[3]{x} - \sqrt[3]{y} = a$, it becomes $\sqrt[3]{(b^3 + y)} - \sqrt[3]{y} = a$ or $\sqrt[3]{(b^3 + y)} = a + \sqrt[3]{y}$. Cubing both sides we have $b^3 + y = a^3 + 3a^2y^{\frac{1}{3}} + 3ay^{\frac{2}{3}} + y$, whence $y^{\frac{1}{3}} + ay^{\frac{2}{3}} = (b^3 - a^3) \div 3a$. This latter equation is manifestly resolvable by completing the square, &c. giving

$$y^{\frac{1}{3}} = \sqrt{\frac{4b^3 - a^3}{12a}} - \frac{a}{2} \text{ or } y = \left(\frac{1}{2}\sqrt{\frac{4b^3 - a^3}{3a}} - \frac{1}{2}a\right)^3;$$

$$\text{and } x = (a + y^{\frac{1}{3}})^3 = \left(\frac{1}{2}\sqrt{\frac{4b^3 - a^3}{3a}} + \frac{1}{2}a\right)^3.$$

The same by Mr. J. H. Harding, Adderbury.

Put $\sqrt[3]{x} = v$, $\sqrt[3]{y} = w$, and the given equation will become $v - w = a$, and $\sqrt[3]{(v^3 - w^3)} = b^3$, or $v^3 - w^3 = b^3$. Divide the last equation by the first so will $v^2 + vw + w^2 = b^3 \div a$, from which taking the square of the first equation we have $3vw = b^3 \div a - a^2$, or $4vw = 4(b^3 - a^3) \div 3a$. This equation added to the fore-mentioned square is $v^2 + 2vw + w^2 = (4b^3 - a^3) \div 3a$; the root of which is $v + w = \sqrt{((4b^3 - a^3) \div 3a)}$. Hence then, having the values of $v + w$, and $v - w$, their half sum and half difference gives

$$v = \frac{1}{2}\sqrt{\frac{4b^3 - a^3}{3a}} + \frac{a}{2}, \text{ and } w = \frac{1}{2}\sqrt{\frac{4b^3 - a^3}{3a}} - \frac{a}{2}. \text{ Theref.}$$

$$x = \left(\frac{1}{2}\sqrt{\frac{4b^3 - a^3}{3a}} + \frac{a}{2}\right)^3, \text{ and } y = \left(\frac{1}{2}\sqrt{\frac{4b^3 - a^3}{3a}} - \frac{a}{2}\right)^3.$$

II. QUESTION 1120, by Mr. Tho. Hopper, Excise Officer.

Four persons playing a game at whist, one of them proposed to take 100 guineas to 1, that he had all the aces in his own hand. Quere his advantage, or disadvantage, and how much?

Answered by the Rev. J. Ewbank, of Thornton Steward.

Let $m = 13$, and $n = 52$; then by the laws of chance, $\frac{m}{n} \times \frac{m-1}{n-1} \times \frac{m-2}{n-2} \times \frac{m-3}{n-3} = \frac{11}{4166}$, the probability that a person playing at whist will have the four aces dealt him. And hence it appears that the odds against him are 4154 to 11, or $377\frac{1}{11}$ to 1.

Mr. Hopper, the Proposer, answers it thus :

Each player receives $\frac{1}{4}$ of the pack, or 13 cards, and there remains 39 among the rest. Also, when the 4 aces are taken from the whole there remains 48 which may be considered as blanks; and the probability of taking the 4 aces with the other 9 cards will be thus (by cor. 2, page 20, *Simpson's Laws of Chance*) $\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{1}$, &c. to 9 factors, divided by $\frac{52}{1} \times \frac{51}{1} \times \frac{50}{1} \times \frac{49}{1}$, &c. to 13 factors, which after reduction becomes $\frac{10 \times 11 \times 12 \times 13}{52 \times 51 \times 50 \times 49} = \frac{11}{4165}$. Therefore the required odds are as 4165 to 11, or as $378\frac{1}{11}$ guineas to 1. Consequently his disadvantage in $378\frac{1}{11}$ bets will be $278\frac{1}{11}$ guineas.

Another answer by the Rev. Tho. Scurr, Hesham.

As it signifies not whether the gentleman receives 13 cards in the regular order of dealing, or together, the question appears similar to Exam. 4, Prob. 6, of *Simpson's Laws of Chance*, where we have a theorem, which becomes in the present instance $(\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{1} \times 13 \times 12 \times 11 \times 10) \div (52 \times 51 \times 50 \times 49) = \frac{11}{4165}$, the probability of his holding all the four aces in his own hand; and that of failing as 4154 to 11, or as 378 to one nearly. Now, because the probability of his receiving 100 guineas is $\frac{11}{4165}$; therefore $\frac{11}{4165} \times 2100 = 5s. 6\frac{1}{2}d.$ the value of his expectation; and $21s. - 5s. 6\frac{1}{2}d. = 15s. 5\frac{1}{2}d.$ his disadvantage as required.

III. QUESTION 1121, by Mr. J. Lockwood.

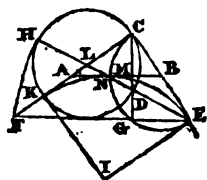
Having given the perimeter of a right-angled triangle $= 77\frac{1}{2}$, to find the sides when a perpendicular from the right angle cuts the hypotenuse in extreme and mean proportion.

Answered by Mr. J. Campbell, Chelsea.

Let abc in the next figure, be the right-angled triangle, cm the perpendicular from the right angle, dividing the hypotenuse ab in extreme and mean proportion. Now, if unity be divided in this proportion, the greater part will be $\cdot 618034$, and the less $= \cdot 381966$; let these be denoted by a and b , and put $ab = x$; then $bm = bx$, $am = ax$. And since $bc^2 = ab \times bm$, and $ac^2 = ab \times am$, we have $bc^2 = bx^2$, $ac^2 = ax^2$; consequently the whole perimeter $= x + x/a + x/b$, or $x + 1\cdot 404185x = 77\cdot 8$ per question. Whence $x = 32\cdot 31964 = ab$; $bc = 20$; and $ac = 25\cdot 44036$. For a construction, see *Simpson's Algebra*, pa. 353.

Another answer by Mr. J. Farey, Westminster.

Construction. Describe a square CEIK, whose side is equal to half the given perimeter of the triangle; on its side KC as a diameter describe the circle CDKH, and on its side CE a semicircle CGE, also on its angle I as a centre describe the circle KNE. Then, through L the middle of KC draw EL cutting the circle CDK in D, and produce it till it cuts the circle in H. On E as a centre, with the radius EH, describe an arc HF, cutting the side CK produced in F: draw EF cutting the semicircle in G. Then parallel to FE draw a tangent to the circle KNE, touching it in N, and cutting the sides of the square in A and B; and ABC is the triangle required.



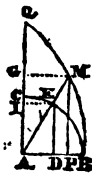
Demonstration. Join CG, cutting AB in M, which will be perpendicular to FE (Euc. 3, 31) and therefore CM is the perpendicular of the triangle from the right angle ACB. Since by the construction, KC + CE = the given perimeter, and by reason of the equality of tangents AK is equal AN, and BE to BN, therefore AB + AC + BC = the given perimeter. Now, because of the similar triangles EFC and ECG, we have EF (= HE) : EC (= HD) :: EC (= HD) : GE; and (Euc. 3, 36) HE × ED = EC² = HD², or HE : HD :: HD : ED, whence GE = ED. Again, since HE = FE, we have HD = FG (= EC), and (Euc. 3, 36) HE × ED (= FE × GE) = EC² (= FG²), or FE × GE = FG²; wherefore the base of the triangle FEC is divided in G, by its perpendicular, in extreme and mean proportion. Also, by reason of the parallels FE and AB, the hypotenuse AB is likewise divided in M by the perpendicular from the right angle in extreme and mean proportion. Q. E. D.

IV. QUESTION 1122, by Mr. Wm. Watkins, Heddon-on-the-Wall.

If from any points, in the arc of a given quadrant, sines be drawn; and if through the same points and the centre there be drawn lines equal to the radius and respective versed sines; a curve line will be formed by the extremities of these, whose equation is required?

Answered by Mr. P. Barlow, Shipdham.

Let ABC be the given quadrant, and BMQ the curve whose equation is required. Draw lines as in the figure and put MP = y, BP = x, and AB = a. Then AE (= AB) : AB (= AB - DB) :: AM (= AB + BD) : AP = AB - BP, or, as AB : BD :: BD : PB, therefore BD = √(ax). Again, AE (= AB) : ED (= √((2AB - DB)(DB))) :: AM (= AB + BD) : MP, or, as a : √(2a√ax - ax) :: a + √ax : y. Hence ay = (a + √ax) × √(2a√ax - ax) or y² = 3ax - x² + 2a√ax, the equation required, supposing the curve to commence at a.



The same by Mr. Tho. Hopper, Manchester.

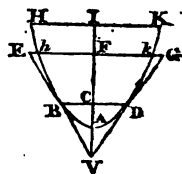
Put radius $AC = \frac{1}{2}AQ = a$, abscissa $AG = x$, and corresponding ordinate $GM = y$: then (Euc. 1, 47), $AM = \sqrt{(x^2 + y^2)}$, and, by similar triangles, $AM : \text{radius } AE \text{ or } CA :: GM : IE = ay \div \sqrt{(x^2 + y^2)} = \text{sine } EAC = \text{cosine } EAB$. And, since $EM = DB = AB - IE$, we have $EM = a - (ay \div \sqrt{(x^2 + y^2)})$. Consequently $AM = AE + EM = 2a - (ay \div \sqrt{(x^2 + y^2)})$. Equating this value of AM , with the former we obtain, after proper reduction, $x^4 + y^4 + 2ay^3 + (2ay + 2y^3 - 4a^2)x^2 - 3a^2y^2 = 0$, which is a central equation to the curve EMQ .

V. QUESTION 1123, by Mr. Peter Barlow, *Shipham*.

If a conical vessel, whose depth is 2 feet, and top or greatest diameter 15 inches, being fixed with its vertex downwards, be filled with water; what quantity will run over, by placing vertically therein a parabolic conoid, vertex downward, and its two dimensions the same as those of the vessel?

Answered by Mr. Gregory.

In the annexed figure, which represents a section of the cone and paraboloid through their common axis IAV , are given $HK = EG$, and $IA = FV$, per question, also $CA = AV$, per conics = FI , per question. Now, by similar triangles, $VF : FG :: VC (= 2AC) : CD$, or $VF^2 : FG^2 :: 4AC^2 : CD^2 = 4AC^2 \times GF^2 \div VF^2$. And, by the property of the parabola, $VF (= AI) : FG^2 (= IK^2) :: AC : CD^2 = AC \times FG^2 \div VF$. Hence, comparing these two values of CD^2 , and striking out like terms, we have $4AC \div VF = 1$, or $4AC = VF = AI$. Thus much is general, for all cones and paraboloids, having equal bases and equal altitudes. The water expelled by the paraboloid will, manifestly, be equal to the content of the part $Ahhk$; to find which, take $AI : KH^2 :: AF : kh^2$, that is $24 : 15^2 :: 18 : \frac{675}{4} = kh^2$. Hence



$kh^2 \times \frac{1}{2}AF \times .785398 = \frac{675}{4} \times 9 \times .785398 = 1192.823516$ inches, or 4.22987 gallons, are measure, the quantity of water that will run over.

The same by Mr. Wm. Mellor:

Let IA or FV be denoted by $a = 24$, $HK = EG = 15 = 2b$, and $VA = AC = x$. Then, by similar triangles, $VF (= a) : EF (= b) :: VC (= 2x) : BC = 2bx \div a$. But, by the nature of the parabola, $\sqrt{AI} (=$

$\sqrt{a} : \sqrt{AC} (\equiv \sqrt{a} :: HI (\equiv b) : BC \equiv b\sqrt{(x \div a)}$. Hence $2bx \div a \equiv b\sqrt{(x \div a)}$, or $x = \frac{1}{4}a = AC \equiv FI$, and $AF = \frac{1}{2}AI$. Again, $\sqrt{AI} (\equiv \sqrt{a}) : \sqrt{AF} (\equiv \frac{1}{2}\sqrt{3a}) :: HI (\equiv b) : HF \equiv \frac{1}{2}b\sqrt{3}$. Conseq. the solidity of the part $AKPAB \equiv \frac{1}{6}b^2 \times \frac{1}{2}a \times 3.1416 \equiv 1192.8264$ inches or 4.2399 ale gallons, the quantity of water that will run over.

VI. QUESTION 1124, by Mr. Matthew Terry, Lancaster.

Suppose a cone of oak, when lying on its slant side on a horizontal plane, be rolled round its vertex as a centre, and in two revolutions of its base to describe a circle; then lifting up its vertex till the axis be in a horizontal position, it required a vertical force of 100 pounds there just to balance it in that position, the edge of the base resting on the plane. The solidity of the cone is required, its specific gravity being 925?

Answered by Mr. Wm. Francis, Tipton.

Since the cone in two revolutions of its base describes a circle, the diameter of its base must be equal to the slant height, which would enable us to determine its dimensions, were they required. But here the solidity only is wanted; therefore, since the centre of gravity of a cone is distant from its base $\frac{1}{4}$ of the altitude, we have from the principles of mechanics $4 \times 100 = 400$ lbs. the weight of the cone. Consequently, $925 \text{ oz.} : 1 :: 400 \text{ lbs.} : 6.918918$ feet, the content of the cone.

The same by Mr. Rob. Nunn, Eye, Suffolk.

The weight of the whole cone being considered as acting at its centre of gravity, and the powers sustaining this at the two ends being reciprocally as their distances from that centre, we have $1 : 3 :: 100 : 300$ lbs. wt. sustained at the base of the cone; whence its whole weight is 400 lbs. And $400 \times 16 \div 925 = \frac{256}{23} \equiv 6\frac{1}{23}$ feet, the content of the cone.

The same by Mr. Geo. Pickering, Newington Green.

If g be the centre of gravity of the cone, v its vertex, and c the centre of its base, we have, by the nature of the question and the property of the lever, $cg : cv :: 1 : 4 :: 100 : 400$ lbs. the weight of the cone. Consequently $925 : 1 :: 400 \times 16 : 6.918918$ feet, the content required.

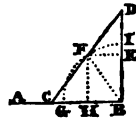
VII. QUESTION 1125, by Mr. J. Chantry, Boston.

A ladder, 60 feet long, lying on the ground, in a position at right

angles with an upright wall, with the end of it touching the bottom of the wall; the ladder then being reared by sliding the touching end directly up the wall, while the other end moved along the ground, till the top of the ladder reached the top of the wall; during this motion it was observed, that the middle round or step in the ladder, of 12 inches long, described an area or space of 24 square feet. What is the height of the wall?

Answered by Mr. J. Chantry, the Proposer.

Let AB be the ladder on the ground, CD its position when leaning against the top of the wall BD, F or G the middle step. Draw FE parallel to AB; then $CF : FD :: BE : ED$, therefore $BE \leq ED$, and $BF = FD = BG$; which being likewise the case in every position of the ladder, the locus of its middle point is a circular arc, whose radius is $BF = \frac{1}{2}AB$. Now, the semi-circumference to the radius 30 is 94.248 . Consequently $94.248 : 24 (= GF) :: 180^\circ : 45^\circ 50'$ nearly, the natural sine of which is $.7173161$: this multiplied by 30 gives $21.519483 = HF = BE$. Whence $DB = 2 EB = 43.03896$ feet, height of the wall.



The same by Mr. J. A. Dotchen, Golden Square.

The curve described by the middle round of the ladder in its ascent, is a quadrant of a circle whose centre is B, and radius $BF = \frac{1}{2}CD = 30$. The circumference of such a circle is 188.496 , and the length of the arc GF is, per question, $= 24$: therefore, $188.496 : 24 :: 360^\circ : 45^\circ 50'$, the measure of arc GF, or angle FBC $=$ FCB. Then, by trigonometry, as radius : CD :: sine DCB : DB $= 43.0392$ feet, the height required.

The same by Mr. T. Eley, Nottingham.

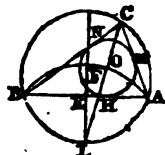
The middle round of the ladder, in its motion, will describe part of a circle, or rather of a circular bandage, which, if it were continued till the ladder became perpendicular, would be $= 47.124$ feet. Therefore $47.124 : 24 :: 90^\circ : 45^\circ 48' 52''$, the arc described by the middle round. Whence, by trigonometry, the height of the wall $= 43.0412$ feet.

VIII. QUESTION 1126, by Mr. Tho. Smith, Hethersett.

In a plane triangle, are given, the difference of the sides, the difference of the angles at the base, and the radius of the inscribed circle; to construct the triangle.

Answered by Mr. John Farey.

Construct a right-angled triangle LFO , having the angle L equal to half the given difference of the angles at the base, and the perpendicular OF , equal to half the given difference of the sides; with LO as radius, and centre L , describe an arc BOA , also on O as a centre describe the given inscribed circle; and perpendicular to FL draw to that circle the tangent HE , intersecting the arc BOA in B and A . Draw the tangents BN and AM , intersecting each other in C , and ABC is the triangle required. The demonstration is too familiar to the geometrical reader to need being stated.



The same answered by Mr. T. Myers, Hammersmith.

Construct the right-angled triangle OHE , making EH = half the given difference of the sides, and OH = the radius of the inscribed circle. Draw EL parallel to OH , also OL making the angle LOH = half the difference of the angles at the base. Then with L (the point of intersection of EL and OL) as a centre, and radius LO describe the arc BOA , cutting EH produced, in B and A . Lastly, describe the circle MNH , having OH for a radius, and draw the tangential lines BC , AC , meeting in C , so will ABC be the triangle required. As is pretty obvious without a formal proof.

IX. QUESTION 1127, by Mr. Wm. Burdon, *Acaster Malbis.*

There is a vessel in the form of the frustum of a cone, the perpendicular depth of which is 36 inches, top diameter 50 inches, and bottom diameter 40 inches. If this vessel be full of water, in what time will it empty itself through an orifice at the bottom of half an inch diameter: and what ought the diameter to be, if made in the greater end, that the vessel, when inverted, may empty itself in the same time as the former: supposing, in both cases, the section of the issuing stream is contracted to two-thirds the area of the orifice, as it commonly is, nearly?

Answered by the Rev. J. Furnass, Ponteland.

To solve this question, put a = 36 inches, the altitude of the vessel, b = 1256.64, the area of the bottom or less end, t = 1963.5, area of the top end, n = 1309 = $\frac{2}{3}$ of the area of the orifice, and m = 386 inches. Then, by Cor. 1, Pr. 2, Art. 1, Dr. Hutton's Mathematical

Miscellany, we have $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{8b + 4\sqrt{bt} + 3t}{15} = 6914.103$ nearly, or $1^h 55^m 14^s$ the time of emptying it in the vessel's first position. Next, putting $b = 1963.5$, $t = 1256.64$, and x for n in the preceding expression, we have $\frac{2\sqrt{a}}{x\sqrt{m}} \times \frac{8b + 4\sqrt{bt} + 3t}{15} = 6914.103$ by the question. Whence $x = \frac{2\sqrt{a} \times (8b + 4\sqrt{bt} + 3t)}{\sqrt{m} \times 15 \times 6914.103} = .151713$ inches, area of the contracted vein; and $\sqrt{(.151713 \times \frac{1}{2} \div .7854)} = .5383$ inches nearly, the diameter of the orifice sought.

The same by Mr. W. Marratt, Sibsey.

Let 36, the altitude of the vessel, be denoted by a , the contracted area of the orifice $= \frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} \times .7854 = .130899$ by n , 386 inches by m , also let $b = 1963.5$ the area of the greater end, and $t = 1256.64$, that of the less end. Then, by cor. 1, page 7, Hutton's *Miscellanea*

Mathematica, we have $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{3b + 4\sqrt{bt} + 3t}{15} = 6915^s = 1^h 55^m 15^s$

the time of exhaustion, the less end being downwards. Again, by the

same theorem, when the greater end is downwards, we get $\frac{2\sqrt{a}}{\frac{1}{3}n\sqrt{m}} \times$

$\frac{8b + 4\sqrt{bt} + 3t}{15}$, for the time of exhaustion, which, by hypothesis =

6915^s , which call s ; then by reduction $n = \frac{2\sqrt{a} \times (8b + 4\sqrt{bt} + 3t)}{10s\sqrt{m}}$

$= .22755148 =$ area of the orifice, and $.53826$ its diameter, as required.

X. QUESTION 1128, by Mr. John Bennet, Redruth.

On the middle of the ascent from the southern extremity of a beautiful hill, in form of a cone, a gentleman has a summer-house; leading to which he has a winding coach road, rising gradually one sixth part of its length, going quite round the hill, which is 5280 feet or one mile in length. Required to determine the dimensions of the hill?

Answered by Mr. John Bennet, the Proposer.

Let s be an indefinitely small part of the circumference of the cone's base, included between two lines drawn from its vertex, m the corre-

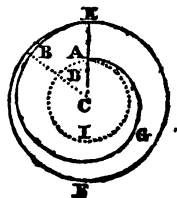
sponding part of another circle indefinitely near the former, z the spiral space intercepted between these two circles, y the vertical ascent made in passing over the spiral space z ; a = altitude of the cone = 1760, per question, r = radius of the base; then will $ry \div a$ = horizontal space advanced towards the axis. By Euc. 1, 47, and the question, $\dot{m}^2 + (r^2 \dot{y}^2 \div a^2) + \dot{y}^2 = \dot{z}^2 = 36 \dot{y}^2$; consequently $\dot{m} = \dot{y} \sqrt{(35a^2 - r^2)} \div a$; or, putting $d = \sqrt{(35a^2 - r^2)}$; $\dot{m} = (d\dot{y} \div a)$. Now y denoting the whole vertical ascent at the end of any time, we have $a : a - y :: \dot{x} : \dot{m}$, whence $\dot{m} = \frac{a - y}{a} \times \dot{x} = \frac{d\dot{y}}{a}$, and consequently $\dot{x} = \frac{d\dot{y}}{a - y}$; its fluent is $x = d \times \text{hyp. log. } \frac{a}{a - y}$, and when $y = \frac{1}{2}a$, $x = d \times \text{hyp. log. of 2}$. Put $p = 3.141593$, $L = \text{hyp. log. of 2}$, and restoring the value of d , there arises $x = L\sqrt{(35a^2 - r^2)} = 2rp$, whence, by reducing, we obtain $r = aL \times \sqrt{\frac{35}{4p^2 + L^2}} = 1141.75$ feet; radius of the base of the hill. From which the slant height $\sqrt{(r^2 + a^2)}$ may readily be found.

The same by Mr. O. Gregory, R. M. Academy.

The question may without difficulty be answered by a fluxional process, but as the curve referred to is one whose properties are well known, it is unnecessary to institute an investigation of that kind. For, conceiving lines drawn from the vertex of the cone to the periphery of the base, it is obvious that the winding-coach-road must intersect each of these at the same angle; or, if these slant lines and the curve of the road be projected by vertical lines on the plane of the base, the slant lines will be projected into radii of the circle, which will always be cut by the projected curve at the same angle: this is a known property of the logarithmic spiral; so that both the coach-road on the hill, and its projection on the base, are spirals of that kind.

In the annexed figure EBCA represents the spiral projected on the base of the cone, the length of which is $5280 \times \frac{1}{2}\sqrt{35} = 5206.1502$. Now, it has been shewn (see Emerson's Miscellanies, page 473) that the tangent of the angle included between a radius and a small part of the curve; that is, tangent $CE =$

$\frac{m \times \text{arc AD}}{ED \times (\log. CE - \log. CB)}$, where $m = .43429548$ the modulus of Briggs's Logarithms. But, in the case before us, CE being $= 2CA$, $\log. CE = \log. CA$



$= \log. 2 = \cdot 3010300$. Hence, tangent $CEB = \frac{43429448}{\cdot 3010300} \times$

$\frac{\text{per ADI}}{\text{CA}} = 1\cdot442695 \times 6\cdot283185 = 9\cdot06472102 = \text{tangent } 83^\circ 42'$

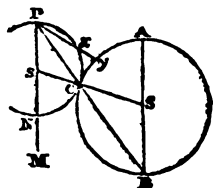
$16''$. Again, from the nature of the curve, $\text{rad.} : \cos. CEB :: BE : CE - CB$, or in the present instance, $\text{rad.} : \cos. 83^\circ 42' 16'' :: BECA (5206\cdot1502) : CE - CA = 570\cdot8914$. Consequently CE , the radius of the base of the hill, is $1141\cdot7828$ feet. Whence, the altitude being 2760 , per question, we have $2097\cdot919$ feet for the slant side.

XI. QUESTION 1129, by Mr. J. G. Cambridge.

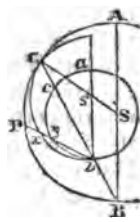
To determine the line of quickest descent from one circle to another, the latter lying wholly within the former.

Answered by Mr. J. G. the Proposer.

Lemma. The line of quickest descent from any point without a circle to the circumference of that circle, passes through the inferior extremity of the diameter perpendicular to the horizon. For, let P be the point, AB the vertical diameter of the given circle ACB . Join PB , cutting the circle in C . Through P draw PNM parallel to AB ; and through C and the centre S draw a line SC meeting PM in N . Because, in the triangles CSN , CSF , the vertical angles at C are equal, and the alternate angles SNB , SNF equal, also angle $CSF = \text{angle } NSC$, therefore these triangles are similar, and SC being $= SB$, $SC = SP$: consequently a circle described with SC as radius will pass through P and touch the circle ACB in C ; and therefore PC is the line of quickest descent. For if any other line PFY be drawn, cutting PCN in X , and ACB in Y , then, by mechanics, the time in $PX = \text{time in } PC$; therefore time in PY is greater than the time in PC . Q. E. D.



Solution. To determine the line of quickest descent from the circle ACB to the circle acb . Let AB , ab , be the vertical diameters. Then, by the lemma, whatever point be taken in the circumference ACB , the line of quickest descent from that point to the circumference acb passes through b . Draw $ccba$ meeting the diameter of the circle in b , a , and cc is the line required. For join cs by a line cutting ab in s . Then, by similar triangles, csb , csa , we have $cs : sb :: cs : sa$; therefore $cs = sb$. If, therefore, a circle be described with sc as radius, it will touch ACB in c , and acb in b : consequently cc is the line of quickest descent from ACB to acb . For, draw any other line $pxyb$, to b . It is well known that



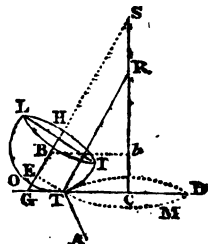
time in xy = time in cc ; therefore the time in cc is less than that in xy . Q. E. D.

XII. QUESTION 1130, by Mr. John Surtees, *Houghton-le-Spring*.

Suppose an ivory spinning top (its shape a paraboloid, absciss equal three inches, and ordinate one) be put in motion on a smooth horizontal plane, its axis making an angle of seventy-five degrees, and the point touching the plane describing a circle whose radius is six inches. It is required to determine the time of one revolution round its own axis, when the motion is an exact balance for gravity.

Answered by Mr. H. Codling, South-Walsham.

In the adjoining figure let $LOTR$ be the top, T the point touching the horizontal plane, TO a tangent to the paraboloid at T , and $TE = y$, an ordinate. Put also $a = 3$ the axis, $n = 1 = HI$, s the sine and c the cosine of angle $TOH = 75^\circ$, $g = 193$ inches, $p = 3.1416$, $d = 2\pi c = 12$, $f =$ the central force, $v =$ the circular velocity generated by that force, $1 =$ force of gravity, or weight of the top. Then, by the nature of the



curve, $n^2 : y^2 :: a : \frac{ay^2}{n^2} = OE$, whence $2OE$

$= \frac{2ay^2}{n^2}$, and, by trigonometry, $s : c :: y : \frac{2ay^2}{n^2}$; wherefore $y =$

$\frac{cn^2}{2as}$, and $2yp = \frac{pcn^2}{as}$, the periphery of the revolving circle of the

top that falls on GN in its motion round RT . Now, if we suppose the top to revolve about its axis in the given position, and the tendency it has to fall taken away for a moment, then it is manifest the top would describe a tangent TA to the circle TWN , equal in length to the product of $pcn^2 \div as$ (the periphery of the revolving circle) into the number of its revolutions; consequently the tendency to fall, is in this case equivalent to f , the central force, and any arc described is equivalent to the above product. Make AC perpendicular to the horizon, to meet RT parallel to the axis, then the solution will be the same whether we conceive the top supported by the plane in the given position, or by a string RT ; the forces exerted in either case being the same. Hence, it follows, that $AC : TC :: s : c :: 1 : f$, whence $f = c \div s$. But the velocity in a circle is equal to that gained by a descent through $\frac{1}{2}$ radius and the velocities in a perpendicular descent, are as the square roots of the spaces; therefore, $\sqrt{g} : 2g :: \frac{1}{2}\sqrt{d} : \sqrt{(dg)}$, the circular velocity by the force of gravity. Again, the forces in the same circle are as the

squares of the circular velocities; consequently $1 : dg :: c \div s (= f) : v^2$, and $v = \sqrt{(cdg \div s)}$: but, when the velocities are equable the spaces described are as the times; hence $\sqrt{\frac{c}{s}} dg : 1'' :: \frac{cn^2 p}{as} : n^2 p \sqrt{\frac{c}{s}} dg = .0112639$ of a second, the time in which the top performed one revolution round its axis.

The same by Mr. John Surtees, the Proposer.

Let the point B (in the preceding figure) represent the centre of gravity of the top, T the point touching the plane, and describing the circle TMD. Also let $r = TC =$ half a foot, $a = \frac{1}{4} = HO$, $n = \frac{1}{18} = HL$, s and $c =$ the sine and cosine of angle sec ($= 75^\circ$) to radius 1.

Then, as in Mr. Codling's solution, TE is found $= \frac{cn^2}{2as} = .0037215$.

And, drawing ab parallel to the horizon, the distance sb may be found thus: By trigonometry, $s : c :: ET : EG :: \frac{cn^2}{2as} : \frac{c^2 n^2}{2as}$,

whence $og = \frac{c^2 n^2}{4as^2}$; also, $s : 1 :: \frac{cn^2}{2as} : \frac{cn^2}{2as} = GT$; consequently

$ec = r + \frac{cn^2}{2as}$. Again, $c : 1 :: r + \frac{cn^2}{2as} : \frac{r}{c} + \frac{n^2}{2as^2} = gs$.

Then, no being $= \frac{3}{4}a$, we have $og = \frac{3}{4}a + \frac{c^2 n^2}{4as^2}$, which taken

from gs leaves $sb = \frac{r}{c} - \frac{3}{4}a + \frac{2n^2 - c^2 n^2}{4as^2}$. Wherefore $1 : s ::$

$es : sb = \frac{rs}{c} - \frac{3}{4}as + \frac{2n^2 - c^2 n^2}{4as} = 1.6712567$.

Now, I conceive the effect will be the same whether the top be supported on the plane TMD, or suspended at s by the line sb , and describing a conical surface about sc as an axis; this being admitted, we have, by page 262, vol. 1, Dr. Hutton's Mathematical and Philosophical Dictionary, $1.108 \sqrt{sb} = 1.432389$ seconds, the time of one revolution about sc , when the motion is an exact balance for gravity. Lastly, $rc : TE :: 1.432389 : .01066127$ of a second, the time in which the top performs one rotation on its axis, as required.

XIII. QUESTION 1131, by Mr. John Morun, Mount Byrnie.

To find three square numbers the sum of every two of which shall make rational square numbers: from the sum of whose roots if you subtract the area of a rational right-angled triangle, the remainder

shall be such a square number, as being decreased by each side of the triangle respectively, the several remainders shall form three rational square numbers in arithmetical progression. Required the three original squares, and the sides of the triangle, with a general method of investigation ?

Answered by Mr. Luke Blakeley, Bailey.

It is well known that 41^2 , 117^2 , and 240^2 , are three square numbers, the sum of every two of which are squares. Let, therefore, $41^2 x^2$, $117^2 x^2$, and $240^2 x^2$ be the squares required : also, let $3y$ and $4y$ be the legs, and $5y$ the hypotenuse of a right-angled triangle. Now, as $401x - 6y^2$ is required to be a square, put it $= a^2$; put, likewise, $a^2 - 3y = b^2$, $a^2 - 4y = c^2$, and $a^2 - 5y = d^2$; then $a^2 = d^2 + 5y$, $b^2 = d^2 + 2y$, and $c^2 = d^2 + y$. For y write $-2dr + r^2$, then $c^2 = d^2 - 2dr + r^2$, $b^2 = d^2 - 4dr + 2r^2$, and $a^2 = d^2 - 10dr + 5r^2$. But when these two last equations are both squares, their product $a^2 b^2 = d^4 - 14d^2 r^2 + 47d^2 r^2 - 40dr^3 + 10r^4$ will be a square; assume it equal $(d^2 - 7dr - r^2)^2 = d^4 - 14d^2 r^2 + 47d^2 r^2 + 14dr^3 + r^4$, it will give $d = \frac{r}{6}$; whence $a^2 = \frac{121r^2}{36}$. Also, $a^2 = \frac{r^2}{36}$, $c^2 = \frac{25r^2}{36}$, and $b^2 = \frac{49r^2}{36}$ are three squares in arithmetical progression. And $401x = 6y^2 + a^2 = \frac{96r^4 + 121r^2}{36}$, $x = \frac{96r^4 + 121r^2}{401 \times 36}$, $y = \frac{2r^2}{3}$; where r may be any number. When $r = 1$, then $44^2 x^2 = \frac{50997769}{13024881}$, $117^2 x^2 = \frac{7958041}{2579806}$, $240^2 x^2 = \frac{18835600}{1447209}$, $3y = 2$, $4y = \frac{8}{3}$, and $\frac{5}{y} = \frac{10}{3}$. When $r = 6$, $\frac{24770982544}{160801}$, $\frac{175149783061}{160801}$, and $\frac{736987910400}{160801}$ are the squares; and 72, 96, and 120 are the sides of the triangle.

XIV. QUESTION 1132, by Mr. John Ryley, Leeds.

Required to find three numbers, such, that when each is severally added to the cube of their sum, the respective sums shall be a cube number?

Answered by Mr. John Bennet, Redruth.

Let nx denote one of the required numbers; and s their sum, then

$s^3 + nx = s^3 + 3s^2x + 3sx^2 + x$, consequently $n = 3s^2 + 3sx + x^2$, and $x = \frac{1}{2}\sqrt{(4n - 3s^2)} - 3s$. Now, take $n = 21s^2$, then $x = 3s$, and $nx = 63s^3$.

Again, take $n = 31s^2$, then $x = 4s$, and $nx = 124s^3$.

Lastly, take $n = 57s^2$, then $x = 6s$, and $nx = 342s^3$.

These three values of nx may obviously be the numbers required; we have, therefore, $629s^3 = s$, whence $s = \frac{1}{23}$. Consequently the num-

bers are $\frac{7^3 - 1}{23^3}$, $\frac{5^3 - 1}{23^3}$, and $\frac{4^3 - 1}{23^3}$, that is $\frac{342}{12167}$, $\frac{124}{12167}$

and $\frac{63}{12167}$, their sum being $\frac{1}{23}$.

The same by J. H. Hearing, Adderbury.

Find, by the Lemma at page 100, book 3, of Kersey's Elements of Algebra, three such cubes that if from every one of them a given number (suppose unity) be subtracted, the sum of the remainders may make a square: such are the cubes $\left(\frac{17}{15}\right)^3$, $\left(\frac{28}{15}\right)^3$, and 2^3 , or

$\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8. Then taking x to represent the sum, and

$\frac{1538}{3375}x^3$, $\frac{18577}{3375}x^3$, and x^3 , the three numbers, and proceeding as in

page 101, of the above-mentioned curious work, we find $x = \frac{5}{18}$, and

the numbers required equal to $\frac{1538}{157464}$, $\frac{18577}{157464}$, and $\frac{23625}{157464}$. And

by a similar method others may soon be found.

Remark. This method is not confined to three numbers only; but as Mr. Kersey justly observes, may, by means of the lemma which he premises, be easily extended to 4, 5, or as many numbers as shall be desired.

Mr. Gregory gave an original investigation, from which he deduced several sets of numbers answering the question; but want of room prevents us from inserting more than four sets of his resulting numbers;

which are these: First set, $\frac{18954}{132651}$, $\frac{4184}{132651}$, and $\frac{271}{132651}$.

Second set, $\frac{23625}{157464}$, $\frac{18577}{157464}$, and $\frac{1538}{157464}$. Third set, $\frac{309519}{2744000}$,

$\frac{8911}{2744000}$, and $\frac{14770}{2744000}$. Fourth set, $\frac{96768}{1157625}$, $\frac{126784}{1157625}$, and

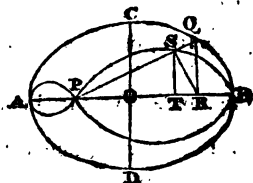
$\frac{41048}{1157625}$. See quest. 288.

PRIZE QUESTION, by Mr. Isaac Rowbottom, Teacher of Mathematics, at West Hallam.

In a gentleman's park (which is a level plain) is an elliptical piece of land; in the area of which is a pond, of such a nature, that if right lines be drawn through a given point p in the transverse axis, to cut the curve in q , and from q an ordinate be drawn at right angles to the transverse axis, cutting it in r , and from r a perpendicular be demitted on pq cutting it in s , the point s will be in the mound of the pond. Now the gentleman (who is a very whimsical one) having ordered his steward to set out a circular piece of land, and to make a pond in the area thereof, of the same nature as the former (the point p to be at the centre) and the area of the pond to be exactly 30 acres; but the steward being no mathematician, requests the ingenious correspondent of the Ladies' Diary to give the equation of the curve in the first case, with the radius of the circle, and to describe the pond, and give its greatest semiordinate in the latter case.

Answered by Mr. Rowbottom, the Proposer.

Let AB , CD (in the first marginal figure) be the transverse and conjugate axes of the ellipse, p the given point; draw the lines pq , qr and rs , as directed in the question; and from s draw st parallel to qr . Put $AB = 2t$, $CD = 2c$, $PO = a$, $PT = x$, and $ST = y$. By similar triangles $PT : ST :: ST : TR = y^2 \div x$; hence $TR + OT = OR = \pm y^2 \div x \pm x \mp a$ (the upper or lower signs taking place, according as qr falls towards B , or A , from DC). Again,



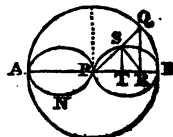
by similar triangles, $PT : ST :: PR : QR = \frac{y^2 + yx^2}{x^2} =$ (by the

prop. of the ellipse) $\frac{c}{t} \sqrt{\left\{ t^2 - \left(\frac{\pm y^2 \pm x^2 \mp ax}{x} \right)^2 \right\}}$. This reduced

gives $t^2 y^6 + (2t^2 + c^2) x^2 y^4 + (t^2 + 2c^2) x^4 y^2 - 2acy^2 x^3 + c^2 x^5 - 2ac^2 x^3 + (a^2 c^2 - t^2 c^2) x^5 = 0$, the required equation of the curve.

When $x = 0$, $y = 0$, and the curve passes through the given point. When $x = t \pm a$, then $y = 0$, and the curve passes through A and B . If the given point is not in the centre, the curve will have two unequal ovals, as in the preceding figure; but when the given point is in the centre, the equation becomes $t^2 y^6 + (2t^2 + c^2) y^4 x^2 + (t^2 + 2c^2) y^2 x^4 + c^2 x^6 - t^2 c^2 x^4 = 0$; and the curve will then have two similar and equal ovals crossing at the centre; but when $c = t$, or the ellipse becomes a circle, as in the second figure, the equation becomes $y^6 + 3y^4 x^2 + 3y^2 x^4 + x^6 - t^2 x^4 = 0$, or $y^2 + x^2 = t^2 x^{\frac{2}{3}}$; whence $y = \sqrt{(t^{\frac{2}{3}} x^{\frac{2}{3}} - x^2)}$. Let $x^{\frac{2}{3}}$

be denoted by v , then $y = v \sqrt{(t^{\frac{2}{3}} - v^2)}$, the fluxion of the area is $y\dot{x} = 3v\dot{v}\sqrt{(t^{\frac{2}{3}} - v^2)} = (3t^{\frac{2}{3}}v\dot{v} - 3v^2\dot{v}) \div \sqrt{(t^{\frac{2}{3}} - v^2)}$, the fluent of which (Prop. 59, page 164, Vince's Fluxions) restoring x , is $\frac{1}{15}t^{\frac{5}{3}}A + (\frac{1}{2}x^{\frac{5}{3}} - \frac{1}{8}t^{\frac{2}{3}}x - \frac{1}{16}t^{\frac{2}{3}}x^{\frac{1}{3}}) \times \sqrt{(t^{\frac{2}{3}} - x^{\frac{2}{3}})}$, where A is the circular arc whose sine is $\frac{1}{2}\sqrt{(x \div t)}$, radius being 1. When q arrives at B , then $x = 1$, and the fluent is $\frac{1}{15}t^{\frac{5}{3}}A = \frac{1}{4}$ of the area of the pond (A being the quadrantal circular arc, radius 1) $= 4840 \times 30 = 145200$ yards, per question. Hence $t^{\frac{2}{3}}\sqrt{(145200 \div 3A)} = 351.069206$ yards, the radius of the circle. Again, making the expression $\sqrt{(t^{\frac{2}{3}}x^{\frac{2}{3}} - x^2)}$, a maximum, and proceeding as usual, we get $x = \frac{2}{3}t\sqrt{\frac{2}{3}}$: hence $y = \frac{2}{3}t\sqrt{3} = 135.1266$ yards, the greatest semi-ordinate.



Remark. The greatest semi-ordinate is to the corresponding abscissa, as the side of a square is to its diagonal.

The same answered by Mr. John Aspland, Soham.

In the first figure to the preceding solution, put $AO = t$, $co = c$, $AP = a$, $BP = m = 2t - a$, $PO = n$, (whence $m - a = 2n$), $PR = x$, $TS = y$, $PR = v$. Then, by the nature of the ellipse $(c \div t)\sqrt{((a + v) \times (m - v))} = (c \div t)\sqrt{(ma + mv - av - v^2)} = (c \div t)\sqrt{(ma + 2nv - v^2)} = QR$: also, $x : y :: v : vy \div x = QR$. But $PR \times PR = PS^2$, that is $vx = x^2 + y^2$; therefore $v = \frac{x^2 + y^2}{x}$, and $\frac{vy}{x} = \frac{x^2y + y^3}{x^2} = QR$. This value of v being substituted for it in the

first expression for QR , we have $\frac{c}{t}\sqrt{(ma + \frac{2nx^2y + 2ny}{x} - \frac{x^2 + 2y^2x^2 + y^4}{x^2})} = QR = \frac{x^2y + y^3}{x^2}$. From which, after proper

reduction, we obtain, $c^2amx^4 + 2nc^2yx^2 + 2nc^2y^2x^2 - c^2x^4 - 2c^2y^2x^2 - c^2y^4x^2 - t^2x^2y^4 - t^2y^6 = 0$, for an equation to the curve in the first case. Or, by farther ordering, we get $c^2amx^4 + 2nc^2yx^2 + 2nc^2y^2x^2 = c^2x^4 + (t^2 + 2c^2)x^2y^4 + (2t^2 + c^2)x^2y^4 + t^2y^6$. Now when the ellipse becomes a circle and the point P is at the centre, $t = c = m = a$, ($= 1$ suppose), and $n = 0$, and our last equation becomes $x^4 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$, and consequently $x^{\frac{4}{3}} = x^2 + y^2$, and $y = x^{\frac{2}{3}}\sqrt{(1 - x^{\frac{2}{3}})}$. Hence $y\dot{x} = x^{\frac{2}{3}}\dot{x}\sqrt{(1 - x^{\frac{2}{3}})} = z$, whose fluent (by Form. 4, Cotes's Harmonia Mensurarum) is $\frac{1}{15}t^{\frac{5}{3}}A$.

$= \frac{1}{16}(-3x^{\frac{1}{2}} - 2x + 8x^{\frac{3}{2}}), \times \sqrt{(1-x^{\frac{1}{2}})} - \frac{1}{16}q$, where $q = \text{arc}$,
 whose cosine is $x^{\frac{1}{2}}$, sine $\sqrt{(1-x^{\frac{1}{2}})}$, rad. 1, or $= \text{arc } BQ$, in the
 second figure. But when $x = 0$, $x = 0$, and the arc becomes the
 quadrant $BC = q'$; hence the correct fluent is $\frac{3q'}{16} - \frac{3x^{\frac{1}{2}} - 2x + 8x^{\frac{3}{2}}}{16} \times$
 $\sqrt{(1-x^{\frac{1}{2}})} - \frac{3q}{16}$. When, therefore $x = PB = 1$, $q = 0$, and the
 area of half the curve $PSBP$, is $= \frac{3q}{16}$: consequently, $\frac{3q'}{4}$ is the area
 of the double oval curve similar to that required, and having its diame-
 ter AP or $PB = 1$. And since the areas of similar curves vary as the
 squares of their diameters, if $r =$ the diameter of the required curve
 we shall have $\frac{3r^2q'}{4} = 30 \text{ acres} = 4800 \text{ perches}$. But $q' = 1.5708 =$
 $.7854 \times 2$; consequently $\frac{1}{2}r^2q' = r^2 \times \frac{1}{2} \times .7854 \times 2 = \frac{1}{2}$ of the
 circle whose diameter $= r$. Hence $r = 40\sqrt{(2 \div .7854)} =$
 63.8073 poles. To determine the greatest ordinate, we have $x^{\frac{1}{2}}$
 $- x^{\frac{1}{2}}$ a maximum. In fluxions $\frac{1}{2}x^{\frac{1}{2}} - 2x^{\frac{1}{2}} = 0$, whence $x =$
 $\frac{1}{4}$. This substituted for x in the general expression, gives $y =$
 $\frac{1}{2}\sqrt{3}$. Since $\frac{1}{2}\sqrt{3}$ is greater than $\frac{1}{2}$, the greatest ordinate of the
 curve PSB , will be nearer the point B , than the greatest ordinate of the
 circle described upon PB as a diameter.

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Questions proposed in 1805, and answered in 1806.

I. QUESTION 1134, by Jos. Woollin, *Smalley, Derbyshire.*

A heavy body is $5\frac{1}{2}$ seconds more in falling down the hypothe-
 nuse of a right-angled triangle, than down the perpendicular: re-
 quired the triangle when the area is a maximum?

Answered by Mr. T. Hopper, Manchester.

Suppose ABC to be the right-angled triangle, and AD drawn perpen-
 dicular to AC , meeting CB produced in D . Then it is well
 known that the time of descent from C to A is equal to the
 perpendicular descent through CD . And, by the question,
 the time through AD must be $= 5\frac{1}{2}$ seconds. Now the
 greater CB is taken, the greater will be the velocity at B , and
 of consequence the greater space passed over in the next $5\frac{1}{2}$



seconds from B to D, and as AB is a mean proportional between CB and BD, AB will likewise continue to increase: so that, the greater BC is taken, the greater will be the area of the triangle ABC. Whence it is evident that the question is not properly limited.

II. QUESTION 1135, by the Rev. Wm. Wright, *Helmsley*.

The two diameters of a vessel in form of a conic frustum, are five and four inches: required its depth when it holds just one quart, ale measure?

Answered by Mr. J. Craven, Bamburg Castle.

By the rule for the frustum of a cone in Bonnycastle's Mensuration, we have $[(5^3 - 4^3) \div (5 - 4)] \times .7854 = 47.9094$, for a product, which multiplied by one-third of the height of the frustum would give the solidity. Consequently, in the present case, the inches in an ale quart ($70\frac{1}{2}$) being divided by the above sum will give for a quotient one-third of the depth of the quart whence that depth is easily found = 4.4145825 inches.

The same by Mr. J. A. Dotchen, Broad Street, Golden Square.

If x denote the height of the frustum, then by Mensuration, $[(5^3 - 4^3) \div (5 - 4)] \times .7854 \times \frac{1}{3}x = 70.5$: whence $x = 4.4145$ the altitude required.

The same, by Mr. J. B. Harris, at the Rev. L. Evans's Academy.

First, we have the content of the conic frustum equal to $70\frac{1}{2}$ cubic inches, by the question. Then, putting x for the depth, we have by the rules of mensuration $(4^3 + 5^3 + 4 \times 5) \times .7854 \times \frac{1}{3}x = 70.5$. Whence x is found = 4.4145 the depth required.

III. QUESTION 1136, by the Rev. Wm. Wright, *Helmsley*.

What time in the forenoon of the 5th of April, 1804, in latitude $54^\circ 15'$ north, was my shadow twice the length of myself?

Answered by Mr. P. Barlow, Shipham.

Here we have $2 : 1 :: \text{rad.} : \text{tang. } 26^\circ 33' 54''$ the apparent altitude of the sun's upper limb; which being corrected for semi-diameter, &c. gives $26^\circ 15' 54''$ the true altitude of the sun's centre. Now, in a spherical triangle szp are given the sides, viz. $zp = 45^\circ 45'$, co-lat. $zs = 63^\circ 44' 6''$, co-alt. and $sp = 83^\circ 54'$, co-declin. to find the angle $p = 54^\circ 12' 21''$, = 3 hours 36 minutes 48 seconds; the time

from noon. Hence 8 hours 23 minutes 12 seconds in the morning is the time sought.

The same by Mr. E. England, Wintringham.

First, in a right-angled plane triangle, as $2 : 1 :: \text{rad.} : \text{tang.}$ $26^{\circ} 33' 55'' =$ apparent altitude of sun's upper limb ; which corrected for semi-diameter, refraction, and parallax, leaves the altitude of the sun's centre $26^{\circ} 16' 10''$. Again, in a well-known spherical problem, are given the latitude $54^{\circ} 15'$, the sun's declination, April 5th, $6^{\circ} 3' 40''$, and the sun's altitude corrected as above, to find the angle at the pole, or the time from noon, $52^{\circ} 10' 55'$, answering to $8^{\text{h}} 31^{\text{m}} 17^{\text{s}}$ in the morning.

The same by Mr. J. H. Hearing, of London.

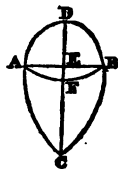
In this question the ratio of the tangent of the sun's apparent altitude to radius, being as 1 to 2, the altitude itself must be $26^{\circ} 33' 54'' 11'''$, and, after correction for semi-diameter, refraction, and parallax, the true altitude is found to be $26^{\circ} 16' 8''$. The sun's declination on the morning of April 5, 1804, was $6^{\circ} 3' 15''$ north, and as the latitude is given there are now sufficient data, *i. e.* the co-altitude, the co-declination, and the co-latitude, whence to find the angle at the pole $= 52^{\circ} 5' 19\frac{1}{2}''$, corresponding to $3^{\text{h}} 21^{\text{m}} 23^{\text{s}}$, and this taken from 12^{h} leaves $8^{\text{h}} 31^{\text{m}} 37^{\text{s}}$ for the time of the morning.

IV. QUESTION 1137, by Mr. John Hawkes, Finedon.

If a person ascends vertically over the parallel of London (latitude $57^{\circ} 37'$ north) till he can see an arc of 120 degrees, or just a third part of the equator ; how many acres of the earth's surface in the southern hemisphere may he view, supposing the whole earth a sphere, and its radius $3978\frac{1}{2}$ miles ?

Answered by Mr. Joseph Gittins, Whitechapel.

Let AEB in the annexed figure represent the part of the equator in view from the supposed elevation, and AEB which intersects the equator at A and B, a portion of the circle which limits the observer's sight ; thro' D draw the great circles DA, DB, and DE. Now, in the right-angled spherical triangle BED there are given the leg BE ($= 60^{\circ}$) and DE $=$ the given latitude, whence the hypotenuse BD is found $= 74^{\circ} 28' 1'' 8$, the angle BDE $= 64^{\circ} 0' 28'' 26$, and the $\angle DBE = 61^{\circ} 13' 21'' 04$. Again, $2 : \text{vers. BD} (= .7322095) :: \text{surface of the whole earth} (198943750 \text{ square miles} = m) : \text{surface of the segment within the observer's view} (= 72834251.8578 \text{ miles} = n)$. Find the area of the triangle BED ($= 9134527.6837 \text{ square}$



miles = s) by Prob. 12, page 149, Hutton's Mensuration, 3rd edit. then $(n - s) \times 2 = 6430743 \cdot 898$ square miles = surface of the part AEBF, which in acres is 4115676094 \cdot 72. In this solution the latitude is taken as printed in the Diary : but if the true latitude $51^\circ 31'$ be substituted for the former, the result will be 4990353920 acres, nearly.

The same by Mr. Richard Nicholson, Liverpool.

Let D in the preceding figure represent the zenith, EB half the part of the equator which is viewed by the observer, and FB a corresponding part of that less circle which terminates his view ; now, in the right-angled spherical triangle EDB we have $ED = 51^\circ 31'$, and $EB = 60'$, to find the angle $EDB = 65^\circ 40' 47''$, $EBD = 55^\circ 27' 15''$, and $DB = 71^\circ 52' 18''$; whence we easily find the area of this triangle = 8602651 \cdot 814407443 square miles. Again, as 2 rad. (= 2) : the whole surface of the sphere :: versed sine DB (= .6888535) : the whole surface viewed ; and 36° : the whole surface viewed :: $68^\circ 40' 47''$: the space $EBF = 12501356 \cdot 81535$ square miles. Hence $2EBF = 49903424027 \cdot 2327$ square acres, as required.

The same by Mr. John Taylor, Bristol.

This question may be answered by having recourse to prob. 14, page 205, Hutton's Mensuration, 2nd edition. Let AB in the preceding figure represent an arc of the equator = 120° , $DE = 51^\circ 31'$, the latitude, $DF = DB = DA$, also, $AE = EB$, then is $AEBF$ the surface viewed in the southern hemisphere. In the two equal right-angled spherical triangles DEB , DEA the legs are given ; hence the angle DBE or $DAE = 55^\circ 27' 15''$, $\angle BDE$ or $ADE = 65^\circ 40' 47'' \cdot 7$. The earth's diameter being 636620 chains, the versed sine of the arc DB to the earth's radius will be found = 219270 chains nearly. Then (the angle BDA being = $131^\circ 21' 35'' \cdot 4$) it is evident from the above cited problem that the surface $ADB = (3 \cdot 1415926 \times 636620 \times 219270 \times 131^\circ 21' 35'' \cdot 4) \div 360 \times 10 = 16001822479 \cdot 2702653$ acres : the area of the spherical triangle ADB , found by prob. 12, page 202, *ibid.* is 11011486277 \cdot 859722 acres ; and this taken from ADB leaves $AEBF = 4990336201 \cdot 4105433$ acres nearly, the answer.

V. QUESTION 1138, *by Mr. Peter Phelan, near Dublin.*

If a circle be inscribed in any plane triangle, and a right line drawn from the vertex to the point where the circle touches its base ; then will the middle of that line, the centre of the circle, and the middle point in the base of the triangle be in the same right line. Required the demonstration ?

Answered by Mr. J. Dawes, Birmingham.

Let x, s, m , be the three points which, by question, must be in a right
D 3

Answered by Mr. M. Mooney, the Proposer.

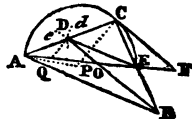
The difference in accuracy, according to the statement in the quest. depends upon the orbits of the respective satellites; and those of Saturn are more eccentric than Jupiter's, consequently the portion of the former that measures the parallactic angle deviates more from a circle than the portion of the latter, and the parallax of Saturn cannot be ascertained with the same degree of accuracy as that of Jupiter.

VII. QUESTION 1140, *by Mr. John Aspland, Soham.*

In a plain triangle are given, the vertical angle, and the lengths of the two lines drawn from each angle at the base, bisecting and terminating in the opposite sides, to construct the triangle *generally*.

Answered by Mr. P. Barlow, Shipdham.

Let ABC be the triangle, AE , BD , the bisecting lines intersecting in O , join ED , and it will be parallel to BA ; therefore, $DE : AE :: EC : BC :: 1 : 2$, and hence $AB = 2ED$; so that, by similar triangles, AO , BO are respectively equal to $2OE$, and $2OD$. Now produce AE to F making $AF = 2OF$, and join CF , then will $CF = 2OD = BO$. Hence this



Construction. On one of the given lines AE describe a segment to contain the given vertical angle, and produce AE to F making $EF = \frac{1}{2}AE$, also take $CF = \frac{2}{3}BD$ and apply it from F to touch the segment in C ; join CE and produce it to B , making $BE = EC$, join AB , AC , and ABC will be the triangle required.

The same answered by Mr. T. Hopper, Manchester.

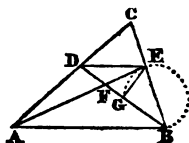
Construction. Take AE , in the preceding figure, equal to one of the given lines, upon which describe a segment of a circle capable of containing the given vertical angle, of which F is the centre. Take $EO = \frac{1}{3}AE$ and upon O as a centre with one third of the other given line describe an arc dd . Join AF , and bisect it in Q , and with radius $AQ = QF$ describe an arc en cutting dd in D . Join AD and produce it to cut the circle in C ; produce CE and DQ to meet in B ; join AB , and ABC will be the triangle required.

Demonstration. Join CF and DQ , then as $DQ = AQ$ (by construction) and $2AQ = FC$, QD is therefore parallel to FC and bisects AC in D . And, since $OE = \frac{1}{3}AE$, it follows, from Emerson's Geometry, prop. 21, cor. 1, that DO will be $= \frac{1}{3}BD$. And the side BC will be bisected in E .

The same by Mr. R. Nicholson.

This question is the same as the 5th in the Gentleman's Diary for 1793, to which two geometrical solutions are given in the Diary following: it may therefore be advisable to give a solution here, when the lines drawn from the angles at the base divide the sides in a given ratio, though the solutions above referred to are applicable to this case.

Analysis. Let ABC be the required triangle, AE , BD the given lines dividing the sides AC , BC , in the given ratio, then will AC , DC , have a given ratio, which will be the same as that of AB to DE , or of AF to FE , or of BF to FD , therefore BF , DF , AF and FE , are given. Draw EG parallel to AC , then GD will have to GB the given ratio of CE to EB , and therefore DG and GB are given.



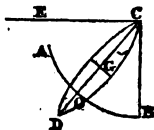
Construction. Upon GB describe the segment of a circle to contain the given vertical angle, to which from F apply FE ; take EA and BD of the given lengths, join AB , and produce BE and AD to meet in C , and ABC is the triangle which was to be constructed.

VIII. QUESTION 1141, by Mr. T. Myers, *Hammersmith.*

A piece of marble in the form of a circular spindle, the greatest diameter of which is 6, and its length 32 inches, is suspended by one end, and it vibrates in an arc of 100° , at the rate of 70 vibrations per minute. Required the horizontal force at its point of suspension, when a maximum, and the position of the axis at the time,

Answered by Mr. T. Myers, the Proposer.

Let c be the centre of suspension, A half the arc and cn the oscillating body, the solidity of which is 533.2279 cubic inches, and its weight 51.71135lbs. = w : also g the centre of the body will be the centre of gravity. And by Hutton's Course, vol. ii, page 207, as $70^\circ : 60^\circ :: 39\frac{1}{2} : 28.5408$ inches = oc the distance of the centre of oscillation o from the centre of suspension c . Let s = the sine of the angle pcx , and c its cosine, then by Hellins's Essays, page 129, scw. $ce \div co$ = the required force. Here, because s and c are the only variable quantities, it is evident that the force will be a maximum when the rectangle of s and c is such, and which is well known to be when the $\angle pcx = 45^\circ$. Consequently 14.49459lbs. is the force in the direction cx .



The same by Mr. Geo. Pickering, Newington Green.

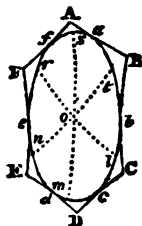
Let nc be the oscillating body, c the point of suspension, a the centre of the body which is also its centre of gravity; then, by Hutton's Mensuration, prop. 21, page 158, edition 3, the solidity is 533.228 cubic inches; and as $1728 : 2700 \div 16 :: 533.228 : 51.711 \text{ lbs.} = w$. Also, by article 255, page 211, vol. 2, Hutton's Course, as $70^\circ : 60^\circ :: 39\frac{1}{2} : 28.7449 = co$ the distance between the centres of oscillation and suspension. Then is $s. c. w. ca \div co$ the force required, where s and c are the sine and cosine of the angle nce ; and since these are the only variable quantities the force will be a maximum when $s \times c =$ a maximum, which is well known to be the case when each equals 45° : therefore 14.494 lbs. the force of the centre c in the direction ce .

IX. QUESTION 1142, by Mr. O. G. Gregory, *R. M. Academy.*

If an ellipse be circumscribed by any polygon, as $ABCDEF$, the sides touching the ellipse in the points a, b, c, d, e, f , respectively: then will the continual product of the segments of the several tangents, (made by the points of contact) taken alternately, be equal, viz. $aa. bb. cc. dd. ee. ff = ab. bc. cd. de. ef. fa$. Required a demonstration?

Answered by Mr. T. Kirton, Peterborough.

From o the centre of the ellipse draw the semidiameters ol, om, on, or, os , and ot , parallel to the tangents AB, BC, CD, DE, EF , and FA respectively, then we have, by the nature of the curve, these analogies, viz. $ab : bb :: ol : mo$; $bc : cc :: om : on$; $cd : dd :: on : or$; $de : ee :: or : os$; $ef : ff :: os : ot$; and $fa : aa :: ot : ol$. These analogies being multiplied together, the two latter terms will be found to be equal, and consequently the two former are equal also; as affirmed in the question.



The same answered by Mr. R. Nicholson.

Let o be the centre of the ellipse, and draw ot, ol, om, on, or , and os , parallel to FA, AB, BC, CD, DE , and EF respectively, to meet the periphery of the ellipse. Then (Emerson's Conics, 32 and 44, I. cor. 1) we have $Af : Aa :: ot : ol$; $ab : bb :: ol : mo$; $cb : cc :: om : on$; $cd : dd :: on : or$; $de : ee :: or : os$; and $ef : ff :: os : ot$. Whence, compoundedly, $Af. ab. cb. cd. de. ef :: Aa. bb. cc. dd. ee. ff :: ot. ol. om. on. or. os :: ol. om. on. or. os. ot$; in which proportion the two latter terms being manifestly equal, the two former must be so also.

The same by Mr. W. Watkins, Haydon Bridge, Northumberland.

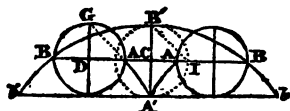
If from o the centre of the ellipse ol be drawn parallel to AB , and ol parallel to AF , then by prop. 22, cor. 3, Hutton's Conics, Ellipse, $ol : ol :: aa : af$. And by proceeding thus with all the pairs, and compounding, we shall have a proportion, the last ratio of which will be found a ratio of equality, therefore the former ratio will be such, and gives us $aa \cdot bb : cc \cdot dd \cdot ee \cdot ff = ab \cdot bc \cdot cd \cdot de \cdot ef \cdot fa$.

X. QUESTION 1143, by Mr. P. Barlow, Shipdham.

There is in a piece of machinery, a wheel five feet in diameter, which is constantly trundled half round and back again, on an horizontal plane, four times in a minute; now admitting this wheel to be kept in continual motion for 365 days; it is required to determine the difference of the spaces passed through in that time, by those two points in its circumference which are in the same horizontal line with its centre, at the beginning and end of each motion?

Answered by Mr. J. Collins, Kensington.

Let AB be the wheel, A and B the points when put into motion, $A'C'B'$ its position when perpendicular to the horizon, and ADB its position when trundled half round. By Emerson on Curve Lines, page 13, the whole cycloidal curve $bb'b' = 4AB = 20$. The curve $BB' = 2AB' = 2\sqrt{(B'C'^2 + C'I^2)} = 7.071$, and therefore $BB'B = 14.142$: hence $bb'b' - BB'B = b'b + b'b' = 20 - 14.142 = 5.858$; which is also $AA' + A'A$, and therefore $14.142 - 5.858 = 8.284$ feet, the distance the point B will pass over more than the point A by being trundled half round. Consequently $8.284 \times 8 \times 60 \times 24 \times 365 = 34832563\frac{1}{2}$ feet, the difference of the spaces passed over in 365 days.



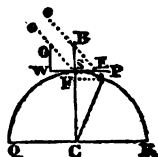
The same by Mr. J. Turner, New Malton.

It is evident that while the point B is ascending from B to B' , the point A is descending from A to A' , and while the said point B is descending from B' to B , the point A is ascending from A' to A . Now, by the properties of the figure, $AA' = A'A = Bb' = bB$, and $2B'I = 2\sqrt{(2.5^2 + 2.5^2)} = 7.071 = \text{arc } BB'$; and $2AB = 10 = \text{arc } B'bb$. Hence $4AB - 4B'I = 20 - 14.142 = 5.858$, the space passed over by the point A in one motion; and again $14.142 - 5.858 = 8.284$, the difference of the spaces passed over by the points B , A , during one motion. Whence $8.284 \times 8 \times 525600 = 34832563\frac{1}{2}$ feet, the difference of the spaces required.

sea, and was then in the zenith of the ship; it was two o'clock P. M. by observation, Sept. 2, 1803, and a cloudless day; and the shadow of the balloon fell exactly upon the port bound to. The ship's direct course to the said port, with its distance, latitude, and longitude, are required: the radius of the earth being reckoned 3979 miles, and notice being taken of the semidiameter, parallax, and refraction?

Answered by Mr. P. Barlow.

First, in a spherical triangle szp (readily conceived without a figure) where s is the sun, z the zenith, and p the pole; there are given $zp = 34^\circ 57'$ co-latitude, $sp = 81^\circ 48'$ co-declination, and angle $p = 30^\circ$, to find $zs = 52^\circ 33' 42''$, the co-altitude, which corrected for semidiameter, &c. give $52^\circ 13' 42''$ apparent co-altitude, and $z = 141^\circ 26'$, the sun's azimuth from the north.



Now let Q , be the sun, p the balloon, r the port, s the ship, and c the earth's centre: then, in the triangle pbc , are given $bc = 4184$, $pc = 3979$, and $\angle pbc = 52^\circ 13' 42''$; to find $\angle pcb = 3^\circ 59\frac{1}{2}'$. The corresponding circular arc is $\cdot 0696677$, which multiplied by radius gives 277.207 miles, the distance sought. Also, $180^\circ - 141^\circ 26' = 38^\circ 24'$ (north-easterly) the course. Whence, these being known, the required latitude is found to be $58^\circ 39' 43''$ N. and the longitude $6^\circ 40' 22''$ E.

The same by the Rev. J. Furness, the Proposer.

In solving this question it will, first, be necessary to determine the sun's altitude and azimuth at 2 o'clock P. M. on Sept. 2, 1803, when the sun's declination is $8^\circ 12'$ north. This performed by a well-known rule gives $142^\circ 28'$ the azimuth from the north, and $37^\circ 44' 4''$ for the apparent altitude of the sun's upper limb.

Let $qsarc$ be a vertical section of the earth's superior hemisphere in a direction with the solar ray passing through the centre c . Draw wsr perpendicular to csb , and w parallel to sb , and join cr ; then will s be the place of the ship, p of the balloon, and r of the port bound to; also $\angle sbe = \angle wos = 52^\circ 15' 56''$, the sun's co-altitude $sb = 205$ miles, the height of the balloon, and $cs = 3979$ miles, the earth's radius. Now in the triangle cbr are given cb , cr , and the angle b , to find the angle $brc = 123^\circ 44'$; whence $bcr = 4^\circ$ nearly, which is the angle at the centre contained between s the ship's place, and r the port bound to. Then, (by Hutton's Mensuration, second edition, p.

123) $4 \times .0174533 \times 3979 = 277.7867228$ miles = *sr* the distance from the ship to the port.

Again, the sun's azimuth from the south being $37^\circ 32'$, or S.W.b.S. $3^\circ 47'$ W. the direct course is N.E.b.N. $3^\circ 47'$ E. And, hence, as radius : distance :: cosine course : diff. latitude = 220.28 miles = $3^\circ 40' 16''$ N. Then, on the principles of Mercator's sailing, as cosine course : mer. dif. lat. :: sine course : diff. long. = 259.7 miles = $4^\circ 19' 42''$ E. Wherefore, the course is N. E. b. N. $3^\circ 47'$ E., the direct distance to the port 277.7867228 miles, the latitude of the port $58^\circ 43' 16''$ N., and its longitude $5^\circ 42' 42''$ E. Agreeing with very nearly Christiansand in Norway.

The same by Mr. T. Towan, Redruth.

In the preceding figure let *s* denote the place of the ship, *a* the balloon, and *r* the port bound to. Also let *sz* be drawn perpendicular to *sr*, and *rp* parallel to *sz*. Then from the given latitude, declination, and hour, the apparent altitude of the sun's upper limb is found = $37^\circ 43' 27''$, and his azimuth from the south $38^\circ 33' 32''$ towards the west, therefore the ship's course is $38^\circ 33' 32''$ east of north. And per trig.; as rad. : *sz* = 205 :: cotang. $37^\circ 43' 27''$: *sr* = 265 . Put *a* = *sz*, *b* = *sr*, *x* = *sr*, and *y* = *rp*; then by similar triangles *a*

: *b* :: *a* + *x* : *y* therefore *ay* = *ab* + *bx*, or $y = \frac{ab + bx}{a}$, and

$\left(\frac{ab + bx}{a}\right)^2 + (r - x)^2 = r^2$, where $r = cr = sc = 3979$ miles.

This equation gives $x = 9.694833$ whence $y = 277.5323$. And, as $3979 : 277.5323 :: 1 : .0697492 = \sin 4^\circ = \sin \angle acp$. Whence the course and distance sailed, and the latitude and longitude of the port are found nearly as in the foregoing solutions.

XIII. QUESTION 1146, by Mr. I. Rowbottom, West-Hadlam.

One morning last summer, in a certain northern latitude, I took a good horizontal dial, made for that lat. and placed it horizontally, so that the shadow of the gnomon fell upon the hour line of 12, and the meridian of the dial made an angle of $83^\circ 32'$ with that of the place: in this position the time of the shadow's passing from the hour line of 12 to that of 2 was the greatest possible. Required the latitude of the place, the time the dial was set, and how long a time was occupied by the shadow in passing from the hour line of 12 to that of 2?

Answered by the Rev. T. Scurr, Hexham.

Here it is necessary to find the ball's velocity after it quits the target; as also its velocity when it strikes the bank of earth. Now, since a body projected perpendicularly upwards with a velocity of 1700 feet in a second, will rise to an altitude of $(1700)^2 \div 4 \times 16\frac{1}{2} = 44922$ feet; hence, by Mr. Robin's experiment, we have the force by which elm retards the ball, to the force of gravity, as 44922, to $\frac{1}{12}$ or as 107813 to 1. Now, put $x = \frac{1}{12}$ of a foot, $l = 16\frac{1}{2}$, $r = 107813$, $b = 44922$ feet, $v =$ the velocity required; and, by Atwood's Rectilinear Motion, page 45, $v = \sqrt{4l \times (b - rx)} = \sqrt{4 \times 16\frac{1}{2} \times (44922 - 107813 \times \frac{1}{12})} = 1520$ feet per second.

Again, put $d = \frac{1}{48} = \frac{1}{16}$, $a = 1520$, $x = 180$ feet, $n = 11\frac{1}{2}$ the density of the ball, and $\pi = \frac{1}{2500}$, the density of the air; then, (by

Dr. Hutton's Math. vol. 2, p. 356) $bx = \frac{3\pi x}{8nd} = \frac{3 \times 3 \times 3 \times 16 \times 180}{8 \times 34 \times 2500}$

$= \frac{243}{2125}$, and $v = 1520 \div e^{\frac{243}{2125}} = 1041.9$ feet, per second, (e being $= 2.718281828$, the number whose hyp. log. is 1.) having lost 478.1 feet per second of its velocity. Lastly, since the specific gravity of lead is 11.333, the diameter of the ball $= .0625$ of a foot, put $d = .0625$, $n = 11.333$, and $\pi = \frac{1}{11}$ (assigning the resistance of the earth to that of the elm as 1 to 11) $v = 1041.9$, the last velocity, and $s =$ the depth to which the ball will penetrate: assuming Mr. Robin's experiment as a standard, we derive a general expression for the value

of s , having by Atwood, page 43, $s = \frac{5}{12} \times \frac{v^2}{g} \times \frac{d}{\frac{1}{16}} \times \frac{n}{11.333} \times$

$\frac{1}{R} = 1.72161$ feet, the depth of penetration, as required.

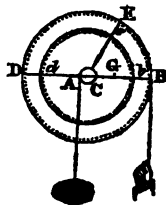
PRIZE QUESTION, by Mr. Gregory, Royal Military Academy Woolwich.

There is a solid wheel and axle made of oak (specific gravity 925) the diameter of the wheel being 3 feet, and of the axle half a foot, the thickness of the wheel 3 inches, and the whole length of the axle 1 foot 9 inches, now, if the power acting at the circumference of the wheel be equivalent to an hundred weight, what must be the weight raised by a cord folding round the axle, so that its momentum may be a maximum? And, if the uniform thickness of the wheel were 6 inches, and the axle as before, then, what would be the weight raised in the case of its maximum effect? In both instances regarding the difference caused by the power having inertia, or being divested of it,

64

Answered by Mr. H. Codrington, South Walsham.

After some remarks on the nature of Rotatory motion, this gentleman proceeds thus: In the wheel where the axis of motion passes through the centre of gravity, the parts composing its mass are in equilibrio, and therefore contribute nothing by means of gravity to effect the motion of the system; but when a weight is applied to the circumference to give motion to the wheel, only those parts of the wheel situate in its circumference move with the same velocity with the part of the mass affected with gravity, consequently these are the only parts of the wheel whose resistance can be estimated by their quantities of matter only; but if the resistance of the rest be found in parts of the resistance made by these and the sum of the whole taken, the velocity of the part affected will evidently be obtained by using the said sum instead of the quantities of matter in the mass.



To this end conceive the wheel to be divided into an infinite number of concentric rings, BED , the outermost whose quantity of matter is represented by n and resistance by n ; bed another at any distance cb from the centre c whose quantity of matter is represented by b , and resistance by r ; as all the rings in the motion of the wheel must move with the same angular velocity, first, suppose this to be effected by means of forces applied to the circumference of each ring, viz. in the ring BED by a force at D , and in bed by a force at d ; then, by the second law of motion, these forces will be as the motion they generate in each ring, that is, by the similarity of the sectors as $cb \times n$ to $cb \times b$, or, as the resistance made by BED to a force at D , to the resistance made by bed to a force at d : but, by the property of the lever, the resistance bed makes to a force at d , is to the resistance it makes to a force at D , as cb to cb ; therefore, $n : r :: n \times cb^2 : b \times cb^2$, whence $r = b \times cb^2 \times n \div n \times cb^2$; but as the breadth of each is supposed indefinitely diminished the quantity of matter in bed moves with the same velocity as the power, consequently its resistance will be estimated by its quantity of matter, so that we may put $n = n$, and $r = cb^2 \times b \div cb^2$. Put p = the power = 1 cwt, w = the weight to be raised on the axle, $2g = 32\frac{1}{2}$, $d = cb = cd$, the radius of the wheel, a = that of the axle, l = its length, t = thickness of the wheel, $cb = cd = x$, $c = 3.141593$, and $925 \div (112 \times 16)$ the weight of the quantity of matter in a cubic foot of oak; then will $b = 2tcgax$ and $r = 2tcgax \times x \div d^2$, and thence $tcgax^2 \div 2d^2$, the resistance from the wheel $ched$, which when $l = l$ and $x = a$ becomes $lca^2 \div 2d^2$, the resistance from the axle; but when $x = d$ it becomes $tcgd^2 \div 2d^2$ the resistance from the wheel $caed$: consequently the resistance from both is $(ld^2 + la^2) \times cg \div 2d^2$. Let w be now suspended on the axle and a weight $= aw \div d$ at the circumference of the wheel,

and the whole mass will be in equilibrio, and no part thereof by means of gravity can contribute any thing to affect the circular motion of the wheel; let a weight h be now added to $aw \div d$ to give motion to the mass, then is h the only part affected by gravity, the resistance to which from its own mass and weight $aw \div d$ will be estimated by the quantity of matter contained by both, viz. by $h + aw \div d = p$: that from the wheel and axle has been found $= (td' + la') \times cg \div 2d^2$, and the resistance from w will be found $= a^2w \div d^2$, by substituting a, d, w , for cb, cb , and b respectively, in the expression $r = (b \times cb^2 \div b \times cb^2) \times R$; and thence the resistance to h from the whole mass is $p +$

$$(td' + la') \times \frac{cg}{2d^2} + \frac{a^2w}{d^2}. \text{ Write } m \text{ for } 2d^2p + (td' + la') \times cg =$$

$$2d^2p + (t + \frac{l}{1296}) \times cg d^2, \text{ and the whole resistance will be expres-}$$

sed by $\frac{m + 2a^2w}{2d^2}$; whence it follows from what has been previously

said, that $\frac{1}{h} : 2gT :: \frac{1}{(m + 2a^2w) \div 2d^2} : 4gTd^2 \times \frac{h}{m + 2a^2w}$, the

velocity generated in h or p in any given time T . But the velocity of w is to that of p as a to d , therefore the momentum of w is $4gTda \times (p -$

$\frac{aw}{d}) \times w \div m + 2a^2w$, which, by the conditions of the question, is

to be a maximum; its fluxion, therefore, made $= 0$, and reduced gives

$$w = \frac{\sqrt{((m + 2dpa)m) - m}}{2a^2}.$$

From this in the first case where p is supposed to make a part of the mass moved,

and $\begin{cases} t = 3 \text{ inches, } w \text{ will be found} = 2.91887 \text{ cwt.} \\ t = 6 \text{ inches, } w = \text{—————} = 2.937409 \text{ do.} \end{cases}$

But when p is divested of inertia, or vanishes in the value of m we have

when $\begin{cases} t = 3 \text{ inches, } w = 2.767942 \text{ cwt.} \\ t = 6 \text{ inches, } w = 2.874526 \text{ do.} \end{cases}$

when the inertia of the wheel and axle is inconsiderable, then $m =$

$$2pd^2, \text{ and } w = \frac{dp}{a^2} (\sqrt{(d^2 + da)} - d) = 6 \times (\sqrt{42} - 6) =$$

$$2.884444 \text{ cwt.}$$

The same by Mr. Gregory, the Proposer.

Let $p = 1$ cwt. the given power, $x =$ the weight whose momentum is to be a maximum, $m = cb$, radius of the wheel, $n = ca$, that of the axle, $w =$ weight of the wheel, $\Delta =$ that of the projecting part of the axle. Now, the absolute force acting at n is p , contrary to which the

weight acts at A with the force $\frac{x \times CA}{CB}$. The difference of these is

$P - \frac{x \times CA}{CB} = \frac{P \times CB - x \times CA}{CB} = \frac{mP - nx}{m}$, for the force actually impelling the system.

Then, G being the centre of gyration of the wheel, g that of the axle, by the established principles of Rotatory motion, the system will have the same angular velocity in fact, as it would if bodies equivalent to $\frac{CG^2}{CB^2} w$, $\frac{CG^2}{CB^2} A$, $\frac{CA^2}{CB^2} x$, and P (supposing it to possess inertia) were all collected in the point B. Hence then, the algebraic expression for the aggregate mass moved, or, the resistance opposing the motion at B will be $(\frac{1}{2}m^2w + \frac{1}{2}n^2A + n^2x + m^2P) \div m^2$; by which dividing the moving force we have $\frac{m^2P - mnx}{\frac{1}{2}m^2w + m^2P + \frac{1}{2}n^2A + n^2x}$, for

the force accelerating B. Multiplying this into $\frac{n}{m}$ there arises

$\frac{mnP - n^2x}{m^2(P + \frac{1}{2}w) + \frac{1}{2}n^2A + n^2x}$, for the force accelerating x in its ascent.

Or, putting s to denote the sum of those terms in the denominator into which x does not enter, the accelerating force referred to x will be

$\frac{mnP - n^2x}{s + n^2x}$. Consequently $\frac{mnP - n^2x}{s + n^2x}$, will represent the motive

force; which being a maximum, by question, if put into fluxions and

reduced we shall obtain $x^2 + \frac{2s}{n^2}x = \frac{m}{n^2}Ps$, and $x = \frac{1}{n^2} \sqrt{(s \times (s +$

$mnP)) - \frac{1}{n^2} s}$, the wt. required.

Now, having the dimensions of the wheel and its specific gravity, we readily find $w = .9121708$ cwt. and $A = .1520285$ cwt. in the first case, whence $s = m^2(P + \frac{1}{2}w) + \frac{1}{2}n^2A = 13.123772$, and $x = 2.918852$ cwt. In the second case, where $w = 1.8243416$, and $A = 2.266904$, we shall have $s = 17.2253735$, and $x = 2.93737$ cwt.

Remark 1. If the moving force be conceived free from inertia, s will be $= \frac{1}{2}m^2w + \frac{1}{2}n^2A = 4.1237722$ in the first case, and 8.2253735 in the second: these being introduced into the final theorem for x will produce 2.767784 cwt. and 2.87444 cwt. for its value in the respective cases.

2. Considering the wheel and axle without inertia, but retaining

that of p, s becomes barely $\doteq mnp$, and $x = \frac{mp}{n^2} \sqrt{(m(m+n))} - \frac{m^2}{n^2} p = 2.884444 \text{ cwt.}$

3. If in the last theorem m were equal to n , we should have $x = (m \div n) p \times .414214$. So that the common solution to this problem can only be true in the extreme case when the wheel and axle is converted into the single pulley.

4. When every thing but x is considered as divested of inertia, the expression for the motive force will become $(mp - nx) \div n$, which admits of no proper maximum.



Questions proposed in 1806, and answered in 1807.

I. QUESTION 1149, by Mr. W. Watkins, Second Usher in the Free Grammar School, Haydon Bridge.

In a plane triangle are given, the rectangle of the base and sum of the sides, the rectangle of the shorter side and difference of the base and longer side, and the rectangle of the longer side and quotient of the two sides : to find the sides of the triangle ?

Answered by Mr. Jas. Adams, Stonehouse.

Put x = the base of the triangle, y = the longer side, and z = the shorter. Then, by the question $xy + xz = a$, and $xz - yz = b$, and $xy \div z = c$, or $xy = cz$. From the 1st equation take the sum of the 2d and 3d, and it gives $cx + yz = a - b$. Next multiply the 1st and the difference of the 1st and 3d by z , and equating them gives $cx^2 + cyz = ay$; find the value of y in this, and substitute in the preceding one, gives $cx^2 \div (a - cz) + cz = a - b$, therefore $z^3 - cz^2 + (2a - b)z = (a - b)a \div c$. Hence z may be found; and then x and y .

II. QUESTION 1150, by Mr. S. Jones, Irondish, near Frodsham.

If P be a point in the diameter AD produced, of a given semicircle $ACBD$, and from P a right line PCB be drawn to cut the semicircle in c and B , from which points raise CH, BI , perpendiculars to PB , to meet

the diameter AD in H and I : then the points H and I will be equidistant from the centre of the circle. Required the demonstration?

Answered by Mr. John Smith, Alton Park.

From the centre E draw EF perpendicular to CB : then the chord CB is bisected in F (by Hutton's Course, Geometry, theorem 41), and CH, FE, BI are parallels (cor. to th. 13); therefore FE bisects HI in E (th. 82); that is the points H and I are equidistant from the centre of the circle E.

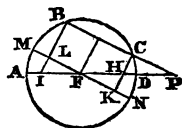


The same by Mr. Geo. Pickering, Islington.

Draw the lines as in the question, and from the centre E draw EF parallel to CH or BI : then, by the nature of the parallels, $IP : EP :: BP : FP$, and $EP : HP :: FP : CP$, therefore $IE : HE :: BF : CF$; but $BF = CF$, therefore $IE = HE$.

The same by Mr. John Dawes, Birmingham.

The annexed figure being drawn as in the question; continue CH till it meet another diameter MN drawn parallel to the chord BC, in K, then BI will cut the same in L. Now the angles at B and C being both right ones, the figure BCKL is a rectangle, and $LF = FK$; hence the right angled triangles FLI and FKH are identical, therefore $FI = FH$.



III. QUESTION 1151, by Mr. Tho. Hopper, Manchester.

Not long ago being one of four who agreed to raffle for a pipe of wine value 90l. under the following conditions, viz. to raffle with an octaedron regularly marked; and the first that cast was to pay 30l. the second 25l. the third 20l. and the last that threw to pay the remaining sum of 15l. and to continue to cast in this order until it happened that any one first brought up the highest number which entitled him to the whole pipe of wine. Quere, the probable expectation of each person?

Answered by Mr. P. Barlow, Shipdham.

The probability of the first person winning the first throw is $\frac{1}{8}$, and of his missing $\frac{7}{8}$; the probability of the 2d man winning is therefore $\frac{1}{8} \times \frac{7}{8} = \frac{7}{64}$; but the chance of his missing is the same as the first,

viz. $\frac{1}{8}$ therefore the probability of the third winning is $\frac{1}{8} \times \frac{1}{8} \times \frac{1}{8} = \frac{1}{512}$: in like manner the probability of the 4th winning is $\frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$: or their expectations are as the numbers 512, 448, 392, 343, the sum of which is 1695. Therefore as 1695 : 90 ::

512 : 27·185£.	} the value of those shares.
448 : 23·787	
392 : 20·814	
343 : 18·212	

The same by Mr. T. Myers, Hammersmith.

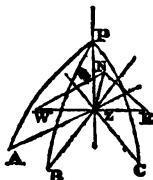
Since, from the nature of the question, the total expectation of each person is as his expectation on the first throw, therefore (Simpson's Laws of Chance, prob. 18) they are as 1, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{24}$, the sum being $\frac{1695}{1695}$. Then each of these divided by this sum gives $\frac{512}{1695}$, $\frac{448}{1695}$, $\frac{392}{1695}$, $\frac{343}{1695}$, the several expectations, which being each multiplied by 90, gives 27·113, 23·821, 20·814, 18·212, the values of the expectations.

IV. QUESTION 1152, by Mr. T. Kirton, Peterborough.

In latitude 52° N. I have a triangular field, in one side of which that runs directly east and west there is a straight tree standing perpendicular to the horizon, the shadow of which on June 21, 1804, at 10h. A. M. was parallel to the N. E. side of the field; at 1h. P. M. the shadow produced passed through the northern angle; and two hours after that it was parallel to the north west side. Required the area of the field, the distance of the tree from the northern angle being 6 chains?

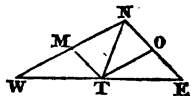
Answered by Mr J. A. Dotchen, of Chelsea.

Suppose WNE to represent the field, z the zenith, or place of the tree, pz the meridian, and p the north pole. In the three spherical triangles PAZ, PBZ, PCZ, there are given the colatitude pz common to all = 38°, the side PA = PB = PC the sun's codeclination = 66° 32', and the included angles ZPA, ZPB, ZPC = 45°, 15°, 30° respectively (the hour angle from noon); to find the angles PZA = 112° 17' 42'', PZB = 152° 20' 23'', PZC = 129° 42' 14''; hence are obtained the angles of the field, viz. $\angle W = 22^\circ 17' 42''$, $\angle N = 118^\circ 0' 4''$, and $\angle E = 39^\circ 42' 14''$; also $\angle NZW = 117^\circ 39' 37''$. Then by plane Trigonometry as $\sin. \angle W : NZ = 6 \text{ chains} :: \sin. \angle NZW : WN = 14.00801$. And as $\sin. E : WN :: \sin. WNE : WE = 19.36102$. Lastly $WN \times WE \times \frac{1}{2} \sin. \angle W = 51.44508$ square chains, or 5 acres 23.1212 poles, the area of the field.



The same answered by Mr. John Smith, Alton Park.

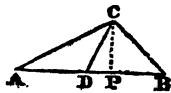
Let WNE be the triangular field, $TN = 6$ chains, the distance of the tree from the northern angle, and TM (parallel to NE), TN , and TO (parallel to WN), the directions of its shadow at the given times. From the data we find the sun's azimuth from the north at those times respectively $= 129^\circ 42' 14''$, and $152^\circ 20' 37''$, and $112^\circ 17' 43''$; hence the angles $E (= MTW)$, and NTE , and NTW , and $w (= OTE)$ and wNE are respectively $39^\circ 42' 14''$, and $62^\circ 20' 37''$, and $117^\circ 39' 23''$, and $22^\circ 17' 43''$, and $118^\circ 0' 3''$. In the $\triangle NET$ are given the angles E and T , and the side TN , to find $NE = 8.32$ chains; and in the $\triangle NWT$ are given the angles T and w , and the side TN , to find $WN = 14.01$. Then $NE \times NW \times \frac{1}{2} \sin \angle N = 51.4592$ chains $= 5$ acres 23 perches, the area of the field.



The same by the Rev. Mr. T. Scurr, of Hexham.

This problem presents us, in 3 several instances, with a case of spherics, viz. in an oblique spherical triangle, two sides and the contained angle; first the co-latitude 38° , the co-declination $66^\circ 32'$, and the angle at the pole or time from noon 30° ; secondly, the same two sides, and angle at the pole 15° ; and thirdly the same two sides with the polar angle 45° ; to find the angles at the zenith, or sun's azimuths from the north, viz. $129^\circ 42' 13''$, and $152^\circ 20' 38''$, and $112^\circ 17' 51''$ respectively.

Again, let ABC represent the triangular field, and $CD = 6$ chains, the distance from the tree D to the northern angle C . Then, from the above, the angles are as follow, viz. $B = 39^\circ 42' 13''$, $D = 62^\circ 20' 38''$, $A = 22^\circ 17' 51''$. Now let fall the perpendicular CP ; then by trig. as $\sin P : CD :: \sin D : CP = 5.33455$; and $\sin A : CP :: \sin P : AC = 14.06$; also as $\sin B : AC :: \sin A : CB = 19.43$. Hence $\frac{1}{2} AB \times CP = 51.82515$ chains $= 5$ acres 29 perches.



V. QUESTION 1153, by Mr. R. Wilkinson, North Shields.

A ship plying to windward sails within $6\frac{1}{2}$ points of the wind (which is then N.N.E.) with her starboard tacks on board 83 miles, she then tacks and runs 62 miles farther. Required her difference of latitude made on both tacks?

Answered by Mr. P. Barlow, Shipdham.

In this question; when divested of its technical terms, there are given the course and distance, to find the difference of lat. which may be found in any transverse table by inspection only.

*The same answered by Messrs. T. Dickinson, M. Hamilton, M. Young
Bomburgh Castle.*

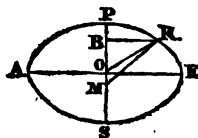
The first course N.W. $\frac{1}{2}$ W. and the 2d E $\frac{1}{2}$ S. Therefore; as radius : 85 :: cosine $4\frac{1}{2}$ points : 52.654 the northing on the 1st course; and as radius : 62 :: sine $\frac{1}{2}$ a point : 6.077, the southing on the second course; the difference of these is 46.577, the difference of lat. sought.

VI. QUESTION 1154, by Mr. W. Wiseman, Hull.

If the earth be an oblate spheroid the ratio of whose axes is 230 to 229, in what latitude will the line of motion in which a heavy body falls to the earth make the greatest angle with one drawn perpendicular to the surface; and how much is that angle?

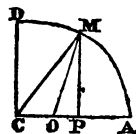
Answered by Mr. W. Wiseman, Hull.

I apprehend that the line of motion of the body would, if continued, pass through the earth's centre o. Let AE represent the equatorial, and sr the polar diameter; also rm perpen. to the surface. By Simpson's Fluxions, page 506, $AE^3 + rs^3 : AE^3 - rs^3 ::$ radius : sine of ORM; which, supposing AE = 7957.75, and rs = 7922.98, gives 438 = sine of ORM = $15' 4''$ the angle required. And, by the same, $45^\circ + \frac{1}{2}ORM = 45^\circ 7' 32''$ is the latitude.



The same by Mr. R. Nicholson, of Liverpool.

This question may be resolved as prob. 8, page 506, Simpson's Flux. But another solution may be as follows: Let ca and cd denote the semitransverse and semiconjugate axes of the earth, and the point m in the given latitude. Draw the ordinate mp, and mo perpendicular to the curve, also join cm. Now by conic sections $ca^2 : cd^2 :: cp : op$, a given ratio by the question, and



the angle CMO a maximum; therefore MP is a tangent to a circle described on CO ; hence $CP \times OP = MP^2$, and the triangles MOP , MCP given in specie; hence the angle MOP is found $= 45^\circ 7' 29''$ the latitude and $CMO = 14' 59''$ the required angle.

The same by Mr. James Adams, of Stonehouse.

Let EF denote the earth's greater, and DP the less axis, GC the line of motion, and GB perpendicular to the surface: then the $\angle CGB$ is to be a maximum. Put $x = \text{tangent } AGC$; then by conics $CE^2 : CP^2 ::$

$AC : AB :: x : \frac{CP^2}{CE^2} \times x = \text{tangent } AGB = mx$,

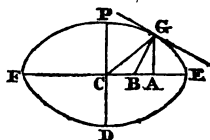
putting $m = \frac{CP^2}{CE^2}$; and by trigonom. $\frac{x - mx}{1 + mx^2}$

$= \frac{1 - m}{1 + mx^2} \times x = \text{tangent } CGB$; their difference or $\frac{x}{1 + mx^2} =$

a maximum; whose fluxion made $= 0$, and reduced gives $mx^2 = 1$, or $x^2 = \frac{1}{m} = \frac{CE^2}{CP^2}$, and hence $x = \frac{CE}{CP}$, or $CP : CE :: \text{radius} :$

$\text{tangent } AGC$; also $mx = \frac{CP^2}{CE^2} \times \frac{CE}{CP} = \frac{CP}{CE}$, or $CE : CP :: \text{radius} :$

$\text{tangent } AGB$; also by substitution in $\frac{x - mx}{1 + mx^2}$, it gives $2CP \times CE : CE^2 - CP^2 :: \text{radius} : \text{tangent } CGB$; or in numbers $2 \times 229 \times 230 : 230^2 - 229^2 :: 1 : \text{tangent } 14' 58'' \frac{1}{4} = \angle CGB$ when greatest: then $45^\circ + \frac{1}{2}CGB = 45^\circ 7' 29''$ is the latitude or angle.



VII. QUESTION 1155, by Mr. J. Collins, Kensington.

Find three square numbers such that if each be taken from the sum of their three roots the remainders may be square numbers?

Answered by Mr. J. Whitley, Gt. Woodhouse.

Let the required squares be denoted by x^2, n^2x^2, m^2x^2 : then the conditions of the question require that $x + nx + mx - x^2 = \square$, $x + nx + mx - n^2x^2 = \square$, $x + nx + mx - m^2x^2 = \square$. Let each be divided by x^2 , then also will $\frac{1 + n + m}{x} - 1 = \square$ (No. 1), $\frac{1 + n + m}{x}$

$-n^2 = \square$ (No. 2), $\frac{1+n+m}{x} - m^2 = \square$ (No. 3). From the equation No. 1 we have $x = \frac{1+n+m}{1+a^2}$, where a^2 denotes the square No. 1; which substituted for x in the equations No. 2 and No. 3, there will be had $1+a^2-n^2 = \square$, and $1+a^2-m^2 = \square$. The former of these two is a square when $a = \frac{1}{2}n^2$, and the latter when $a = m = \frac{1}{2}n^2$; hence then we have $x = \frac{1+n+m}{1+a^2} = 2 \times \frac{2+2n+n^2}{4+n^2}$; consequently the roots of three square formulæ answering the proposed conditions are these, $2 \times \frac{2+2n+n^2}{4+n^2}$, and $2n \times \frac{2+2n+n^2}{4+n^2}$, and $n^2 \times \frac{2+2n+n^2}{4+n^2}$, where n may be taken at pleasure.

Exam. 1. Let $n = 3$, then the squares are $\frac{4}{25}$, $\frac{36}{25}$, $\frac{81}{25}$.

Exam. 2. Let $n = 4$, then we obtain the numbers $\frac{16}{25}$, $\frac{16}{25}$, $\frac{64}{25}$, which are perhaps the smallest fractions answering the conditions of the question.

It is worthy of remark that the sum of each of these sets of numbers is a rational square, and indeed this will universally be the case, whatever be the value of n . For since $m = \frac{1}{2}n^2$, therefore $x^2 + n^2x^2 + m^2x^2 = x^2 \times (1 + n^2 + \frac{1}{4}n^4) = x^2 \times (1 + \frac{1}{4}n^4)^2$ a square universally.

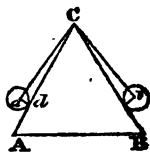
VIII. QUESTION 1156, by Mr. T. Myers, *Hammersmith*.

Two 12lb. cast iron balls connected together by a string 30 inches long, the middle of which is fixed to the vertex, rest upon the sides of an equilateral triangle, whose plane is perpendicular to the horizon, and its area in square inches to the perpendicular pressure on its vertex in avoirdupois ounces, as seven to three. What is the length of each side?

Answered by Mr. T. Myers, the Proposer.

Let ABC be the triangle, a and b the two balls, the diameters of which (pa. 257, Dr. Hutton's Course, vol. 2, 4th edition) are 4.4026 inches; hence $ad = 2.2013$, and $ac = 17.2013$, and the $\angle adc$ a

right angle. Then, as $ac : \sin \angle d :: ad : \sin \angle acd = 7^\circ 21' 9''$; hence $60^\circ - 7^\circ 21' 9'' = 52^\circ 38' 51'' =$ the angles of elevation of the equal planes on which the balls act. Then (prop. 35, page 189, *ibid.*) as $\sin 90^\circ : \sin 52^\circ 38' 51'' :: 24 : 19.07781b.$ or 305.2448 ounces, the perpendicular pressure on its vertex c . Then as $3 : 7 :: 305.2448 : 712.2379$ inches, the area of the triangle, hence each side is easily found $= 40.5567$ inches.



IX. QUESTION 1157, by W. Z. of Kirby Moorside.

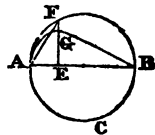
A gentleman bequeathed to his son and daughter a circular piece of land worth 20l. per acre; out of which his son is to have 300l. worth more than the daughter, and is to have it laid off from one end of the diameter to the other by a curve line whose ordinate is every where equal to the difference between its respective abscissa, and corresponding circular chord: required the area of the field, and their respective fortunes?

Answered by Mrs. Eliz. Bryhon, Helmsley.

By the question, as $20 : 1$ acre or $100000 :: 300 : 15$ acres or 1500000 links, double the area between the diameter and curve; for the half of which put a , also x the diameter of the field. Then we have (from art. 26, page 169, Dr. Hutton's Mathematical Miscellany), $\frac{1}{2}x^2 = a$; hence $x = \sqrt{6a} = 2121.32$ links; then $212.32^2 \times .7854 = 3534300$ links or 35.343 acres, the area of the field; and their fortunes are 503l. 8s. 7d. and 203l. 8s. 7d.

The same by Mr. Geo. Pickering, of Islington.

Let $AFBC$ be the circular field, and AGB the dividing curve line. Then, as the son is to have 300l's worth, or 15 acres more than the daughter, the part $AGBEA$ will be the half of that, or $7\frac{1}{2}$ acres. Put $d^2 = AB$ the diameter, and $x^2 = AE$ the absciss; then, by property of the circle, the chord $AF = dx$, therefore the ordinate $EG = dx - x^2$, and the fluxion of the area $AEG = 2xx(dx - x^2)$, the fluent of which gives $\frac{1}{2}d^4 = 7\frac{1}{2}$ area of $AGBEA$: hence $d^4 = 45$ the square of the diameter; then the area of the circle $= 4b \times .7854 = 25.343$ acres. The half of which increased and decreased by $7\frac{1}{2}$, gives 35.1715 and 10.1725 acres the shares of the son and daughter, and their fortunes are 503l. 8s. 7½d. and 203l. 8s. 7½d.

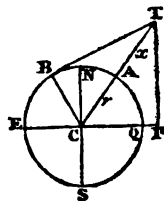


X. QUESTION 1158, *by the Rev. J. Furnass, Ponteland.*

Admit a watch-tower reared up perpendicularly over Newcastle-upon-Tyne, as high as it could just stand: Quere, the utmost north latitude, with its direct distance from the bottom of the said tower, which a man placed at the top could see; supposing the earth's radius 3980 miles, and no obstruction from the medium?

Answered by Mr. James North.

By the question it seems the proposer intends that the tower should be of such height, that the centrifugal force at its top may be equal to the power of gravity there. Now it has been determined by Newton, and other philosophers, that at the earth's surface, the centrifugal force at the equator is equal to the 289th part of gravity there, or it opposes and diminishes the gravity there by the 289th part of it. In the annexed figure, let c denote the earth's centre, eq the equatorial diameter, ns the axis, A the place Newcastle, in exactly 55° lat. AT the height of the tower, and TP perpendicular from its top to radius cq produced. Put the height $AT = x$, the radius CA or CN or $CQ = r$, and $c = \cosine$ of the lat. or sine of the angle τ : then is $CT = r + x$, and $CP = c(r + x)$. Now, 1 being considered as the centrifugal force at q , and 289 as the gravity at A , and this diminishing as the square of the distance from the centre



increases, therefore as $CT^2 : CA^2 :: 289 : \frac{289r^2}{(r+x)^2}$ the gravity at T the top of the tower. And since the centrifugal force, when the time of rotation is the same, increases as the radius, therefore as $cq : CP :: 1 : \frac{(r+x)c}{r}$ the centrifugal force at r or at T , in direction CP ; and as

$$CT : CP :: \frac{r+x}{r} c : \frac{r+x}{x} c^2 \text{ the centrif. force at } T \text{ in direction } CT.$$

Then making this equal to the gravity at T , gives the equation $\frac{r+x}{r} c^2$

$$= \frac{289r^2}{(r+x)^2}; \text{ hence } (r+x)^2 c^2 = 289r^2, \text{ and } r+x = r \times$$

$\sqrt{\frac{289}{c^2}} = CT = 9.5772r$, or the height of the tower AT , or x is 8.5722 times the radius AC , or $= 38117$ miles.

Now draw the tangent TB : then is B the place of the utmost distance on the earth which a person at T may see. Join BC . Then, in

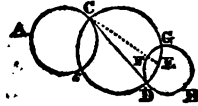
the triangle BCT, as $CT : CB :: 9.5772 : 1 :: \text{radius} : \cosine \angle BCT = 84^\circ = \text{the arc ANB}$; then $ANB - AN = 84^\circ - 35^\circ = 49^\circ = BN$ the distance of B beyond the north pole, also ANB or $84^\circ \times 3980 \times .01745 = 5834$ miles is the length of the arc ANB, or distance on the surface from A the foot of the tower.

XI. QUESTION 1159, *by Mr. J. G. of Cambridge.*

To determine the line of longest descent between the circumferences of two given circles?

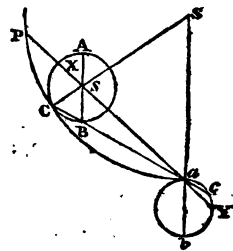
Answered by Mr. W. Putsey, Master of Pickering Free School.

Let acc , dgb be two given circles, to which draw the common tangent cd , on which as a diameter describe the circle egd ; then cd will be the line of longest descent on the outsides of the circles. For draw any other line cf . Then the time in cd is equal to the time in ce , which is longer than the time in cf .



The same by the Proposer, Mr. J. G. Cambridge.

Let acb , acb be the two given circles. By the solution of question 11, last year, it appears that whatever point be taken in ACB , the line of longest descent from that point passes through a . Draw ca , cutting the lower circle again in c ; then cc is the longest descent required. For, through the centre s draw cs meeting ab produced in s . Then by the similar triangles sca , scb , $sc : sa :: sc : sb$, therefore $sc = sa$, and a circle described with sc as radius, will touch ACB in c , and acb in a ; and cc is the line of longest descent. For draw any line pxa through a . Then the time in pc = the time in cc , and greater than the time down xc .

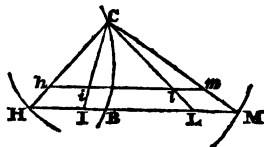


XII. QUESTION 1160, *by Mr. W. Mellor.*

Given the radii of two concentric circles, with the centre c , and position of two right lines nc , oc ; required the position and magnitude of the right line hm whose extremities h and m are in the peripheries of the lesser and greater circles, and passing through the points i and l in the lines given by position, so that $ih = il = lm$?

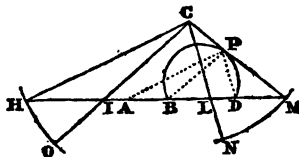
Answered by Mr. R. Nicholson, Liverpool.

In any line, assumed at pleasure, take the three equal parts hi , il , lm ; and on one of them il , describe the segment of a circle icl capable of containing the given angle made by the lines given in position; then (by Simpson's Exercises, pr. 19, page 109, or Eucl. 15, 4) describe the circular arc bc , such, that if to any point of it lines be drawn from h and m , they shall have the ratio of the given radii; the point c , where this arc cuts the circle icl , will be the common centre of the two given circles: then is $hilm$ the position of the required line: and if in ch and cm there be taken, ch and $cm =$ the given radii, and hm joined, it will evidently be the position and magnitude of the required line, divided into the three equal parts hi , il , lm , by the lines ci , cl . And the same way will do, whatever be the ratio of the parts hi , il , lm .



The same by Mr. Samuel Ryley.

In any line take AB , BD , DM , in the given ratio of hi , il , lm , in the required line: on BD describe a circular segment to contain an angle, $= ocn$, made by the two radii co , cn , given in position; to the circumference of which draw AB and MP in the ratio of co to cn ; produce MC to c making $mc = cn$; draw ch , ci , cn parallel to pa , pb , pd respectively, meeting ma in the points h , i , l , and the parts of the required line hm will be hi , il , lm . All which is quite evident from the nature of the several parallel lines.



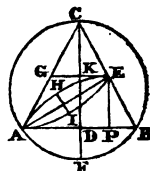
XIII. QUESTION 1161, by Mr. Tho. Towan, Redruth.

Required to divide the greatest cone that can be inscribed in a globe whose diameter is 6 inches, into two equal parts by a plane passing from the extremity of its base and terminating in a point in the opposite side: to determine the distance of the said point from the vertex of the cone, and likewise the area of the section.

Answered by Mr. Tho. Towan, Redruth.

Draw the lines as in the annexed figure; where $ACBF$ denotes the globe, and ABC the cone; also AE the plane which divides the cone

into two equal parts. Put $CF = 2r = 6$, $c = 3.1416$, $CD = x$, $AD = DB = y$. Then, by the nature of the circle, $y^2 = 2rx - x^2$, and $(2rx - x^2) \times \frac{1}{3}cx$ is the solid content of the cone a maximum, or $2rx^3 - x^3$ a maximum, by the question; the fluxion of which made $= 0$, gives $x = \frac{2}{3}r = 4$: hence $y = \sqrt{8} = 2.828427 = AD$, the double of which is $AB = 5.6568542$; hence $BC = AC = \sqrt{(AD^2 + DC^2)} = 4.8989795$, and the solidity of the cone $ABC = 33.5104$ the half of which is the oblique cone $ACE = 16.7552$: then by Dr. Hutton's Mensuration, corol. 4, pa. 228, first edit.



or pa. 170, 3d edit. as $ABC^3 : AEC^3$ or as $4 : 1 :: AB^3 : GE^3$, hence $GE = 3.5634$. By similar triangles as $AB : BC :: GE : EC = 3.0859$, the distance of the point from the vertex of the cone. Again, $BE = BC - CE = 1.8130$; and by trigonometry as $BC : \sin \angle D :: CD : s. \angle B 54^\circ 44'$; hence in the triangle ABE , are given the two sides AB , BE , and the included angle B , to find $AE = 4.842$: then $HI = \sqrt{(GE \times AB)} = 4.489$ the conjugate axis of the ellipse; hence $AE \times HI \times .7854 = 17.0712$ inches, the area of the section.

The same by Mr. J. H. Hearing, Pall Mall.

Put the diameter of the globe $CF = 6 = a$, $3.141593 = b$, and the altitude CD of the greatest inscribed cone $= x$. Then will $DF = a - x$, and $AD^2 = ax - x^2$, hence the solidity of the cone is $(ax - x^2) \times \frac{1}{3}bx = a$ maximum by the question, or $ax^2 - x^3$ is a maximum also; this put into fluxions and reduced, gives x , or the altitude of the cone, $= \frac{2}{3}a = 4$ inches; consequently $AB = 2\sqrt{(ax - x^2)}$ is $= \frac{2}{3}a\sqrt{2} = 4\sqrt{2}$, the diameter of the cone's base, and $\frac{1}{3}b \times 4 \times (2\sqrt{2})^2 = 33.510325$ the content of the whole cone, and its half $= 16.755163$ the content of each part. If a cone be cut by a plane passing through the extremity of its base, and terminating in the opposite side, the figure of the section will be an ellipsis. Let $AHEI$ denote that ellipsis, and draw GE parallel to AB ; also put $AB = 4\sqrt{2} = c$, $CD = 4 = d$, half the solidity of the cone $= 16.755163 = e$, and $GE = y$. Then, by the rule for Conic Ungulas (Dr. Hutton's Mensuration, prob. 26) the content of the lower half AEB is $\frac{c^3 - y\sqrt{cy}}{c - y} \times \frac{d}{c} \times (c - y) \times c \times 0.2618 = (c^2 - \sqrt{cy^3}) \times 0.2618d = e$, per question hence $y = \sqrt[3]{\left\{ (c^2 - \frac{e}{.2618d})^2 \div c \right\}} = 3.5636$ inches, $HI = \sqrt{cy} = 4.4899$, the conjugate axis of the elliptic section, also $BC = \sqrt{(CD^2 + DB^2)} = \sqrt{24} = 4.8990$, and $AB : BC :: GE : EC = 3.08617$ inches, the distance of the required point from the cone's vertex. Then $4.8990 - 3.0862 = 1.8128 = BE$; hence AE is found $= 4.8420$ inches, the transverse diameter, and the area is $4.4899 \times 4.8420 \times .78539 = 17.0746$ square inches.

The same by the Rev. Mr. T. Scurr, Hexham.

This question is compounded of 3 distinct problems, viz. To find the greatest cone that can be inscribed in a globe of 6 inches diameter. 2d, To divide the cone so found into 2 equal parts by a plane passing from the extremity of its base; and lastly the area of the elliptic section.

1. *For the greatest cone.* Put $CF = a = 6$ inches; $CD = x$, and $c = .7854$; then $DF = a - x$. Now, by the property of the circle, $CD \times DF = AD^2$, that is $x \times (a - x) = ax - x^2 = AD^2$; hence $4acx - 4cx^2$ is the area of the cone's base, and $\frac{4}{3}acx^2 - \frac{4}{3}cx^3$ is its solidity, a maximum, or $ax^2 - x^3$ a maximum, the fluxion of which made $= 0$, gives $x = \frac{2}{3}a = 4$ inches $= CD$. Hence $AB = 2\sqrt{(ax - x^2)} = 2\sqrt{8} = 5.6568$, and the solidity $= 33.5104$.

2. *For the division of the cone.* Here put $b = AB = 5.6568$, $d = CD = 4$, $s = 16.7552$ half the cone's solidity, $CK = y$; then $KD = d - y$, and as $d : b :: y : by \div d = GE = ey$ suppose; hence (by Dr. Hutton's large Mensuration, edition 3, page 165), $\frac{b^3 - by\sqrt{bey}}{b - ey} \times .2618b \times (d - y) = s$, from which, by approximation, is easily found $y = 2.519$; then by similar triangles $CD : CB :: CK : CE = 2.8339$ nearly, the distance of the point from the vertex.

3. *For the area of the section.* Now $HI = \sqrt{(AB \times GE)} = 4.48905$, the conjugate diameter of the elliptic section; and $\sqrt{(AF^2 + FE^2)} = AE = 4.84165$ the transverse diameter. Hence $4.14165 \times 4.48905 \times .7854 = 17.0702$ is the area of the elliptic section,

The same by Mr. R. Wilkinson, North Shields.

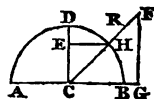
Put $CF = 6$ inches $= b$ the diameter of the globe, $x = CD$ the altitude of the cone, $a = .7854$. Then, by the nature of the circle, $bx - x^2 = AD^2$; and the solidity being as $AD^2 \times CD$, $bx - x^2$ is a maximum. Hence $x = \frac{2}{3}b = CD = 4$, and $AB = 2\sqrt{8}$, and $CA = CB = 2\sqrt{6}$, and half the cone's solidity, $= \frac{1}{3}a \times 64$. Let now $AB = D$, and $CD = H$, and $GE = d$; then (by Dr. Hutton's Mensuration, page 174) the solidity of the top part $ACE = ABE = \frac{1}{3}adh \sqrt{dd} = \frac{1}{3}a \times 64$; hence $d = 4 \div \frac{6}{2} = GE = 3.563594$; then $AB : BC :: GE : CE = 3.086163$ the distance of the dividing point from the vertex of the cone. Again, $CB : CD :: CE : CK = 2.519841$, therefore $CD - CK = EF = 1.480159$, and $AD + KE = AP = 4.610225$, and $DB - KE = FB = 1.04663$; hence $\sqrt{(AF^2 + FE^2)} = AE = 4.842006$ the transverse diameter, and $\sqrt{(AB \times GE)} = HI = 4.489847$ the conjugate; then $AE \times HI \times .7854 = 17.074492$ is the area of the section.

XIV. QUESTION 1162, by Mr. J. Skewes, *Gænnap*.

Suppose a slender rod 500 miles long fixed perpendicularly to the earth's surface in the latitude of 50° ; from the top of which, a ring of heavy metal is left to descend by its own gravity; how long will it be before it arrives at the earth's surface, supposing the force of gravity uniform, and that the earth turns round its axis in 24 hours?

Answered by Mr. J. Skewes, the Proposer

Put the earth's radius in feet $= r$, $n = 3.1416$, $c = \text{EH}$ the cosine of the latitude to the radius r , $m = 86480$ the seconds in 24 hours, $s = 32\frac{1}{8}$ feet the force of gravity, $a = \text{EF}$ the length of the rod in feet, $x = \text{HR}$ the distance of the ring from the earth's surface at the end of any time t , and v the velocity along the rod at that instant. Then as $m : 2rn :: 1'' : \frac{2rn}{m}$ the velocity per



second of a particle at the equator, and $\frac{4rn^2}{m^2} =$ its centrifugal force there, and $r : c :: \frac{4rn^2}{m^2} : \frac{4cn^2}{m^2} =$ the centrif. force at H , the given place, in direction EH , and $r : c :: \frac{4cn^2}{m^2} : \frac{4c^2n^2}{rm^2} =$ the centrifugal force at H in direction of the rod HR , and $r : r + x :: \frac{4c^2n^2}{r^2m^2} : \frac{4c^2n^2}{r^2m^2} \times (r + x) =$ the force at H to hinder the descent of the ring down the rod; consequently $s - \frac{4c^2n^2}{rm^2} \times (r + x) =$ the force that urges the ring down the rod. Then, by the principles of motion, $v\dot{v} = (s - \frac{4c^2n^2}{r^2m^2} \times (r + x)) \times -\dot{x} = -s\dot{x} + b\dot{r}\dot{x} + b\dot{x}\dot{x}$, putting b for $\frac{4c^2n^2}{r^2m^2}$; and the fluent gives $v^2 = -2sx + 2brx + bx^2$; but when $v = 0$, $x = a$, therefore the fluents corrected are $v^2 = 2sa - 2bra - ba^2 - 2sx + 2brx + bx^2 = p + qx + bx^2$ by putting $2sa - 2bra - ba^2 = p$, and $2br - 2s = q$; hence $v = \sqrt{(p + qx + bx^2)}$, and $t = \frac{-\dot{x}}{v} = \frac{-\dot{x}}{\sqrt{(p + qx + bx^2)}}$, the correct fluent of which, when $x = 0$, is $t =$

$\sqrt{\frac{1}{b}} \times$ hyperbolic logarithm of $\frac{q+2ab+2\sqrt{b} \times \sqrt{(p+qa+ba')}}{q+2\sqrt{(bp)}}$
 ≈ 406 seconds nearly, the time sought.

The same by Mr. James North.

Let c represent the earth's centre, AB the equatorial diameter, CD the semiaxis, the $\angle HCB = 50^\circ$ the latitude; continue the earth's rad. CH ($= 3980$ miles) to F , making $HF = 500$ miles the position of the rod revolving with CG ; R the place of the ring at the end of any time t , and let v be its velocity there in the direction FR . Let $FC = 4480$ miles $= r$; hence $CG = 2879.688448 = p$. Put $32\frac{1}{2}$ feet $= \frac{1}{3125}r = q$, and 24 hours $= 86400$ seconds $= s$, also $3.1416 \approx \frac{1}{2}c$; and $FR = x$, or $CR = r - x$.

Then is $(pc^2 \div rs') \times (r - x)$ the centrifugal force of the ring R in the direction CG , and its effect in direction CF is $(p^2c^2 \div r^2s'^2) \times (r - x)$. Also the force of gravity in the direction FC is q ; therefore $q - (p^2c^2 \div r^2s'^2) \times (r - x)$, is the force accelerating the ring down the rod at $R = q - m \times (r - x)$, putting $m = p^2c^2 \div r^2s'^2$. By the laws of motion, $qs' - ms' \times (r - x) = v\dot{v}$, and the fluent gives $v = \sqrt{(2qs' - m \times (2rx - x^2))} = \sqrt{(nx + mx^2)}$, putting $n = 2q - 2mr$. Then $\dot{x} = \dot{x} \div v = \dot{x} \div \sqrt{(nx + mx^2)}$; the correct fluent of which is $2\sqrt{a} \times \text{hyp. log. of } \frac{\sqrt{x} + \sqrt{(an + x)}}{\sqrt{an}}$ (when R arrives at H or x becomes $= 500$ miles) 405.9163 seconds $= 6$ minutes 45.9163 seconds the time required.

Note. The time of descending the 500 miles, with the constant gravitation, but without allowing for the centrifugal force, is 404.682 seconds $= 6$ minutes 44.682 seconds only 1.234 seconds less than the former.

PRIZE QUESTION, by Mr. James North.

Required the expence of gilding an oblique cone at $10s.$ per square foot, the diameter of the circular base being 3 feet, and the greatest and least slant sides 8 and 6 feet respectively?

Answered.

We were aware, when this question was proposed last year, that it was not a new one, as a similar question had been solved elsewhere by several persons; as Simpson, Euler, D'Alembert, &c. But the intention was, if possible, to procure some more easy modes of solution, and at the same time accurate. We have indeed received numerous solutions, but mostly referring to the former modes, when they are accurate. Some new ways are attempted, but not correct. Besides

which, several neat approximations are given, some of which we have selected as below.

1. *By Mr. Wm. Brown, of Bamburgh Castle.*

I made a cone or model of exactly equal dimensions to that in the question from a smaller scale, and covered the convex surface nicely with paper; which being taken off, and unfolded, it gave a shape the half of which is exhibited in the annexed figure. I then measured the paper as an irregular figure, and found its area come out 28·47916 feet; amounting, at 10s. a foot, to 14l. 4s. 9½d.



2. *By Mr. Thomas Myers, Hammersmith.*

The greatest and least slant sides being 8 and 6, and the diameter of the base 3 feet, the perpendicular alt. is easily found $= 5·0963128 = b$. Then the curve surface is nearly equal to the area of a circle, the radius of which is a mean proportional between r and $\frac{1}{2}\sqrt{(h^2 + r^2)} + \frac{1}{4}s + \frac{1}{4}s$ (Leybourn's Mathematical Repository, New Series, No. 3, page 113); where r denotes the radius of the cone's base, h the perpendicular height; and s and s the greatest and least slant sides. Here $r = 1\frac{1}{2}$, and $\frac{1}{2}\sqrt{(h^2 + r^2)} + \frac{1}{4}s + \frac{1}{4}s = 6·16094$, therefore $\sqrt{(6·16094 \times 1·5)} = 3·039774$, or $6·079548$ the diameter; hence the area is $29·029 =$ the curve surface of the cone, which at 10s. the square foot comes to 14l. 10s. 3½d.

Note. The method of approximation used in question 369, Ladies Diary, gives 28·9388 for the surface, probably nearer the truth.

3. *By the Rev. Tho. Scurr, Hexham.*

This gentleman gives the solution by two methods of approximation, viz. both by that used in the Diary question 369, and by that above mentioned in Leybourn's Repository. By the former he finds the surface is 29·266, and by the latter 28·775.

4. *By Mr. John Surtees, Houghton-le-Spring.*

By the former of the same approximations, viz. that in the Diary question 369, Mr. Surtees brings out the surface $= 29·25$ square feet.

Secondly by another approximation which is this, $\frac{1}{2}ps \times \left\{ d + \frac{s + s}{2s} \right\}$.

$\sqrt{\left[\frac{s}{s} \times (d^2 - (s - s)^2) \right] \left\{ \right\}}$, (where $p = 3·1416$, d the diameter of the cone's base, and s and s the greatest and least slant sides) he brings it out 29·4958.

Questions proposed in 1807, and answered in 1808.

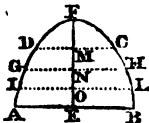
1. QUESTION 1164, by Mr. Wm. Moore, London.

Three lovely sisters uncle left
 To them a piece of gold,
 Its true dimensions are annex'd,
 The form of which is told *.
 To part it equally and right,
 You must cut parallel
 To its smooth base, and then the height
 Of each piece truly tell.
 If this you do 'twill them oblige,
 For 'tis the old man's wish;
 And where they live, if there you ride,
 You'll get from each a kiss.

* The frustum of a parabolic conoid, whose diameters are as 2 to 1, its altitude 3 inches, and solidity 94.248.

Answered by Mr. W. Crane, of Boston.

Let $2x = AB$, and $x = DC$; then, by mensuration, $5x^3 \times 3 \times .3927 = 94.248$ the content; hence $x = 4 = DC$, and $2x = 8 = AB$, the top and bottom diameters. Put $y = FM$ the height of the part wanting at the top; then by conics, $y : 4 :: 3 + y : 16$, hence $4y = 3 + y$, and $y = 1 = FM$. Again, put $z = MN$ the height of one part; then as $1 : 4 :: 1 + z : 4 + 4z = 6z^2$; hence, by mensuration again, $(16z^3 + 32z) \times .3927 = 31.416$, or $z^3 + 2z = 5$, and $z = \sqrt{6} - 1 = 1.449 = MN$ the height of one part. In like manner $NO = .866$ the height of the second part, and consequently $.6844$ the height of the third.



The same answered by Mr. Edward England, Wintringham.

By the question $AB : CD :: 2 : 1$; and by the parabola $AB^3 : CD^3 :: FE : FM$; therefore, by equality, $4 : 1 :: FE : FM$, and by division $4 - 1 = 3 : 4 :: FE - FM = 3 : FE = 4$; hence also $FM = 1$. Again, sim. segments being as the squares of their altitudes, as $16 : 1 :: AFB : DFC$, and by division, as $15 : 16 :: ABCD = 94.248 : AFB = 100.5312 = d$ the solidity of the segment AFB . And by continually subtracting one of the equal shares, we have seg. $IFL = 69.1152 = c$, and $GPH = 37.6992 = b$, and $DFC = 6.2832 = a$. And again, sim. seg. being as the squares of their altitudes, as $\sqrt{a} : 1 :: \sqrt{b} : 2.449 = FN$; and $\sqrt{a} : 1 :: \sqrt{c} : 3.3166 = FO$; then $FM - FN = 1.449 = MN$, and $FO - FN = .86713 = NO$, and $FE - FO = .683375$, the three heights required.

The same by Mr. Peter Phelan, of Carlow.

As the two diameters of the frustum are as 2 to 1, the corresponding altitudes are as 4 to 1, whose difference being 3, the same as in the question, therefore 4 and 1 are the altitudes themselves, the height of the top segment cut off being 1. And supposing the solidity of that top part to be 1, then the solidity of each division, counting from the vertex of the paraboloid, will be equal to the square of the corresponding altitude. Now the frustum contains 15 feet when the top segment is 1 foot; and the third part of 15, or 5, the content of each portion, and the contents of the segments, counted from the vertex, are $5 + 1 = 6$, and $6 + 5 = 11$; hence $\sqrt{6} = 2.4495$ the altitude from the first division to the vertex, and $\sqrt{11} = 3.31662$ the altitude from the second division, the whole altitude from the bottom being 4. Then $4 - 3.31662 = .68336$ is the height of the lowest portion, and $3.31662 - 2.4495 = .86712$ is the height of the second, and $2.4495 - 1 = 1.4495$ is the height of the upper portion.

II. QUESTION 1165, by Mr. Wm. Wiseman, Hull.

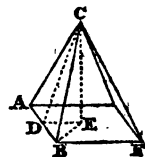
It is said that the largest Egyptian pyramid, or rather frustum, is 481 feet high; that each slant side is a trigon, and the base a square. Suppose the perpendicular height of the pyramid, if completed, be 490, which I believe is pretty near the truth; how many acres is occupied by the base?

Answered by Mr. Joseph Brewer, of Preston.

Suppose the side of the square base of a similar pyramid be $= 1$: then $1^2 - (\frac{1}{2})^2 = \frac{3}{4}$ is the square of the perpendicular of one of the faces, and $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ the square of the altitude; then, by similar figures, as $\frac{1}{2} : 1 :: 490^2 : 2 \times 490^2 = 480200$ feet $= 11$ a. 0 r. $223\frac{1}{2}$ feet, the area of the base required.

The same by Mr. Jos. Hall, of Hurdfield.

Join BE, which by the principles of geometry is $= CE = 490$ feet; and because BEC is a right angle, $BE^2 + CE^2 = 2CE^2 = 2 \times 490^2 = 480200 = BC^2$; but by the question, $AB = BC$, therefore 480200 is the area of the pyramid's base, which divided by 43560, the square feet in an acre, brings out $11\frac{223}{1080}$ acres.



The same by Mr. Wm. Wiseman, Custom House, Hull.

Let ACF be the pyramid, CE its axis; draw CD to the middle of AB,

and let $x = AB = BC = AC = BF$, &c. Now CD being perpendicular to AB , therefore $x^2 - (\frac{1}{2}x)^2 = \frac{3}{4}x^2 = CD^2$; also CE being a right angle, and $DE = \frac{1}{2}AB$, therefore $\frac{3}{4}x^2 - (\frac{1}{2}x)^2 = \frac{1}{4}x^2 = CE^2 = 490^2$; hence $x^2 = 490^2 \times 2$, and $x = 490\sqrt{2} = 692.9647$ the side of the base, and $490^2 \times 2 = 480200$, which divided by 43560, the square feet in an acre, gives 11 a. 0 r. 3.8 p. the area.

III. QUESTION 1166, by Mr. W. Putsey, Master of Pickering School.

Required the dimensions of a cone, its content being 282 cubic inches, and its slant side being to its base diameter as 5 to 4?

Answered by Mr. John Bennett, of Redruth.

Assume a cone whose base diameter is 4, and slant side 5; then its altitude will be found $= 4.582575$, and its solidity 19.195493 . Hence by similar solids, as $19.195493 : 4^3 :: 282 : 940.22$, the cube root of which is 9.796 the base diameter; and as $19.195493 : 5^3 :: 282 : 1836.36$, the cube root of which is 12.246 the slant side. Also $12.246^2 - 9.796^2 = 125.974112$, its root 11.223 is the altitude.

The same by Mr. Charles Brady, Schoolmaster, Norwich.

Let $5x$ and $4x$ denote the slant height and base diameter: then $\sqrt{((5x)^2 - (2x)^2)} = x\sqrt{21}$ is the altitude. Then, by Hutton's Mensuration, $16x^2 \times .7854$ is the area of the base, and $16x^2 \times .7854 \times \frac{1}{3}x\sqrt{21} = 282$ the content; hence, by reduction, $x = 2.44915$; then the slant height, the base diameter, and the altitude, are 12.2457, and 9.7966, and 11.22318.

The same by Mr. Alex. Raby, Hushwaite.

If we find the content of a cone whose side is 5, and base diameter 4, it is 19.1954, and its height 4.5825: then, by proportion,

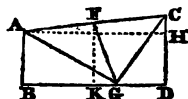
as $\sqrt[3]{19.1954} : \sqrt[3]{282} :: \begin{cases} 4.5825 : 11.223 \text{ the height.} \\ 4 \quad ; \quad 9.7966 \text{ the diameter.} \end{cases}$

IV. QUESTION 1167, by the Rev. Wm. Wright, of Snaith.

There are two stone pillars in a garden at Flaxton, whose perpendicular heights are 20 and 30 feet, and the distance between them 60 feet. A ladder is to be placed at a certain point in the line of distance, of such a length, that from thence it may just reach the top of both the pillars. Where is that point, and what is the length of the ladder?

Answered by Mr. J. A. Dotchen, of Chelsea.

1st. *Geometrically.* Connect the tops of the two pillars AB, CD, by the line AC; at the middle point F of which erect the perpendicular FK cutting the distance BD in G, the place of the ladder's foot; and AG or GC is its length. The reason is obvious by Theor. 1, Geom, of Dr. Hutton's Course.



2nd. *By Calculation.* Draw AH parallel, and FK perpendicular to BD. Then the two triangles ACH, FGK are similar, and AF being = FC by construction, FK is $= \frac{1}{2}(AB + CD) = 25$; then as AH : HC :: FK : KG $= 4\frac{1}{2}$; therefore BG = BK + KG $= 34\frac{1}{2}$ which determines the point G where the ladder is to be placed, also $\sqrt{(AB^2 + BG^2)} = \sqrt{(20^2 + (34\frac{1}{2})^2)} = 39.5899$ its length.

The same calculated by Mr. Geo. Garth, of Cadeby;

By the right-angled triangles $AG^2 = CG^2 = AB^2 + BG^2 = CD^2 + GD^2$, therefore $BG^2 - DG^2 = CD^2 - AB^2 = 500$; hence $BG - GD = 500 \div BD = 500 \div 60 = 8\frac{1}{3}$; therefore $\frac{1}{2}(60 + 8\frac{1}{3}) = 34\frac{1}{6} = BG$, and $\frac{1}{2}(60 - 8\frac{1}{3}) = 25\frac{5}{6} = GD$; then AG or $GC = \sqrt{(20^2 + (34\frac{1}{6})^2)} = \sqrt{(30^2 + (25\frac{5}{6})^2)} = 39.5899$ the length of the ladder.

Algebraic Solution by Mr. Sam. Titlow, of Hurlston.

Put a = the height of the lower pillar, b = that of the higher, c = their distance, and x = the distance of the ladder from the former, then $c - x$ = the distance of the latter; hence $a^2 + x^2 = b^2 + (c - x)^2 = b^2 + c^2 - 2cx + x^2$; therefore $2cx = b^2 + c^2 - a^2$ and $x = (b^2 + c^2 - a^2) \div 2c = 34\frac{1}{6}$; hence the length of the ladder is 39.5899.

v. QUESTION 1168, by Mr. Joseph Woollen, of Smalley.

Divide the number 16 into two parts in two different ways, so that the square of the less multiplied by the greater, and the square of the greater multiplied by the less, shall each of them be equal to 360.

Answered by Mr. John Collins, Schoolmaster, Kensington.

Put x the less number, then $16 - x$ is the greater, and in the first way $x^2(16 - x) = 16x^2 - x^3 = 360$, in which $x = 6$, and $16 - x = 10$, the two parts in the first way. Again, in the second way, $x(16 - x)^2 = x^3 - 32x^2 + 256x = 360$, the only two roots of

which, answering the question, are $5 + \sqrt{85}$, and $11 - \sqrt{85}$, or 14.2195 and 1.7805.

The same by Mr. Johnson Cotton, of Hull.

If x be put for the less number, then $16 - x$ will denote the greater; hence, by the question, $x^2(16 - x) = 16x^2 - x^3 = 360$, and $x(16 - x)^2 = x^3 - 32x^2 + 256x = 360$. The roots of the first equation are 6, and $5 + \sqrt{85}$, and $5 - \sqrt{85}$, and those of the second equation are 10, and $11 + \sqrt{85}$, and $11 - \sqrt{85}$. Hence it is evident, that in the first equation the root 6 only can satisfy the conditions of the question, for the second root would make x too large, and the third would make it negative; therefore 6 and 10 are the numbers in the first pair. It is also evident, from the second equation, that the root 10 cannot be taken for x , for then $16 - x$ would not be the greater part, and if the second root be taken, $16 - x$ will be negative; therefore $11 - \sqrt{85}$ is the root to be taken, and it gives $x = 11 - \sqrt{85}$, and $16 - x = 5 + \sqrt{85}$, for the second pair of numbers required.

The same by Mr. H. Lightbown, Schoolmaster, Edgeworth.

Put one of the greater parts $= x$, then the less part will be $16 - x$, and by the question, $x^2(16 - x) = 16x^2 - x^3 = 360$; here it is obvious that 6 is one of the roots; therefore transpose the terms to one side, and divide the equation $x^3 - 16x^2 + 360$ by $x - 6$, and it gives $x^2 - 10x - 60 = 0$, or $x^2 - 10x = 60$, one root of which is $5 + \sqrt{85}$, and then $16 - x = 11 - \sqrt{85}$. Consequently 6 and 10, and $11 - \sqrt{85}$, and $5 + \sqrt{85}$ are the parts required.

VI. QUESTION 1169, by Mr. Rd. Nicholson, of Liverpool.

If 52500 pounds can be sunk, by equal annual installments, in 14 years, when interest is allowed on the yearly balances; what is the yearly payment?

Answered by Mr. P. Barlow, of Woolwich.

This question does not appear to be clearly stated. It is evidently to find an annuity for 14 years, which shall be worth 52500 pounds, but whether this is to be considered as the ultimate value, or the present worth, does not clearly appear. In this latter case, the solution will be thus: Let x denote 1.05 the amount of 1l. for one payment, $n=14$, $A = 52500$ the amount, and a the annuity required; then, by

Dr. Hutton's Course, vol. 1, p. 260, edit. 5, $a = \frac{x-1}{x^n-1} \times Ax^n =$

5303l. 15s. 9½d. the annual payment in that case. But on the former supposition it will be $\frac{R-1}{R^n-1} \times A = 2657l. 8s. 3\frac{1}{2}d.$ for the annual payment.

The same by Mr. J. H. Hearing, Pall Mall.

Supposing r to denote the amount of 1l. in a year at a given rate of interest, a the given sum, and x the required annual payment, then $a \times \left(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \dots \frac{1}{r^{14}} \right) = a$, and $x = a \div \left(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \dots \frac{1}{r^{14}} \right) = 5303l. 15s. 7\frac{1}{2}d.$ at 5 per cent. interest,

The same by Mr. W. Moore, Woolwich.

If this 52500l. be considered the purchase of an annuity for the continuation of 14 years ; then $52500 \div 9.89864$ (the present value of an annuity of 1l. for that time, by table 2, page 119, Dr. Hutton's Dict.) gives 5303l. 15s. 2d. for the yearly payment required.

The same by Mr. Geo. Pickering, Hammersmith.

Put $52500 = a$, 14 years $= t$, the amount of 1l. for 1 year $= r$, and x the annual payment. Then, $a - x$ is the first year's balance, the interest of which is $ar - xr$, and the second year's balance $= ar - xr - x$, its interest is $ar^2 - xr^2 - xr$, and so on to the time t . Therefore, $ar^t - xr^t - 1 \dots - xr - x = 0$, and $x + xr + xr^2 + xr^3 \dots xr^t = ar^t$; hence the sum of this series $= \frac{r-1}{r^t-1} \times ar^t = 5303.7584 = 5303l. 15s. 2d.$ taking $r = 1.05$.

The same by Mr. Henry Simpson, Preston.

Put x for the part taken from the principal of the first year, and a for the amount of 1l. annuity forborn 14 years at 5 per cent. comp. interest ; then as $1 : a :: x : 52500$, which gives 2678l. 15s. 2d., and $52500 \times .05 = 2625$, this added to the former, gives 5303l. 15s. 2d. the yearly payment.

The same by Mr. John Whitley, Attercliff Academy.

Let $p = 52500$, $n = 14$, $a = 1.05$, $x =$ the yearly payment, Then the principal remaining

after 1 year $= ap - x$

after 2 years $= a^2p - ax - x$

after 3 years $= a^3p - a^2x - ax - x$

after n years $= a^np - a^{n-1}x - a^{n-2}x - a^{n-3}x \dots - ax - x$.

This principal consists of two parts, the one being a^np , and the other inverted the geometrical series $x + ax + a^2x + a^3x \dots + a^{n-1}x$, the

sum of which geometrical progression is $\frac{a^n - 1}{a - 1} \times a$, which must,

by the question, be $= a^np$; hence then $x = \frac{a - 1}{a^n - 1} \times pa^n =$

5303l. 15s. 3d. the annual payment.

VII. QUESTION 1170; by Mr. German Buxton, *London*.

To find two numbers such, that if either of them be added to the square of the other, the sum shall be a square number.

Answered by Mr. Tho. Myers, of Woolwich.

Let $r - x$ and $4rx$ denote the two numbers. Then $(r - x)^2 + 4rx = r^2 + 2rx + x^2$ is evidently a square. And to make the other condition or $16r^2x^2 + r - x$ a square, assume it $= (4rx + n)^2 =$

$16r^2x^2 + 8nr x + n^2$; hence is found $x = \frac{r - n^2}{8rn + 1}$, where r and

n may be any two numbers, provided r be greater than n^2 . By taking $r = 2$, and $n = 1$, then $x = \frac{1}{17}$, and the two numbers $\frac{17}{17}$ and $\frac{8}{17}$. If r and n be each $= \frac{1}{2}$, the numbers are $\frac{1}{2}$ and $\frac{3}{2}$. Also $\frac{1}{2}$ and $\frac{1}{2}$ will answer the conditions.

The sums by Mr. Charles Pritty, of Thornton.

Let x and $2x + 1$ denote the two numbers. Then the square of the first added to the second, makes $x^2 + 2x + 1$, a square: And the first added to the square of the second makes $4x^2 + 5x + 1$, which must be a square: assume it $= (2x - r)^2 = 4x^2 - 4xr + r^2$, and

the equation gives $x = \frac{r^2 - 1}{4r + 5}$. So that $\frac{r^2 - 1}{4r + 5}$ and $2 \times \frac{r^2 - 1}{4r + 5}$

+ 1 are the two numbers. If $r = 3$, the numbers are $\frac{8}{17}$ and $\frac{17}{17}$. If $r = 2$, the numbers are $\frac{1}{2}$ and $\frac{3}{2}$; &c.

VIII. QUESTION 1171, by Mr. I. Rowbottom, *West Hallam*.

If 100 pounds be due one year hence, 200 pounds two years hence, 300 pounds three years hence, and 400 pounds four years hence; at

what time must all be paid together, without loss to debtor or creditor, allowing five per cent. compound interest?

Answered by Mr. Peter Dobson, of Preston.

The present worth of the four sums is $\frac{100}{1.05} + \frac{200}{1.05^2} + \frac{300}{1.05^3} + \frac{400}{1.05^4} = 864.875$. And by theor. 4, Hutton's Course, vol. 1, page 264, fifth edition, the $(\log. 1000 - \log. 864.875) \div \log. 1.05 = 2.9754$ years, the time in which 864.875 will amount to 1000, which therefore is the true equated time.

IX. QUESTION 1172, *by the Rev. J. Furnass, Ponteland.*

Having succeeded in dragging a globe of lead, of six inches diameter, up an inclined plane, whose perpendicular altitude is 50 yards, and inclination to the horizon 30 degrees; but just when it rested the chain came off close by the ball, and consequently it fell. Required the time and velocity with which it reached the ground, the plane's resistance being such as just made the globe roll?

Answered by the Rev. J. Furnass, Ponteland.

The height of the inclined plane being 1800 inches = h , and its angle 30° , whose sine is half radius, the length of the plane will be double its height, or 3600 inches = l . Now, if the sphere were to slide, $\frac{1}{2}$ would be its accelerating force; but as the centre of oscillation is $\frac{1}{2}$ of the radius distance from the surface of the sphere, therefore as $7 : 5 :: \frac{1}{2} : \frac{5}{7}$ the accelerating force in rolling. Then, putting $g = 193$ inches, v = the velocity and t the time, by Dr. Hutton's Conics, page 169, we have $v = \sqrt{2gh} = 996.2787$ inches = 83.023225 feet, the velocity, and $t = \sqrt{(7l \div 5gh)} = 7.2268$ seconds, the time required.

The same by Mr. Tho. Smith, of Hethersett.

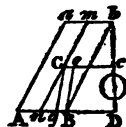
As above the accelerating force is $\frac{5}{7}$. But the plane's resistance causing the body to roll, it will lose $\frac{1}{4}$ of its gravity (Mr. Gregory's Mechanics, art. 316) hence the accelerating force is $\frac{5}{7} \times \frac{3}{4} = \frac{15}{28}$, which put = f ; also put $s = 300$ feet the plane's length, $g = 16\frac{1}{2}$, v = velocity, and t = time required. Then, by theor. 6, Dr. Hutton's Practical Exercises, page 169, or vol. 2, of his Course, page 381, we have $v = \sqrt{4gfs} = 83.0232$ feet, the velocity, and (theor. 7) $t = \sqrt{(s \div gf)} = 7.2268$ seconds, the time of descent.

X. QUESTION 1173, by Mr. Wm. Burdon, *Acaster Malbis.*

A piece of oak timber, in the form of an oblique cylinder, stands by one of its ends on a horizontal plane, the inclining side making an angle of 60° with the horizon; its perpendicular height is 4 feet, and the diameter of its base 3 feet. Required the diameter of the greatest globe of oak that the piece will bear, without falling, when suspended by a string from the top, as in the annexed figure?

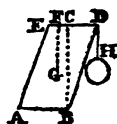
Answered by Mr. Brice Bronwin, Harleston.

By plane trigonometry BD or $oc = 2.3088$ feet. And nc or cm being $= \frac{1}{2}mn$, therefore $ng = \frac{1}{2}BD = 1.1544$, and $ga = .3455$. As the cylinder and globe must be in equilibrium about the point a , ao being perpendicular to AD , o will be the common centre of gravity, co and oc being the two arms of the lever, coc , which must be reciprocally proportional to the weights of the sphere and cylinder. Hence as $2.3088 : .3455 :: .7854 \times 3^3 \times 4 : 4.2311$, the solidity of the globe; whence its diameter is found $= 2.0066$ feet nearly.



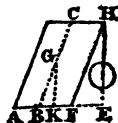
The same by Mr. John Smith, Alton Park.

a being the centre of gravity of the cylinder $ABDE$, let GF and BC be perpendicular to DE . By trigonometry as radius $1 : BC = 4 :: \text{tangent } CBD (30^\circ) : CD = 2.3094$; therefore $FC = \frac{1}{2}EC = \frac{1}{2}(ED - CD) = .3453$. Put $a = AB = 3$, $b = BC = 4$, $c = FC = .3453$, $d = CD = 2.3094$, $n = .78539$ &c. and x = the globe's diameter; then a^2nb and $\frac{2}{3}\pi x^3$ are the solidities of the cylinder and globe, which are to each other as $3a^2b$ to $2x^3$, hence by statics $3a^2bc = 2x^3d$, and $x = \sqrt[3]{(3a^2bc \div 2d)} = 2.00615$ the diameter of the globe.



The same by Mr. John Surtees, Houghton le-Spring.

Let BC be the axis of the cylinder AB , g its centre of gravity; also EK and HE perpendicular to AE . Then $BG = GC$, $AB = BF = \frac{3}{2}$, and F is the fulcrum; hence, by trigonometry as $\text{sine } FBK : GK :: \text{sine } BOK : BK = 2\sqrt{\frac{1}{3}}$, and $FE = 2BK = 4\sqrt{\frac{1}{3}}$, also $KF = BF - BK = \frac{3}{2} - 4\sqrt{\frac{1}{3}}$. Then by mechanics $KF \times \text{the cylinder} = FE \times \text{the globe}$, or $(\frac{3}{2} - 4\sqrt{\frac{1}{3}}) \times 36 \times .7854 = 4\sqrt{\frac{1}{3}} \times \frac{3}{2} \times .7854 \times \text{cube of the globe's diameter}$; therefore the diameter $= \sqrt[3]{\frac{3}{2} \div (6\sqrt{3} - 8)} = 2.00616$ feet.

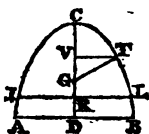


XL. QUESTION 1174, by Mr. Tho. Myers.

A piece of standard gold, in the form of a paraboloid, the greatest double ordinate of which is 10, and the altitude 6 inches, is to be divided into two parts by a section parallel to the base, and the distance between them equal to half the nearest distance of its centre of gravity from its surface: What is the value of each part, supposing the gold to be worth 4l. per ounce?

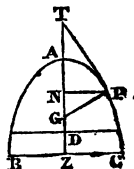
Answered by the Rev. J. Furnass, Ponteland.

The annexed figure being a vertical section through the axis of the paraboloid, and G its centre of gravity; in which the base $AB=10$, and the axis $CD=6$; also IL the dividing line, GT the nearest distance to the curve, and TV an ordinate. Then, by Dr. Hutton's Course, vol. 2, page 325, ex. 8, we have $CG = \frac{2}{3}CD = 4$. Put now $CD = a$, $AD = b$, $CG = \frac{2}{3}a$, $CV = x$; then (ibid. theor. 1 parab.) as $CD : AD^2 :: CV : VT^2$, and $VT = \sqrt{(b^2x \div a)}$; and $GT = \frac{2}{3}a - x$; hence $GT^2 = GV^2 + VT^2 = \frac{4}{9}a^2 - \frac{4}{3}ax + x^2 + b^2x \div a$; a minimum, or $ax^2 + b^2x - \frac{4}{3}a^2x$ a minimum; the fluxion put = 0 gives $x = \frac{2}{3}a - b^2 \div 2a = 1.916$; hence $GT = 3.51089$, and $DR = \frac{1}{2}GT = 1.75545$, and $CR = 4.24455$; also as $CD : AB^2 :: CR : IL^2 = 70.74425$. Then (ib.) $IL^2 \times \frac{1}{2}CR \times .7854 = 117.916447 =$ the content of the segment ICL ; and (ibid. theor. 19 parabola) $(AB^2 + IL^2) \times \frac{1}{2}DR \times .7854 = 117.703315$ the content of the frustum $AILB$. Then, taking 18888 for the specific gravity of standard gold, we find, 1286.562623 and 1288.892274 for the weights in avoirdupois ounces, or 1172.648224 and 1174.7416039 troy ounces; which, at 4l. the ounce, amount to $4690.592896l.$ and $4698.9664156l.$ or $4690l. 11s. 10\frac{1}{4}d.$ and $4698l. 19s. 3\frac{1}{4}d.$ the values of the two parts.



The same by Mr. John Surtees,

Let AZ be the axis, and G the centre of gravity of the paraboloid; TP a tangent to the curve at P , perpendicular to PG the shortest distance to the curve. Then, by conics, $p = BZ^2 \div AZ = \frac{25}{6}$, $NG = \frac{1}{2}p = \frac{25}{12}$, $AG = \frac{2}{3}AZ = 4$, $AN = AG - NG = \frac{23}{12}$, $NP = \sqrt{(p \times AN)} = \sqrt{(\frac{25}{6} \times \frac{23}{12})}$, and $PG = \sqrt{(NP^2 + NG^2)} = \frac{1}{12}\sqrt{71}$, $DZ = \frac{1}{2}PG = \frac{1}{24}\sqrt{71}$ and $AD = 4 - \frac{1}{24}\sqrt{71}$. Hence the solidity of the upper part = $4p \times AD^2 \times .3927 = 117.9159$, and that of the lower part = $4p \times (AZ^2 - AD^2) \times .3927 = 117.7035$ inches.

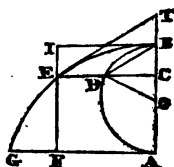


XII. QUESTION 1175, by Mr. Peter Barlow, *Shipdham*.

A person, dying, left a field, in form of a semicycloid, containing 12 acres, to be divided among his three sons in the following manner; viz. to his eldest son the greatest rectangle that can be described in the field; to his second son the remaining part on the base; and to his youngest the remaining part on the axis. What were the three shares?

Answered by Mr. Thos. Myers, Woolwich.

Let ABG be the semicycloid, AE the greatest inscribed, rectangle ABG half the rectangle generating circle, and o its centre, TE a tangent, and the other letters as in the figure. Then by the cycloid, the area of the semicircle $ABD = 4 = \frac{1}{2}ABG$, and therefore $AB = 10.095$ chains. Now put $AB = a$, and $BC = x$, $AC = a - x$, and by the circle $CD = \sqrt{(ax - x^2)}$, and $BD = \sqrt{(ax)}$, therefore (rule 4, p. 193, Hutton's Mensuration, 3d. edit.) $\frac{1}{2}\sqrt{ax} - \frac{1}{2}\sqrt{(ax - x^2)}$ = the arc BD , which is also = DE ; consequently $CD + DE = \frac{1}{2}\sqrt{ax} + \frac{1}{2}\sqrt{(ax - x^2)} = CE$. Then, as BD is parallel to TE , and $AC = CT$ when the rectangle AE is a maximum, by similar triangles $CB : CD :: CT : CE$, or $x : \sqrt{(ax - x^2)} :: a - x : (a - x) \sqrt{(ax - x^2)} \div x = CE$. These two values of CE being equated, and reducing a quadratic equation, gives $x = 3.09834$: therefore $AC = 6.99666$, and $CE = 10.502$. Then by the common rules, the area of the rectangle $AE = 73.47892$, that of $CI = 32.53877$; $CD = 4.6057$, and the arc $BD = 5.8963$. The area of the sector $oBD = 14.8808$, that of the triangle $oCD = 4.4888$, therefore that of the semisegment $BDC = 10.392$ = (by the nature of the cycloid) the external cycloidal space BIE ; hence the area of BEC , the part on the axis, = 22.14677 , and of $GEF = 24.37331$. Thus the three parts are 7 acres 1 rood 15.66272 p., and 2 ac. 0 r. 34.34832 p., and 2 ac. 1 r. 29.97296 p.



The same by Mr. Wm. Wiseman, Land Surveyor, Hull.

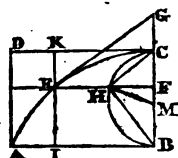
Let ΔEC be the cycloid, CB the diameter of its generating circle, EB the greatest rectangle, and EG a tangent at the point E . Put $z = CB$ $c = 3.1416$; then $\frac{1}{2}cz = AB$, $\frac{1}{2}cz \times z = \frac{1}{2}cz^2$ = the rectangle DB , and $\frac{1}{2}cz^2 \times \frac{1}{2} = \frac{1}{4}cz^2$ = 12 acres = 1200000 links; hence $z = \sqrt{(3200000 \div c)} = 1009\frac{1}{4} = CB$: the half of which call a , and put $b = .01745$, y = number of degrees in arc CH , $x = cR$, $2a - x = DB$, $\sqrt{(2ax -$

x^2) = FH, aby = arc CH, and by prop. cycloid $aby + \sqrt{(2ax - x^2)}$ = EF. Now it is well known, that when the inscribed rect. is a max. FB must = FG, and that

$$\frac{EF \times FC}{FH} = FG = FB, \text{ theref. } \frac{aby + x\sqrt{(2ax - x^2)}}{\sqrt{(2ax - x^2)}} \\ = \frac{aby}{\sqrt{(2ax - x^2)}} + x = 2a - x. \text{ Now let } (1 +$$

$$\frac{x}{12a} + \frac{3x^2}{160a^2} + \&c.) \sqrt{ax}, \text{ expressing the length}$$

of arc CH be substituted for aby , or y may be found by the common method of approximation = $67^\circ 6'$, and $x = 308.38$. Hence the length, breadth, and content of the rectangle EB, are 1055.76, and 700.86 and 739929 = 7 acres 1 rd. 24 p. the eldest son's portion. Now the area CEK = area CHF, therefore area EKCF = area CHF = area CEF = 2 ac. 0 r. $35\frac{1}{2}$ p. the youngest's lot, consequently 2 ac. 1 r. $20\frac{1}{2}$ p. is the portion of the second son.

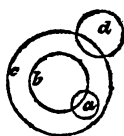


XIII. QUESTION 1176, by Mr. Peter Dobson, Preston.

Suppose a wheel A turns another wheel B, on whose axis is a wheel C, that turns a wheel D; now it is required to find the least number of teeth in each of these 4 wheels, so that the wheels A and D may make equal revolutions; when the wheel D is supposed to roll round the wheel C in 365 turns of the wheel A?

Answered by Mr. Joseph Brewer, Preston.

Let a, b, c, d , represent the 4 wheels, these letters denoting the number of teeth in those wheels. Then $c \div b$ are the turns b or c has for 1 of a , and $c \div d$ the turns d has for 1 of c , consequently $ac \div bd$ are the turns d makes for 1 of a , which would be = 1 if the wheel d was stationary, because a and d are to revolve in equal times. But $c \div d$ are the turns which d gains or loses in 365 turns of the wheel a , according as it is carried round the contrary way, or the same way as the motion of the wheel c ; and as $365 : c \div d :: 1 : c \div 365d$ the part of a turn d gains or loses in 1 turn of a . Hence arises this equation $\frac{ac}{bd} \pm \frac{c}{365d} = 1$, or $\frac{a}{b} = \frac{d}{c} \pm$



$\frac{1}{365}$ Here 365 is divisible into the two factors 73 and 5; theref.

take $c = 73$, and multiply the equation by 5, then it becomes $\frac{5a}{b} =$

$\frac{5d \pm 1}{73}$. Now to find the least value of d so that $\frac{5d \pm 1}{73}$ may be a whole number, multiply it by 29, which gives $\frac{145d \pm 29}{73} = \text{a whole number}$; but $\frac{73d}{73} \times 2 = \frac{146d}{73}$ is a whole number; the diff. of these two is $\frac{d \pm 29}{73} = n$, or $d = 73n \pm 29$. If d roll round c the same way that c revolves, then the upper sign must be used; but if the contrary way, the lower sign. In the first case take $n = 0$, then $d = 29$, and $5a \div b = 2$, or $2 : 5 :: a : b$ the ratio of a to b : hence the numbers may be, $a = 10$, $b = 25$, $c = 73$, $d = 29$. In the latter case take $n = 1$, then $d = 44$, and $5a \div b = 3$, or $3 : 5 :: a : b$, the ratio of $a : b$, in this case; therefore the numbers may be, $a = 15$, $b = 25$, $c = 73$, $d = 44$.

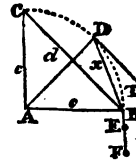
XIV. QUESTION 1177, by the Rev. Mr. T. Scurr, Hexham.

If a leaden ball, 2 inches in diameter, be supposed to strike perpendicularly a pendulous piece of oak, in form of a cylinder, and impel it through an angle of 60° , the radius of that arc being 36 inches. It is required to assign the velocity of the ball, and the cylinder's greatest perpendicular force on the centre of suspension, the whole solidity being 15552 cubic inches: the axis of motion being in the vertex, and the ball supposed to strike the cylinder 6 inches from the lower end?

Answered by Mr. W. Saint, of Norwich.

From Dr. Hutton's Course, vol. 2, prob. 2, p. 260, the weight of the cylinder, or pendulum = 8325 oz. = $520\frac{5}{8}$ lb. of the ball = $1\frac{3}{4}$ oz. Now this question may be easily solved by the application of prop. 51, page 212, vol. 2, of the course; for taking the figure and letters there used, we have $p = 8325$ oz. $b = 1\frac{3}{4}$ oz. $g = 18$ inches = so , $o = 24$ inches = so , $i = 30$ inches = sr , v = velocity of the ball, $c = 24$ inches = chord of the arc of 60° to radius so , $a = 16\frac{1}{2}$ feet = 193 inches; then $v = \frac{bii + gop}{bio} \times c \sqrt{\frac{2a}{o}} = 1470.69$ feet per second the velocity of the ball. And, by Dr. Keil's Nat. Philos. th. 5, p. 292, the centrifugal force (if that be what is meant in the question by the greatest perpendicular force on the centre of suspension) will be equal to its gravity, or the force on the centre equal to its weight.

the same when opened to a right angle CAB , and when CBE is the cord by which the door is drawn, with the wt. x at its end, and passing freely over the pulley fixed at B ; also let AD be any other position of the door, DBF being the string in that case. Now put $a = 2lb$. the balancing weight in the position AC , acting in the direction CB ; $b = 3lb$. the constant urging weight x , acting in direction of the cord DB when the door is in the position AD ; $c = AC$ or AD or $AB = 3$ feet, the breadth of the door; $d =$ the cord $CB = c\sqrt{2} = 3\sqrt{2}$; $e = a\sqrt{\frac{1}{2}} = \frac{1}{2}a\sqrt{2} = \sqrt{2}$, the force or weight which, acting at c in a direction perpendicular to AC , will have the same effect as the force a acting in the direction CB , being the perpendicular resistance of the door: also put $g = 32\frac{1}{2}$ feet the natural force of gravity; $v =$ the velocity of the descending weight x when the door is in any position AD , or v the velocity of the point D in the direction of the cord DB ; $u =$ the velocity of the point D in the direction of the arc at D or of the tang. DT , these two always being to each other as the sine of the angle ADB to radius; and become ultimately $=$ at the point B , or when the door is just shut; $x =$ the varying or decreasing cord DB ; and $z =$ the sine of the angle ADB or ABD to the radius 1 ; also $t =$ the time in motion at any position AD ; and $\tau =$ the tension of the string DBF .



Then, by mechanics, $bg =$ the absolute force of the weight x , $bg - \tau$ the efficacious force of the same; τ is the force to move D in direction of the chord DB , and τz the same in direction of the arc or tangent DT ; also $e = a\sqrt{\frac{1}{2}} = \frac{1}{2}a\sqrt{2}$ is the force acting in opposition to τz in the direction TD . Hence we have these four original equations.

$$(1) b\ddot{v} = (bg - \tau) \dot{t} = bgt - \tau \dot{t}, \quad (2) e\ddot{u} = (\tau z - eg) \dot{t} \\ (3) v\dot{t} = B'F = -D'B = -\dot{x}, \quad (4) v = uz.$$

From the first and second equations $\tau \dot{t} = bgt - b\ddot{v} = \frac{eu + e\dot{g}\dot{t}}{z}$,

or $bgt\dot{t} - b\ddot{v}\dot{t} = eu + e\dot{g}\dot{t}$. Write $\frac{v}{u}$ for z , then $\frac{bgt}{u} \dot{t} - \frac{bv\ddot{v}}{u} = eu + e\dot{g}\dot{t}$, or $bgt\dot{t} - bv\ddot{v} = euu + e\dot{g}u\dot{t}$, or $bv\ddot{v} + eu\dot{u} = (bgt - egu) \dot{t}$.

For $\dot{t}$ write $-\frac{\dot{x}}{v} = \frac{\dot{y}}{u}$, then $bv\ddot{v} + eu\dot{u} = (-bx - cy)g$, putting $y =$ the arc CD ; the fluent is $bv^2 + eu^2 = (-bx - cy)2g$. But when $x = d$, $y = 0$, $v = 0$, $u = 0$, therefore $bv^2 + eu^2 = 2g \times (b(d - x) - cy)$. When $x = 0$, $v = u$, and $y = CDB$, then $(b + e) \times v^2 = 2g(bd - cy)$, and $v = \sqrt{2g \times \frac{bd - cy}{b + e}}$ for the last velocity.

Then $b = 3$, $d = 3\sqrt{2}$, $e = \sqrt{2}$, $y = 4.7124$, therefore $v = \sqrt{(2g \times \frac{bd - cy}{b + e})} = 9.3760$ the last velocity.

The time may be found by an infinite series.



Questions proposed in 1808 and answered in 1809.

I. QUESTION 1179, by Mr. J. Jackson, Sherburn.

The solidity of a cone is 2513.28 inches, and the slant side of a frustum of it, whose solidity is 2474.01, is 19.5 inches. Required the dimensions of the cone?

Answered by Mr. J. Brewer, of Preston.

First $2513.28 - 2474.01 = 39.27$ the solidity of the top part. Then if the slant side of the whole cone be called x , it will be $\sqrt[3]{2513.28} : \sqrt[3]{39.27} :: x : \frac{1}{4}x$ the slant side of the top part, and $x - \frac{1}{4}x = \frac{3}{4}x = 19.5$ by the question, hence $x = 26$ the slant side of the cone.

Now if y be put for its altitude; then $26^2 - y^2$ is the square of the radius of its base, and the solidity $\frac{1}{3}cy(26^2 - y^2) = 2513.28$, where $c = .7854$, or $676y - y^3 = 2400$; hence $y = 24$ the altitude, and then the diameter of the base is found = 20.

The same answered by Mr. John Davey, of St. Just, Cornwall.

It is manifest that the difference of the solidities of the cone and frustum ($= 39.27$) is a cone, and similar in all respects to the greater cone; dividing the greater by the less cone, we have 64, or the solidities as 64 to 1; therefore all their dimensions are as 4 to 1, and consequently the difference of their slant heights, &c. to that of the less, as 3 to 1; therefore $19.5 \div 3 = 6.5$ is the slant height of the less, and $19.5 + 6.5 = 26$ is that of the greater; which put $= a$, and $x =$ its perpendicular height, also $c = .2618$, then is $2\sqrt{(a^2 - x^2)}$ the diameter of the base, and $(4a^2 - 4x^2)cx = 2513.28$ the solidity; hence $x = 24$ the altitude, and $2\sqrt{(a^2 - x^2)} = 20$, the base diameter.

The same by J. H. Hearing, Adderbury.

Put the given solidity of the whole cone $2513.28 = a$, that of the

frustum $2474.01 = b$, the slant side of the frustum $19.5 = c$, and the slant side of the upper part $= x$, whose solidity is therefore $a - b$, and the slant side of the whole cone $c + x$. Then, by similar solids, $\sqrt[3]{a} : \sqrt[3]{a-b} :: c+x : x$, or $\sqrt[3]{a} : \sqrt[3]{a-b} :: c : x$, which gives $x = 6\frac{1}{2}$, and hence $c + x = 26$, the slant side of the large cone.

Again, putting $26 = d$, $3.1416 = p$, and the cone's altitude $= y$, we have $p(d^2 - y^2)$ the area of the base, and $\frac{1}{3}py(d^2 - y^2)$ the cone's solidity $= a$; hence in numbers $y^3 - 676y + 2400 = 0$, the positive roots, of which equation are 24 and $2\sqrt[3]{61} - 12 = 3.620499$, either of which may be taken for the cone's altitude. If $y = 24$, the radius of the base will be 10; but if $y = 3.620499$, the radius of the base will be 25.7467 ; and the solidity in both cases is 2513.28 , as in the question.

The same by Mr. John Smith, Alton Park.

Put $x =$ the slant side of the top part of the cone, the solidity of which is $2513.28 - 2474.01 = 39.27$. Then, by similar cones, $\sqrt[3]{29.27} : \sqrt[3]{2513.28} :: x : 4x$ the slant side of the whole cone; hence $3x$ is the slant side of the frustum $= 19.5$, and $x = 6.5$, and $4x = 26$ that of the whole cone; which let be represented by a , and the perpendicular height by y ; then the radius of the base will be $\sqrt{(a^2 - y^2)}$, and the solidity $(a^2y - y^3) \times \frac{1}{3}c = 2513.28$, where $c = 3.1416$; hence, $y = 24$ the perpendicular, and $\sqrt{(a^2 - y^2)} = 10$, the radius of the base.

II. QUESTION 1180, by Mr. Peter Dobson, Preston.

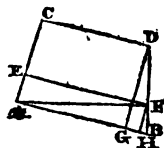
A circular vessel, whose top and bottom diameters are 70 and 92, and perpendicular depth 60 inches, is so elevated on one side, that the other becomes perpendicular to the horizon; required what quantity of liquor, ale measure, will just cover the bottom when in that position?

Answered by Mr. Rob. Nunn, Eye Academy.

The side $BD = \sqrt{(BG^2 + DG^2)} = 61$. Now $BD : BG :: AB : BF = 16.59$; and $BD : BG :: BF : BH = 3$ very nearly; hence $BF = 86$; and $BD : DG :: BF : FH$. Hence, in Dr. Hutton's Mensuration, page 220, is this theorem,

$$\frac{92^2 - 86\sqrt{(92 \times 86)}}{92 - 86} \times .2618 \times 92 \times 16.318$$

$= 53334$ cubic inches $= 189$ gallons very nearly, the content required.



The same answered by Mr. G. Pickering, Hammersmith.

Let ABCD be the vessel, AD the side perpendicular to the horizon, and AF the surface of the liquor. Now, AB = 92, CD = 70, and DG = 60. Then $\frac{1}{2}AB - \frac{1}{2}CD = BG = 11$; and as DG : BG :: radius : tangent $\angle BDG = 10^\circ 23' 20''$, its complement $79^\circ 36' 40'' = \angle DBG$. Then in the right-angled triangle ABF are given AB and all the angles, to find BF = 16.59016. Again, in the right-angled triangle BFH, are given all the angles and the side BF, to find FH = 16.3182. Then, by similar triangles, DG : BG :: FH : BH = 2.99167; and AB - 2BH = EF = 86.01666. Now put AB = D, EF = d, and FH = h;

then by prob. 26, page 165, Dr. Hutton's Mens. $\frac{D^2 - d\sqrt{Dd}}{D - d} \times .2618Dh = 53347.163$ cubic inches = 189.174 ale gallons, as required.

The same by Mr. John Smith, of Alton Park.

Let ABDC represent the vertical section through the axis of the vessel, and the surface of the liquor requisite to cover the bottom when the side BD is perpendicular to the horizon. Then BG = $\frac{1}{2}(AB - CD) = 11$, and BD = $\sqrt{(DG^2 + BG^2)} = 61$; and by similar triangles,

$$BD : BG :: AF : BF = 16.59016,$$

$$BD : DG :: BF : FH = 16.31819,$$

$$DG : BG :: FH : BH = 2.99167,$$

Then AB - 2BH = EF = 86.01666. Put D = AB, d = EF, and h = FH, then, by problem 24, part 3, section 1, of Dr. Hutton's Mensuration,

$\frac{D^2 - d\sqrt{Dd}}{D - d} \times .2618Dh = 53346.93386$ inches = 189.1735 ale gallons, the content of the ungula AFB required.

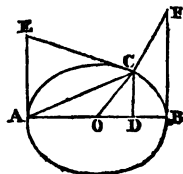
III. QUESTION 1181, by Mr. W. Burdon, Acaster Malbis.

There is a circular fishpond, at the extremities of a diameter of which are two pillars, the one 5 yards and the other 9 yards high. Now a ladder placed in the periphery at 30° of the arc from the higher pillar, will just reach to the top of each of them. Required the diameter of the pond?

Answered by Mr. Jos. Brewer, of Preston.

It is easily proved that the diff. between the squares of the two pillars, is equal to the diff. between the squares of the two seg. AB, DB, which in

the present case is 56. Now if the radius $oc = ob$, be called 1, then the sine $cd = \frac{1}{2}$, because the arc bc is 30° ; hence $od = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$; this added to, and subtracted from the radius ao and ob , gives $ad = 1 + \frac{1}{2}\sqrt{3}$, and $db = 1 - \frac{1}{2}\sqrt{3}$, the difference between the squares of which is $2\sqrt{3}$; then, by similar figures, $2\sqrt{3} : 56 :: 1^2 : \frac{56}{2}\sqrt{3} = 16.165807$ the square of the rad. ob , the root of which is $4.02067 = ob$, the double of which is 8.04134 yards, the diameter of the pond required.



The same by Mr. John Collins, Schoolmaster, Kensington.

Let ABC represent the pond, and AE , BF , the two pillars; of which put $BF = a$, $AE = b$, and the diameter $AB = x$, c being the point in the circumference where the foot of the ladder is to be placed. By theor. 51, Dr. Hutton's Geom. Course, the $\angle CAB = \frac{1}{2}\angle COB = 15^\circ$, the sine and cosine of which let be s and c ; then, by trigonometry, $bc = sx$, and $AC = cx$; hence by the nature of the question, and right-angled triangles, $s^2x^2 + a^2 = c^2x^2 + b^2$, which gives $x = \sqrt{[(a^2 - b^2) \div (c^2 - s^2)]} = 8.04134$ the diameter.

The same by Mr. John Davey, of St. Just, Cornwall.

Suppose $2x = AB$ the diameter of the pond, $cb =$ the chord of 30° , $BF = 9$ yards, and $AE = 5$. Then, by trigonometry, $bc = x\sqrt{2 - \sqrt{3}}$ and $AC = x\sqrt{2 + \sqrt{3}}$. Then, by the question and right-angled triangles, $AC^2 + AE^2 = BC^2 + BF^2$, that is, $2x^2 + x^2\sqrt{3} + 25 = 2x^2 - x^2\sqrt{3} + 81$; hence $2x = 4\sqrt{(7 \div \sqrt{3})} = 8.04134$, the diameter sought.

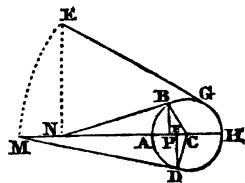
IV. QUESTION 1182, by Mr. W. Watkins, Heydon Bridge.

Admit a lighted candle placed, horizontally with the centre, 10 feet from the surface of a globe, whose superficies in feet is expressed by the same number as its solidity; at what perpendicular height from the candle must an eye be placed, so that the difference of the convex surfaces of the two segments, made by tangents from the eye and the candle to the globe, shall be equal to the fourth part of the illuminated segment?

Answered by Mr. J. A. Dotchen, Exeter Street, Chelsea.

It is well known or easily discovered, that the globe's diameter

must be six feet, hence the question may be easily solved geometrically as follows : Describe the circle $ABHD$ on the diameter $AH = 6$, on which continued set off $AN = 10$; draw the indefinite perpendicular NE , also the tangent NB , and the perpendicular ordinate BP ; make $PF = \frac{1}{4}AP$, erect the perpendicular FD , and draw the tangent DM ; from the centre C , with the radius CM describe the arc ME ; then will E be the place of the eye required; which easily appears from the well known property, that the convex superficies of spherical segments are as their altitudes.



If the height NE be desired in numbers, it may be found thus : $CN : CB :: CB : CP = \frac{9}{13}$; $AP = AC - CP = 2\frac{4}{13}$; $AF = \frac{1}{4}AP = 2\frac{3}{13}$; $FC = AC - AF = \frac{2}{5}$; $FC : CD :: CD : CM = 78 = CE$; $NE = \sqrt{(CE^2 - CN^2)} = 76.9093$ feet.

The same by Mr. Thomas Myers, R. M. A.

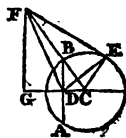
Let N be the candle, and E the place of the eye, c the centre of the globe, and $ABHD$ a section of it in the plane CNE . Draw the tangents NB , and EG , or its equal MD , having taken $CM = CE$, and drawn the other lines as in the figure. Then the diameter of the globe answering the given conditions, being 6 feet, we have $CN = 13$, and from the similar triangles CNB , CBP , $CP = \frac{9}{13}$; therefore $AP = 3 - \frac{9}{13} = \frac{30}{13}$ the height of the illuminated segment. And as the heights of segments are to each other as their surfaces, $\frac{30}{13} \times \frac{5}{8} = \frac{15}{8}$ the height of the segment seen from E ; hence $3 - \frac{15}{8} = \frac{9}{8} = CF$; then in the similar triangles CDF , CMD , or CGE , $\frac{9}{8} : 3 :: 3 : 78 = CE$ or CM ; consequently $NE = \sqrt{(78^2 - 13^2)} = 76.909$ feet, the height required.

V. QUESTION 1183, by Metro, of Bristol.

In the circle BAE , let AB be any given chord bisected in D , and DE any line drawn from D to the circumference, at right angles to which, from the centre C , let CF be drawn, and intersected by EF , a tangent at E : the locus of the point F shall be a right line. Required the geometrical demonstration?

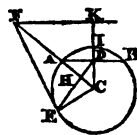
Answered by Mr. P. Barlow, R. M. A.

Let ABE be any circle, and AB , DE , CF , FE , lines drawn as per question; join CD , and produce it to meet the perpendicular FG ; also join FD . Now since CD and CE are constant, $CE^2 - CD^2 = FE^2 - FD^2$ is constant, and $FC^2 - FE^2$ being also constant, GC is constant, and consequently the point F is always found in the line FG parallel to AB .



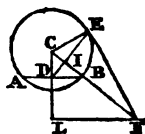
The same answered by Mr. J. A. Dotchen, of Chelsea.

Join CE , CD , and through F draw FK perpendicular to CD produced. Then the triangles CHD , CFK being similar, $CD : CH :: CF : CK$, whence $CD \cdot CK = CH \cdot CF =$ (theor. 87, Geom. Dr. Hutton's Course) CE^2 . But CD and CE are given, therefore CK is given, and the angle at K is also given, being a right one by the construction. Hence the locus of F is the right line FK given in position.



The same by Mr. Wm. Moore, R. M. A.

The lines being drawn as in the question, join CD , which produce to meet the perpendicular FL in L , and join CE . Then by the similar triangles CEI , CEF , we have $CE^2 = CI \cdot CF$; and in the similar triangles CID , CLF , as $CI : CD :: CL : CF$, therefore $CI \cdot CF = CD \cdot CL$; but $CE^2 = CI \cdot CF$, therefore $CD \cdot CL = CE^2$; and since CD , CE are given, CL is also given, and consequently LF is a right line given in position, and the locus of the point F .



VI. QUESTION 1184, by Mr. Tho. Smith, *Hethersett*.

Admitting the latitude of Warsaw to be $52^\circ 14'$ north, and its longitude $21^\circ 10'$ east; and the latitude of Paris to be $48^\circ 50'$ north, and its longitude $2^\circ 25'$ east; also the earth to be a perfect sphere, and its circumference 25000 miles; what is the distance between Paris and Warsaw. And if a messenger should go directly from Paris to Warsaw, and return from the last place again, bearing exactly on the opposite point of the compass, to what he first set out on from Paris, and shall travel in this direction exactly as many miles as he did on his first journey; how far will he then be from Paris; and on what point of the compass must he then travel, going direct, to arrive there?

Answered by Mr. A. Nesbit, Master of Preston Free School.

Let N represent the North Pole, P Paris, W Warsaw, also A the place arrived at when returned from the same. Then, in the spherical triangle PNW are given the two co-latitudes PN , WN , viz. $PN = 41^\circ 10'$, and $WN = 37^\circ 46'$, also their difference of longitude, the $\angle PNW = 18^\circ 45'$; to find $PW = 12^\circ 21' = 858$ miles, the distance between Paris and Warsaw = also AW . Also in the same triangle we find the $\angle NWP = 98^\circ 31'$, and the $\angle NPW = 66^\circ 57'$ the messenger's course from P to W ;



consequently the opposite course, from w to A, will be $113^{\circ} 3' =$ the $\angle NWA$; consequently the $\angle PWA = 14^{\circ} 32'$. Then, in the triangle PWA, we have PW and AW each $= 12^{\circ} 21'$, and the $\angle w$ to find $AP = 3^{\circ} 6' = 215.4$ miles, the distance to go from A. to Paris. Also from the same data we find the $\angle APW = 82^{\circ} 54'$, therefore the $\angle APN = 149^{\circ} 51'$; from which, and the two sides AP and PN, is found the $\angle PAN = 28^{\circ} 30'$ the course to Paris.

The same answered by Mr. W. Wiseman, Hull.

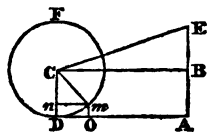
Let P, N, w denote Paris, the North Pole, and Warsaw, &c. as above. Then are given $PN = 41^{\circ} 10'$, $WN = 37^{\circ} 46'$, and $\angle PNW = 18^{\circ} 45'$, to find $PW = 12^{\circ} 21' 14'' = 857.9$ miles, and $\angle NPW = 66^{\circ} 56' 54''$, and $PWN = 98^{\circ} 31' 6''$. Let WA be the track of the messenger on his return: then Awn will be the sup. of $NPW = 113^{\circ} 3' 6''$, and $113^{\circ} 3' 6'' - 98^{\circ} 31' 6'' = 14^{\circ} 32' = PWA$; therefore $AW = PW$, and PWA are given to find $PA = 3^{\circ} 6' 5'' = 215.374$ miles, and $WPA = 82^{\circ} 53' 48''$; hence $NPA = 149^{\circ} 50' 42''$; lastly PN, PA, and NPA are given, to find $NAP = 28^{\circ} 29' 53''$, the required course.

VII. QUESTION 1185, by Dr. Mic. S. O'Riordan, Carlow, Ireland.

In a horizontal yard is an immoveable flag-stone, $1\frac{1}{2}$ inch high, over which a heavy laden carriage is to be drawn by the least possible force. That part of the horse's breast to which the traces are to be applied is 5 feet high, and the distance thence to the axle is 8 feet; what must be the diameter of the wheels?

Answered by Mr. Hugh Hutchinson, Sunderland.

Let $r = CD = cm$ the radius of the wheel, $om = \frac{1}{2}$ foot, the height of the flagstone. Now the horse will draw with the greatest effect when the direction of the traces CE is parallel to the tangent at m, by Dr. Hutton's Course, vol. 2, pa. 35, on mechanics; or Mr. Gregory's Mechanics, vol. 1, art. 144. Draw CB and mn parallel to AD; then by right-angled triangles, $\sqrt{(CE^2 - BE^2)} = \sqrt{[64 - (5 - r)^2]} = CB$: and by similar triangles $CE : CB :: cm : cn = \frac{1}{2}r\sqrt{[64 - (5 - r)^2]}$. But $cn + nd = CD$, that is, $\frac{1}{2}r\sqrt{[64 - (5 - r)^2]} + \frac{1}{2} = r$; this equation solved gives $r = 2.504$ the radius of the wheels, or their height 5 feet and $\frac{1}{10}$ of an inch, nearly.



The same by Mr. W. Marrat, of Boston.

Draw the lines as in the figure, and put the radius of the wheel $DMF = x$, also put the altitude om of the obstacle $= 1\frac{1}{2}$ inch $= a$.

Now the power requisite to draw the wheel over the obstacle om , will evidently be the least when its direction ce is perpendicular to cm ; then, by similar triangles, cbe , cnm , we have $cm = x : cn = x - a :: ce = 8 : cb = 8(x - a) \div x$. But $be = 5 - x$, and $bc^2 + be^2 = ce^2$, that is $8^2 \times (x - a)^2 \div x^2 = 8^2$; whence, by reduction, $x^4 - 10x^3 + 25x^2 - 16x + 1 = 0$, and $x = 2\frac{1}{2}$ feet very nearly.

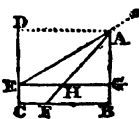
Remark. If that part of the horse to which the traces is fastened be only 4 feet high, as it is in horses of the common size, the height of the wheels will be much less (the height of the obstacle being the same), when the power is the least possible; whence it is plain that the fore wheels of our carriages are high enough; contrary to what is asserted by most authors on the subject.

VIII. QUESTION 1186, by Mr. L. Rowbottom, *West Hallam.*

In latitude of $53^\circ 3'$ N. is a cylindrical cistern, standing truly horizontal: on June 16, 1805, in the morning, it was observed that the shadow of the top A of the cistern fell on the opposite side at E , when the cistern was empty, and its length AE was just 63 inches: instantly the cistern was filled with water, and then the shadow of A fell on a point F in the base, its length AF being then 65 inches. Required the greatest number of ale gallons the cistern will contain, and the time of the day when this curious observation was made?

Answered by Mr. J. H. Hearing.

Let $ABCD$ represent the cistern, AE and AF the lines drawn as in the question, and EHG a parallel to CB drawn through E . Also put $AE = a$, $AF = b$, and $EG = CB = x$. Then $AG = \sqrt{(a^2 - x^2)}$, and x is evidently the sine of the $\angle EAG$ to the radius AE ; therefore the sine of $\angle GAH$, to the same radius, is $= \frac{3}{4}x$, the sine of the angle of incidence being to the sine of refraction as 4 to 3, as is well known. Now the sines being as their radii, we



shall have $a : b :: \frac{3bx}{4a} = BF$; consequently $\sqrt{(b^2 - \frac{9b^2x^2}{16a^2})}$

$= AB$ the depth of the vessel. Then because the content of the cistern must be the greatest possible, $BC^2 (= EG^2) \times AB$, or $x^2 \sqrt{(b^2 - \frac{9b^2x^2}{16a^2})}$, or $x^4 - \frac{9x^6}{16a^2}$ must be a maximum, and its fluxion $4x^3 - \frac{54x^5}{16a^2} = 0$. The reduction of this equation gives $x = \frac{2}{3}a\sqrt{6} =$

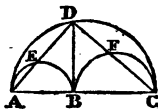
68.5857 feet, for the value of bc ; which shews that, with the numbers given, the solution is impossible.

IX. QUESTION 1187, by *Mr. Wm. Saint, of Norwich.*

If ADC be a semicircle; and if from any point B , in its diameter AC , the perpendicular BD be erected, and the two semicircles AEB , BFC , be described on the two segments AB , BC within the first semicircle: then the curvilinear area $AEBFCDA$, contained within the circumferences of the three semicircles, will be equal to the area of a circle whose diameter is BD . Required the demonstration?

Answered by Mr. Jos. Hall, of Hurdsfield.

Join AD , DC . Then because ADC is a right-angled triangle, the sum of the semicircles on AD , DC is equal to the semicircle ADC , that is, the sum of the semicircles AEB , BFC , and the circle on the diameter BD is equal to the semicircle ADC ; take away the semicircles AEB , BFC , and there remains the curvilinear area $AEBFCDA$, equal to the circle on the diameter BD .



The same by Mr. Sam. Tatlow, of Harleston School.

Join AD and DC . Then is the semicircle on AC equal to those on AD and DC in one sum: in like manner, the sum of the semicircles on AB and BC is equal to those on AD and DC plus the whole circle on BD . Hence the curvilinear area $AEBFCDA$ is equal the circle on BD .

Mr. Robert Rogers, jun. remarks that this property is proved at page 384, vol. 1, Dr. Hutton's Recreations.

X. QUESTION 1188, by the *Rev. Tho. Scurr, Hexham.*

Suppose a weight at the foot of Mount Blanc, 15662 feet high, be found 112lb. What will be the weight at top; supposing the height of the barometer at bottom 30 inches, and the specific gravities of air and quicksilver $1\frac{1}{2}$ and 13600, also the earth's radius 3978 $\frac{1}{2}$ miles?

Answered by Mr. John Bennett, Redruth.

I am of opinion that this question is compounded of three separate problems, viz. On the top of the mountain to find the attraction.

2d. To find the density of the atmosphere. 3d. To find the difference in the centrifugal force. First, as to the force of attraction, this being inversely as the square of the distance from the earth's centre, above the surface, putting r = the earth's radius, a = the mountain's height, and b = 112lb. the weight; then $(r + a)^2 : r^2 :: b : 111\text{lb. } 13\frac{1}{2}\text{ oz.}$

Again, for the weight of the atmosphere, it seems the gentleman should have given the specific gravity of the body in order to ascertain its bulk. For want of that we cannot precisely say what that weight is: we can only say, the increased weight will be equal to the difference in the weight of air at the top and bottom of the mountain. Lastly, the data for the centrifugal force is also defective, because the latitude of the place should be given.

The same by Mr. W. Marratt, of Boston.

The number of feet in the earth's radius is $21007946\frac{2}{3}$, to which adding 15662 gives $21023608\frac{2}{3}$ for the distance of the top of the mountain from the centre; therefore $(21023608\frac{2}{3})^2 : (21007946\frac{2}{3})^2 :: 112 : 111\cdot83318\text{lb.}$ what the body weighs on the top of the mountain, allowing for the decreased gravity. Now to determine the density of the air at that altitude, put l for the log. of the rarity at any alt. a , r for the earth's radius, h = 27600 the height of a homogeneous atmosphere, and m = $0\cdot4342944819$; then $l = \frac{m}{h} \times \frac{ar}{r + a} = 0\cdot2463362$, the number answering to which is $1\cdot7633$; therefore $\frac{1}{1\cdot763} \times 1\frac{1}{2} = \frac{110000}{158697}$ is the air's density at the top. Now supposing the body to be a cube, of one foot each side, (for its figure should have been given) the difference between the weight of a cubic foot of air at the top and bottom of the mountain = $1\frac{1}{2} - \frac{110000}{158697} = \cdot52907\text{ oz.}$ = $\cdot03307\text{lb.}$ which added to $111\cdot83318$, gives $111\cdot86625\text{lb.}$ for the weight sought.

XI. QUESTION 1189, by Mr. P. Barlow, R. M. A.

Required the probability of a person holding 29 at the game of cribbage?

Answered by Mr. P. Barlow, the Proposer.

That a person may hold 29 at cribbage, it is necessary that he has in his hand any three 5's, and the knave of the other suit, and also that the remaining 5 is cut up. Now the number of combinations of 6 that

may be made out of 52 cards, is $\frac{52. 51. 50. 49. 48. 47}{1. 2. 3. 4. 5. 6} = 20358520$;

and the number of these combinations of 6, which contain any specified combination of 4, is $\frac{48}{2} \times \frac{47}{2}$; but there are 4 different combinations of 4, that will answer the first condition of the question; there are, therefore, $4 \times \frac{48}{2} \times \frac{47}{2} = 4512$ of the above number of combinations of 6 that will give the necessary cards, and consequent-

ly the probability of holding 1 of them is $\frac{4512}{20358520} = \frac{12}{54145}$; and

now admitting this to have happened, the other 5's must necessarily be one of the 48 other cards, and that it is one of those remaining still

in the pack, is therefore $\frac{40}{48}$; and that it is cut up is $\frac{1}{40}$; therefore,

that all these happen is $= \frac{12}{54145} \times \frac{40}{48} \times \frac{1}{40} = \frac{1}{216580}$; that is, it is 216579 to 1, that a person does not hold 29.

Corol. When 4 persons are playing, the probability of holding the the necessary card is $\frac{48}{649740}$; the probability of the other 5 being

in the pack is $\frac{28}{48}$, and of its being cut up is $\frac{1}{28}$, and that all these

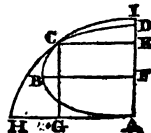
happen is $\frac{1}{649740}$; that is, it is 646739 to 1, that any specified person does not hold 29 at the double game.

XII. QUESTION 1190, by Mr. John Whitley, *Attercliff Academy.*

A sheet of fine silver, of uniform thickness, in the form of a cycloid, whose axis is 12 inches, is worth as many shillings as there are square inches in the surface of its greatest inscribed ellipse. Quere the value of the silver?

Answered by Mr. P. Barlow, R. M. A.

Lemma. The greatest ellipse that can be inscribed in a cycloid, is that which passes through that point where the greatest inscribed rectangle meets the curve. For let ABD be any ellipse, and GA be drawn parallel to BF, the transverse axis, and let it be required to find the rectangle ACCE, of which one angle c shall fall in the ellipse, so that its area may be a maximum. Put AF = a, BF = b, and EF = x; then by the property of the ellipse, $(b \div a) \sqrt{(a^2 - x^2)} = CE$, and we want to

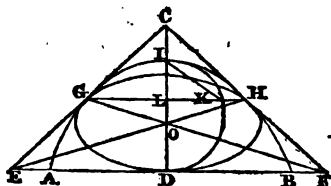


find $(b \div a) \sqrt{(a^2 - x^2)} \times (a + x)$ a maximum, or $(a + x)^3 \times (a - x)$ a maximum; this being put into fluxions and reduced, gives $x = \frac{1}{2} a$; and consequently the rectangle $AGCE = \frac{2}{3} ab \sqrt{3}$. The rectangle therefore being a constant function of the rectangle of the axes of the ellipse, the area of the ellipse must be the greatest when the area of the rectangle is the greatest.

Corol. The area of the whole ellipse $= 3.1416 ab$, and of the whole rectangle $\frac{2}{3} ab \sqrt{3} = 2.5981 ab$ nearly, therefore the area of the ellipse is to that of the rectangle as 3.1416 to 2.5981 nearly. Now the area of the greatest rectangle that can be inscribed in the cycloid $= 209.219$, as may be found by proportion from question XII. last year's Diary; therefore as $2.5981 : 209.219 :: 3.1416 : 252.99$ the required area of the ellipse, and hence the value of the cycloid is found $= 12l. 13s.$ very nearly.

The same by Mr. Putsey, Master of the Free School, Pickering.

It appears from the Schol. to Theor. 8, Simpson on the Max. and Mim. that the ellipse will be the greatest when the tangents CE, CF , are bisected in the points of contact, G, H , and consequently when the subtangent CL is $= DL$; and by writers on Conics and Geom. if the lines EH, FG be drawn, they will intersect each other in O , the centre of the ellipse, and that $EO = 2HO$; and therefore in the present case, $CO = \frac{2}{3} CD$. Now put $a = 12$, and $x = IL$, then $DL = a - x$, and by the circle $KL = ax \sqrt{(ax - x^3)}$; and the chord $IK = \sqrt{ax}$; also by the cycloid, KI is parallel to CH , therefore $IL : KL :: CL : HL = \sqrt{\frac{a^3 - 3a^2x + 3ax^2 - x^3}{x}}$, and by another property,



HL is also $=$ the chord $KL + \text{arc } IK = \frac{2}{3} \sqrt{ax} + \frac{2}{3} \sqrt{(ax - x^3)}$; from these two values we get $x = 3.67652$; therefore $CD = 16.64696$, and the conjugate diameter $= 11.09798$. Whence $CH = 25.0882$, and the transverse diameter of the ellipse $= 28.9694$; consequently its area $= 252.5075$ square inches, and its value $12l. 12s. 6d.$

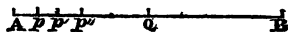
XIII. QUESTION 1191, by Mr. T. Myers, R. M. A.

It is required to divide half a given right line into any assigned number of parts, such that each part, and that part added to its distance from the farther extremity of the undivided half line, may be in

geometrical proportion. This question is practically useful in the division of the monochord, or musical string.

Answered by Mr. J. H. Hearing.

Let AB, the given line whose half is to be divided, be denoted by unity; let q be its middle point, and let $x = bp$. Then by the question $AB : bp :: bp' : x$, i. e. $1 : x :: x : x^2 = bp'$: in like manner $bp'' = x^3$, $bp''' = x^4$, &c. And

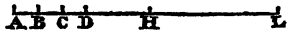


hence $p q$ ($\frac{1}{2}$) will be found to equal x^n , or $x = \sqrt[n]{\frac{1}{2}}$. Therefore, as the parts may readily be thence determined when the values of AB and n are given. I shall omit being more explicit here, and refer those who wish to see the solution at large, and how the question may be applied to practice, to Emerson's Algebra, or some of his other works.

The same by Mr. W. Moore, R. M. A.

Let AL be the given right line bisected in H, and AH divided into the assigned number of parts (n) in points

B, C, D, &c. so that $AB : AL :: BC : BL :: CD : CL$, &c. Then, putting a for the



whole line, and x for AB, we have $\frac{a-x}{a}$ for the ratio of any two suc-

ceeding parts; and $\left\{ \left(\frac{a-x}{a} \right)^n - 1 \right\} x \div \left(\frac{a-x}{a} - 1 \right) = \frac{1}{2} a$, by the nature of geometrical progression, and the question, which equation reduced, gives $x = a - a \sqrt[n]{\frac{1}{2}}$, from which the other parts become known.

XIV. QUESTION 1192, *by Mr. Wm. Wiseman, of Hull.*

In latitude $53^\circ 50'$ north, on a horizontal plane, is an upright pole, 40 feet high. Required the area bounded by the shadow of the pole in the prime vertical, and the curve described by the shadow of the top of the pole on the 6th of June, 1807?

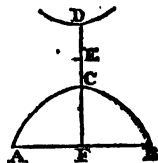
N. B. No notice is to be taken of the change of declination, parallax, refraction, or semidiameter.

Answered by Mr. J. Collins, of Kensington.

Here we have the latitude of the place and the Sun's declination, to find his altitude, when on the prime vertical, and his meridian al-

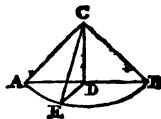
titude on the given day, and on another day when the sun has the same south declination; whence $28^{\circ} 55' 33'' =$ altitude, on the prime vertical, also $58^{\circ} 46'$ and $13^{\circ} 34'$ the two meridian altitudes on the two days.

Now it is shewn, pr. 15, Simpson's Flux: than when the sun is in the equator, the shadow will be a right line, when the declination is equal to the colatitude it will describe a parabola, when near the poles an ellipse, but otherwise, as in the present case, the curve will be an hyperbola, whose transverse axe will be equal to the difference of the sun's meridional shadows on the above two days. Let the annexed fig. be the hyperbola; the lengths of the shadows of the pole are found by trig. 165.7622 and 24.25668 ; their difference $= 141.50552$, and 73.89955 the altitude when on the prime vertical $= AF = BF$, therefore 147.8 nearly $= AB$; hence the base 147.8 , the altitude $FC = 24.25668$, and the transverse axe $= 141.50552$, to find the area; which reduces the question to the Ex. p. 369, Dr. Hutton's Mensuration, therefore putting $a = 70.75276$ semitransverse, $y = 73.9$ the half base, and $v = 95.00944 = EF$; hence by that rule $ay \div \sqrt{(v^2 - a^2)} = 82.4574 = CD$; which by the same rule gives 2318.06865 feet, the area required.



The same by the Rev. J. Furnass, Ponteland.

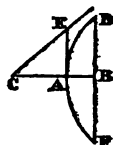
To find the sun's altitude, when east or west, we have given his declination north $22^{\circ} 36'$, the latitude north $53^{\circ} 50'$: therefore by spherics as s. latitude : s. declination :: rad. : s. altitude when east or west $= 28^{\circ} 25' 33''$. Also co-latitude + declination $= 58^{\circ} 46'$ the meridian altitude. In the annexed fig. let AB be the horizontal plane, $CD = 40$ feet, $\angle CAB = \angle CBA = 28^{\circ} 25' 33''$, and $\angle CED = 58^{\circ} 46'$. Hence in the right-angled triangle $ADC = CBD$, we have all the angles and a side CD given, to find $DA = DB = 73.89842$, its double $147.79684 = AB$.



And in the right-angled triangle EDC , we have all the angles and a side (CD) given, to find $DE = 24.25667$, the length of the pole's shadow at noon. Put $c = \cos.$ lat. $d = \sin.$ and $p = \cos.$ of the declination, also $m =$ the length of the pole. Then (by the Diary quest. 980, 1795,) we have $2pdm \div (c^2 - d^2) = 141.4053$ the transverse. But the amplitude $= 40^{\circ} 38'$; therefore $\cotang. \text{ ampl.} \times \text{transverse} = 164.9028 = \text{conjugate}$. Now put $a = \text{trans. axis}$, $s = \text{semiconj.}$ $y =$ the ordinate, and $v =$ its dist. from the centre. Then (Dr. Hutton's Mensuration; 2d. edition, p. 414) $v y - a s \times \text{hyp. log. of } ((ay + sv) \div as) = 2318.8774$, the area.

The same by Mr. J. H. Hearing.

Here the colat. $\pm$ dec. ($= 22^\circ 36'$) gives $58^\circ 46'$ and $13^\circ 34'$ for the meridian altitudes, corresponding to $22^\circ 36'$ north and south declination. Then the height of the pole, 40, multiplied by the cot. of these altitudes, gives the distances of the hyp. vertexes, described by the top of the pole $= 24\ 2567$ & $165\ 7621$, and their diff. $141\ 5054$, is the transverse axis of the hyperbola. Hence we find $AC = 70\ 7527$, $AE = 82\ 4537$, $CB = 95\ 0094$, and $ED = 73\ 899$; then by Dr. Hutton's Mensuration, p. 315, 3d. edition, the area described by the shadow of the pole $= 2318\ 644$ or $257\ 627$ square yards.



The same by Mr. T. Myers, R. M. A.

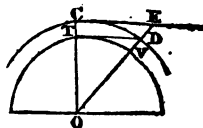
From the latitude of the place, and the sun's declination being given, his altitude when on the prime vertical, is found $= 28^\circ 55\frac{1}{2}'$, also his meridian altitude on the given day $58^\circ 46'$, and when he has the same degree of south declination $= 13^\circ 34'$. Now it is shewn in Simpson's Fluxions that the curve described by the top of the pole, will be an hyperbola, ; the transverse axis of which will be $=$ the difference of the meridian shadows of the pole, with the above altitudes. It is also evident, that its absciss will be equal to the meridian shadow of the pole on the given day ; and its double ord. $=$ double the shadow of the pole on the prime vertical. And as the height of the pole is given $= 40$, and the altitudes of the sun determined above, these shadows are easily obtained by plane trigon. viz. the transverse axis $= 141\ 506$, its absciss $= 24\ 257$, and its double ord. $147\ 8$ nearly. Then (prob. 6, p. 314, Dr. Hutton's Mensuration, 3d edition) the required area is found $= 2318\ 0692$ square feet.

PRIZE QUESTION, by Mr. W. Moore, R. M. A.

Being one day on a very extensive plain, the wind at N. E. and blowing with a velocity of 60 miles an hour ; I observed a cloud perpendicularly overhead, and after remaining in my situation till I saw it immerge into the horizon, I found that twelve minutes had elapsed from the time of the first observation. Required the height of the cloud above the earth's surface ; the latitude being 52° , and the earth's radius 3979 miles.

Answered by Mr. Moore, the Proposer.

As the earth with its atmosphere revolve together, the cloud has no apparent motion, but simply that occasioned by the force of the wind, and since the direction of gravity is always towards the earth's centre, the locus of the cloud's motion is a great circle; consequently the time of observation would be the same in any other latitude. Let cn be the orbit of the cloud; tv a corresponding parallel circle on the earth; τ the place of the observer, to which tn is a tangent; and the other lines which appear in the figure. Then the arc $cd = 12$ is the track described by the cloud. Now the readiest way to find the altitude cr or dv , is by the rule of Double Position; which being accordingly repeated several times, it at last comes out = .0180959 miles, true to the last place of decimals.



The same by Mr. T. Morley, Moor-Street, London.

Let τ be the place of the observer, and c that of the cloud at first; cd the arc described in $12^m = 12$ miles; td a tangent, and d where the cloud immersed. Put $x = cr = dv$, and $r = or$; then by the property of the circle $\sqrt{(2rx - x^2)} = td$, and by sim. tri. $or : td :: oc : r + x$; $ce = \frac{r+x}{r} \sqrt{(2rx - x^2)} = \text{tang. of arc } cd \text{ to the radius}$ $r + x$. Then finding the series for the arc cd in terms of the radius and tangent, and putting it = 12, the root of the equation being found by Dr. Hutton's method of Double Position, it brings out $x = .018095$ miles, = 95.5416 feet, the height of the cloud. Or, since the tangent of an arc of $12'$ is nearly = the arc itself, therefore $\sqrt{(3979^2 + 12^2)} = 3979.0180949$, from which take 3979, leaves .0180949, nearly the same as before.

The same by Mr. Sam. Ryley, Armley Mills.

Let c be the first position of the cloud, τ that of the observer, and o the earth's centre. Draw the tangent td , and the other lines as in the figure. Put $cr = x$, and $or = a = 3979$. Then $od = a + x$, and $td = \sqrt{(a+x)^2 - a^2} = \sqrt{(2ax + x^2)}$; also the chord $cd = \sqrt{(2ax + 2x^2)}$. Hence by rule 4, Dr. Hutton's Mens. the arc $cd = \frac{2}{3} \sqrt{(2ax + 2x^2)} - \frac{1}{3} \sqrt{(2ax + x^2)} = \text{twelve miles}$, or $4\sqrt{(2ax + 2x^2)} - \sqrt{(2ax + x^2)} = 36$ miles. Hence by trial and error is found $x = \frac{1}{3}$ miles, = 32 yards nearly.

The same by Mr. John Skewes, of Guennap.

Put the earth's radius $or = ov = 3979$ miles $= a$, $cd = 12 = d$, and $cr = dv = x$. Then by the property of the circle $rd = \sqrt{(2ax + x^2)}$; and, by Huygen's theorem, the arc $cd = \frac{2}{3}\sqrt{(2ax + 2x^2)} = \frac{1}{3}\sqrt{(2ax + x^2)} = d$. But since $\sqrt{(2ax + x^2)}$ is very nearly $= \sqrt{(2ax + 2x^2)}$, x being very small in respect of a ; therefore $\sqrt{(2ax + 2x^2)} = d$ or 12 very nearly, and $x^2 + ax = \frac{1}{2}d^2 = 72$; hence $x = \sqrt{(\frac{1}{2}a^2 + \frac{1}{2}d^2)} - \frac{1}{2}a = 31.82$ yards nearly the cloud's height.



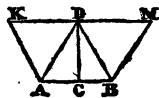
Questions proposed in 1809, and answered in 1810.

I. QUESTION 1194, *by Mr. Jas. Wood, of Dudley.*

Given a vessel, in form of a conical frustum, full of liquor, standing on its less end, the diameter of which end is 36 inches; into which suppose a cone to be immersed, of equal base and altitude, whose convex surface is 1696.464, which is also equal to the area of the top of the vessel. Required the content, and the quantity of liquor, also measure, that overflowed by the immersion?

Answered by Mr. D. Anderson, Royal Mil. Acad. Woolwich.

Since $AB = 36$ inches, therefore $3.1416 \times 36 =$ circumference of the bottom; and $1696.464 \times 2 \div 3.1416 \times 36 = 30 = AD$; also $\sqrt{(AD^2 - AC^2)} = \sqrt{(30^2 - 18^2)} = CD = 24$. Then $36^2 \times .7854 = 1017.8784 =$ area of the bottom; hence $(1696.464 + 1017.8784 + \sqrt{(1696.464 \times 1017.8784)}) \times \frac{1}{3}$ of $24 = 32227.3472$ the content of the frustum. And $1017.8784 \times 8 = 8143.0272$ the content of the cone $= 28.875$ ale gallons that overflowed.



The same by Mr. J. A. Dotchen, Exeter-street, Chelsea.

The area of the top of the frustum being 1696.464, its diameter will be $12\sqrt{15}$; also $1696.464 \div 56.5488$ (the semicircumference of the base) gives 30 for the slant side of the cone, and $\sqrt{(30^2 - 18^2)} = 24$ its altitude. Then, by Mensuration, the content of the frustum is

$$\frac{(12\sqrt{15})^3 - 36^3}{12\sqrt{15} - 36} \times 8 \times .7854 = 32227.3424 \text{ cubic inches} = 144.2813$$

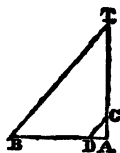
ale gal. and the content of the cone is $.7854 \times 36 \times 8 = 8143.0272$ cubic inches, or 28.8759 gallons, the quantity of liquor that overflowed.

II. QUESTION 1195, by *Mr. Wm. Morley, Conyngthorpe.*

Being on the bank of a river, that runs due west, I perceived that the top of a tree, standing on the other side directly opposite, was reflected from a point on the water six feet from the place where I stood; and that after walking along 20 yards westward, the distance of the new point of reflection from me was then eight feet. Required the height of the tree; my eye being five feet above the surface of the water?

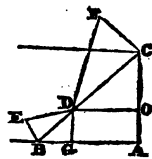
Answered by Mr. Wm. Cole, of Colchester.

Make $AB = 60$ feet, and draw AT perpendicular AB , in which take $AC = 6$, and from c to AB apply $AD = 8$. Draw BT parallel to CD , to meet AT in T , so will T be the place of the tree. Now $CD^2 - AC^2 = AD^2 = 28$; therefore, by similar triangles, as $AD (\sqrt{28}) : AC (6) :: AB (60) : AT = 360 \div \sqrt{28}$. Then $CT = 360 \div \sqrt{28} - 6 = 62.0336$; and as $AC (60) : \text{the height of the eye} (5) :: CT (62.0336) : \text{the height of the tree} = 51.6946$.



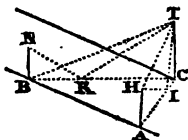
The same answered by Mr. J. A. Dotchen, of Chelsea.

Let A be the spectator's first station, and B his second on the bank AB , distant 60 feet from each other by the question; and let c be the place of the tree on the opposite side, and o the first point of reflection; then, the height of the spectator's eye and of the tree being constant, it is manifest from the principles of catoptrics, that if OD be drawn parallel to AB , meeting BC in D , then D will be the new point of reflection. Hence letting fall DC perpendicular to AB , we have $BG = \sqrt{(BD^2 - DG^2)} = \sqrt{(8^2 - 6^2)} = 2\sqrt{7}$; therefore $BG : BD :: DO : DC = 240 \div \sqrt{7} - 8$, and $BD : BE (\text{the height of the eye}) :: DC : FC = 150 \div \sqrt{7} - 5 = 51.694671$ feet, the height of the tree.



The same answered by Miss Mary Weston, of Gainsborough.

In the annexed figure let $AB = 60$ feet, $AI = 6$, $CT = x$ the height of the tree, $BR = 8$, $BN = AH = 5$ feet. Then, the angle of incidence being equal to the angle of reflection, we have $\angle AIH = \angle CIT$, and $\angle BRN = \angle CRT$, consequently the triangles AIH , CIT , and BRN , RCT are similar; therefore $HA (5) : AI (6) :: CT (x) : CI = \frac{5}{3}x$, and as $NB (5) : BR (8) :: CT (x) : CR = \frac{8}{5}x$; consequently $BC = \frac{8}{5}x + 8$, and $AC = \frac{5}{3}x + 6$. But $BC^2 - AC^2 = AB^2$, that is $(\frac{8}{5}x + 8)^2 - (\frac{5}{3}x + 6)^2 = 60^2$ hence $x = 51.69$ feet nearly.

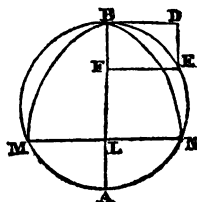


III. QUESTION 1196, by Mr. Wm. Wiseman, of Hull.

A tangent, and a perpendicular to it, from a globe, in the plane of one of its great circles, being given, $= 5\frac{1}{2}$ and 4 inches; required the dimensions and content of the greatest semispheroid that can be contained within the superficies of this globe?

Answered by Mr. J. A. Dotchen, of Chelsea.

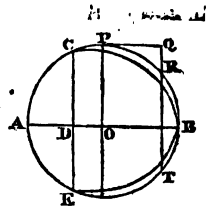
In this question we have, first, by the nature of the circle, DE or $BF : BD$ or $FE :: FE : FA$, or $4 : 5\frac{1}{2} :: 5\frac{1}{2} : 7\frac{9}{16} = FA$; then $FA + FB = 4 + 7\frac{9}{16} = 11\frac{9}{16} = AB$ the sphere's diameter. Now, in order that the semispheroid MBN be a maximum, it appears by mensur. that $LM^2 \cdot LB$ must be a maximum, that is, (because $LM^2 = AL \cdot LB$) $AL \cdot LB^2$ must be a maximum, which, by Simpson on the maxima and minima, will be the case when $AL = \frac{1}{2} LB$; therefore $LB = \frac{2}{3} AB = 7\frac{1}{3} = 7.7083$ &c. the height of the semispheroid, and $LM = \sqrt{AL \cdot LB} = \frac{1}{3} AB \sqrt{2} = 5.450615$ the radius of its base. Hence the content is $\cdot 5236 \times BL \times 4 LM^2 = \cdot 5236 \times \frac{2}{3} AB \times \frac{8}{3} AB^2 = \cdot 5236 \times \frac{16}{9} AB^3 = 479.63524$.



The same answered by Mr. J. H. Harding, Pall Mall.

Let $APBQA$ represent the globe, pq the given tangent, parallel to the diameter AOB , and QR the perpendicular which produce till it meet the surface again in T . Then $pq^2 \div QR = 7.5625 = qt$, and hence the

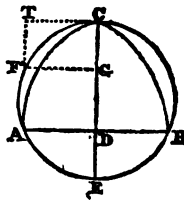
diameter $AB = 11.5625$. Let $CBED$ be the greatest inscribed semi-spheroid then its solidity, by prob. 9, pa. 244, Dr. Hutton's Mensuration, is $.52359877 \times 4 DC^3 \times DB = a$ maximum per question. But, by the nature of the circle, $DC^2 = AD \times DB$; expunge therefore the constant factors, and write this value for DC^3 , and it will appear that $AD \times DB^3$ must be a maximum. But their sum AB is given; therefore $DB = \frac{2}{3} AB = 7.7083$, by Simpson on the maxima and minima theorem 17, consequently $CD = \frac{1}{3} AB \sqrt{2} = 5.4506148$, and the solidity $.52359877 \times 4 \times \frac{2}{3} \times \frac{2}{3} AB^3 = 479.6343$.



Cor. The sphere is to its greatest inscribed semispheroid as 27 to 16.

The same answered by Mr. John Smith, Alton Park.

Put $CE = CG + GE = TF(4) + TC^2 \div TF = 4 + \frac{1}{4} \times 5.5^2 = 11.5625 = a$ (the axis of the globe), $.5236 = c$, and the axis of the semispheroid $CD = x$; then will $DE = a - x$, and, by the nature of the circle, $CD \times DE = AD^2$; that is, $ax - x^2 = AD^2$; hence $4 acx^3 - 4 cx^3 =$ the solidity of the semispheroid a maximum, or $ax^2 - x^3$ a maximum, the fluxion of which made $= 0$ gives $x = \frac{2}{3} a = 7.7083 = CD$. Then $AB = 2\sqrt{(ax - x^2)} = 10.9012$: consequently the solidity $= 479.6359$.

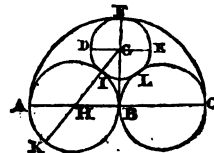


IV. QUESTION 1197, by Mr. Robert Nunn, *Eye Academy*.

If circles be described as in the annexed figure; the diameter DE will be two thirds of AB or BC . Required the demonstration?

Answered by Mr. W. Bardou, Acaster Malbis.

Thro' GH , the centres of the two circles FIL , AIB , draw GK , passing through the point of contact I , and meeting the latter circle again in K . Then $GK = GI + IK = GI + 2 IH$, and $GB = BF - FG = 2 IH - GI$: but by cor. 1, theor. 61, geom. in Dr. Hutton's Course, $KG \cdot GI = GB^2$, that is, $(GI + 2 IH) \cdot GI = (2 IH - GI)^2$, which gives $3 GI = 2 IH$:



The same by Mr. T. Clifton, Nottingham.

Let $BF = 1 = 2 BH$, and $FG = r$; then $GH = BH + r = \frac{1}{2} + r$;
H 3

and $BC^2 + BH^2 = GH^2$, or $(1-r)^2 + \frac{1}{4} = (\frac{1}{4} + r)^2$, that is $1\frac{1}{4} - 2r + r^2 = \frac{1}{4} + r + r^2$; hence $1 - 2r = r$, or $1 = 3r$, and $r = \frac{1}{3}$, or $2r = \frac{2}{3} BF$.

The same by Mr. Ja. Cunliffe, jun. Bolton-le-Moors.

Let $AB = BF = R$, and $GI = r$: then $GH = \frac{1}{2}R + r$, and $BG = R - r$; then $GH^2 = BG^2 + BH^2$, that is, $(\frac{1}{2}R + r)^2 = (R - r)^2 + \frac{1}{4}R^2$, or $\frac{1}{4}R^2 + Rr + r^2 = R^2 - 2Rr + r^2 + \frac{1}{4}R^2$; by equal subtractions $Rr = R^2 - 2Rr$, and by equal additions $3Rr = R^2$, and by equal divisions $3r = R$.

V. QUESTION 1198, by Mr. A. Nesbit, Master of Preston Free School.

Discoursing lately with an intimate friend, concerning the weight of bodies; I affirmed that if a body weighed 14lb. on the earth's surface, I could determine a place where it would weigh only 12lb.; but as he will not believe this on my word only, I hope the ingenious correspondents to the Ladies' Diary, will prove the truth of my assertion, and determine that point, taking the earth's radius at 3995 miles.

Answered by Mr. H. Codling, South Walsham.

To make the assertion good, according to the common acceptance of the term weighing, we must conceive the place for suspending the weight, to be the surface of the earth. Then as the force of gravity above the earth, diminishes in a duplicate ratio of the distances from the earth's centre, the place for suspending the body will be had by the following proportion, viz. $\sqrt{12} : \sqrt{14} :: 3985 : 4304.319$ miles from the earth's centre.

The same answered by Mr. W. Crane, of Boston.

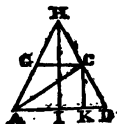
There may be two places answering the assertion in the question, one above the earth's surface, and the other below it. As to the former, gravity decreasing inversely as the square of the distance, we shall have $12 : 14 :: 3985^2 : \text{the square of the distance from the centre}$, or $\sqrt{12} : \sqrt{14} :: 3985 : 4304.291$; the difference gives 319.291 miles above the surface. But for the latter case, as gravity decreases in the direct proportion of the distance, it will be $14 : 12 :: 3985 : 3415\frac{1}{2}$ the distance from the centre, or 569 $\frac{1}{2}$ below the surface for the second place.

VI. QUESTION 1199, by Mr. J. Rowbottom, *West Hallam*:

A and B bought a piece of lead in form of a right cone, the diameter being 6, and altitude 9 feet; which they agree to divide between them by a plane passing through the extremity of the base diameter and terminating in the slant side, so that the convex surfaces may be equal. What must each person pay towards it, supposing A to have the oblique cone, and B the elliptic ungula, and the lead valued at 2l. the cwt.?

Answered by Mr. J. A. Dotchen.

Suppose AC to be the plane dividing the convex surface of the cone into two equal parts. Draw CG parallel, and CK perpendicular to the base AB. Put $AB = 6 = D$, $CG = x$, and $n = .7854$. Then, by the rules at p. 174, 175, Dr. Hutton's Mensuration, 3d edition, the convex sur-



faces of AHC and of ACB are $\frac{n}{D-x} \sqrt{(4h^2 + (D-x)^2)}$

$\times \frac{D+x}{2} \sqrt{Dx}$ and $\frac{n}{D-x} \sqrt{(4h^2 + (D-x)^2)} \times (D^2$

$-\frac{D+x}{2} \sqrt{Dx})$; and equating these two expressions, agreeable to

the question gives $(D+x) \sqrt{Dx} = D^2$; or, by substituting $Dx^2 = z$, it is $(z^2 + 1)z - 1 = 0$. Now, in this equation z may be easily discovered by the tables of tangents and secants: for we have to find the natural tangent z of an arc, which multiplied by the square of its secant $z^2 + 1$, shall be equal to the radius 1; that is $\sec^2 \times \tan$. — rad. = 0, or in logs. $2 \log. \sec. + \log. \tan. - 30 = 0$. Hence, by turning over the tables of log. secants, &c. z is very soon found to be the nat. tang. of an arc between $34^\circ 18'$ and $34^\circ 19'$; and by the method of proportional difs. the accurate value of z is determined = .6823278; consequently $x = Dx^2 = 2.7934272 = CG$, and as $KB : KB :: KB : KC = 4.8098592 = b$. Then, by Dr. Hutton's Mensuration, aforesaid, pa. 166, the solidity of the oblique cone AHC = 26.9459703, and the content of the ungula = 57.8772297. Lastly, the value of the oblique cone, or A's share, at 2l. per cwt. is 340l. 11s. 7.1197d. and that of the ungula, or B's share, is 731l. 10s. 9.5427d.

The same by Mr. Ed. Riddle, Whitburn, near Sunderland.

By equating the two theorems for the surfaces of the elliptic ungula

and the oblique cone, at page 180, Dr. Hutton's Mensuration, first edition, we have $(d + d) \sqrt{bd} = d^3$, where d and d are the bottom and top diameters of the elliptic ungula. This in the present case is $(6 + d) \sqrt{6d} = 36$; whence $d = 2.9342739$. Hence the altitude of the ungula is easily found = 4.8098589; its solidity = 57.8770924, and that of the oblique cone = 26.9459092 feet. Then taking the weight of a cubic foot of lead = 11325 ounces, it is readily determined that A must pay 340l. 11s. 7½d. and B 731l. 10s. 9d.

The same by Mr. John Ryley, jun. Leeds.

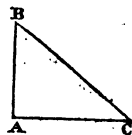
The half of the curve superficies is $6 \times .7854 \times 3 \sqrt{10}$. Put $cx = x$ then by similar triangles $gc = 6 - \frac{2}{3}x$. Now by the theorem, page 180, Dr. Hutton's Mensuration, the superficies of the oblique cone ACH will be $\frac{.7854 \times 3}{2x} \sqrt{(4x^2 - \frac{4}{9}x^2)} \times \frac{36 - 2x}{6} \sqrt{(36 - 4x)} = 6 \times .7854 \times 3 \sqrt{10}$; which reduced becomes $(18 - x) \sqrt{(9 - x)} = 27$, and hence x is found = 4.80985, and $gc = 2.7934\frac{1}{2}$. Hence by the theorem, page 169 of the Mensuration, the content of the elliptic ungula ABC is 57.8771; and then the content of the oblique cone ACH = 26.9461 cubic feet. Therefore the value of A's share will be 340l. 11s. 8½d. and that of B's share 731l. 10s. 9d.

VII. QUESTION 1200, by Mr. B. Bronwin, of Woodbridge.

Two heavy bodies projected from the top of a tower, one at an elevation of 60°, with the velocity of 24 feet per second, and the other of 30°, and 17 feet velocity, they were observed to fall on the same spot in the horizontal plane of the tower's base. Query its height?

Answered by Mr. J. A. Dotchen,

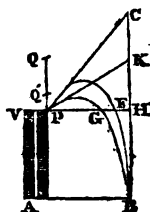
Suppose AB to be the tower, c the point where the two projectiles meet the horizon; join bc : then, putting x and z for sine and cosine of $\angle c$, by trigon. and the data in the question, the sine of the angle of direction of the first projectile above the oblique plane bc , is $= \frac{1}{2}(z\sqrt{3} + x)$, and that of the second $= \frac{1}{2}(x\sqrt{3} + z)$. Hence the two equal ranges on the plane bc , at the elevations of 60° and 30°, are found by article 103, page 164, vol. 2, Dr. Hutton's Course, to be $(z\sqrt{3} + x) \times 24^2 \div 4gt^2$ and $(x\sqrt{3} + z) \times 17^2 \div 4gz^2$ and, substituting $\sqrt{(1 - x^2)}$ for z , there results $287 \sqrt{(3 - 3x^2)} = 291x$, or



$1.7082425 \sqrt{1-x^2} = x$. Hence $x = .8630027$; and then the common range $bc = 60.9704294$, which $\times \sin \angle c (=x = .8630027)$ gives 52.617645 feet, for the height of the tower.

The same by Mr. J. H. Hearing.

Let AV be the tower; $rq, r'q'$ the perpendicular heights corresponding to the given velocities; pc, pk the given directions of projection; $pfb, p'gb$ the two parabolic curves described by the bodies, intersecting at b , a point in the horizontal plane AB of the tower's base. Draw bkc perpendicular and $p'gfh$ parallel to AB . Then in the right-angled triangles hpc, hpk , one angle of each being 30° , the sine of which is $= \frac{1}{2}$ radius, $pc = 2ph$, and $pk = 2kh$; therefore $ck = \frac{2}{3}\sqrt{3} \times ph$, by writing $\frac{1}{2}$ and $\frac{1}{2}\sqrt{3}$ for the sines of 30° and 60° to the radius 1. Now, by the nature of projectiles, Dr. Hutton's Course, prop. 20, cor. 3, bc is $= pc^2 \div 4pq$, and $bk = pk^2 \div 4p'q'$; but from above, $pc^2 = 4ph^2$, and $pk^2 = 4kh^2$; therefore by substitution $bc = ph^2 \div pq$ and $bk = ph^2 \div 3p'q'$; consequently $ck (= bc - bk)$ is $= ph^2 \left(\frac{1}{pq} - \frac{1}{3p'q'} \right) = \frac{2}{3}\sqrt{3} \times ph$ from above; hence $ph = \frac{2\sqrt{3} \times pq \times p'q'}{3p'q' - pq} = 30.80219$ feet; then $bc = ph^2 \div pq = 105.9683$, and $ch = ph \sqrt{3} = 53.35096$; and hence $bc - ch = bh = 52.61734$ feet, is the height of the tower.



VIII. QUESTION 1201, by Mr. T. Myers, R. M. A.

What is the area of a right-angled triangle, the radius of its circumscribing circle being twenty, and the area of the inscribed circle a maximum?

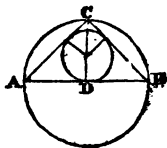
Answered by Mr. W. Marrat, of Boston.

Because an angle in a semicircle is a right angle, the diameter of the circumscribing circle is evidently the hypotenuse of the right-angled triangle; and the inscribed circle will be a maximum when the perpendicular from the vertical angle is so, that is, when the perpendicular bisects the hypotenuse: in which case the other two sides of the triangle are equal to each other, and its area $= 20 \times 20 = 400$.

The same answered by Mr. Noah Wilmot, S-s, near Newcastle.

With the given radius $AD = 20$ describe the circle ABC , the diameter

AB forming the hypotenuse of the required triangle, which, to contain the greatest circle must also be a maximum and this it is well known is when the two unknown sides are equal. Therefore bisecting the semicircle in c, and drawing AC, BC, these will complete the triangle; the area of which will be $AB \times CD = AD^2 = 20^2 = 400$.



Algebraic Solution of the same, by Mr. S. Titlow, Harleston.

Let ABC be the required triangle, the hypotenuse AB of which is given $= 40 = d$; and put the legs AC and BC $= x$ and $\sqrt{(d^2 - x^2)}$. Then it is easily perceived that the radius of the inscribed circle is $\frac{1}{2}(AC + BC - AB) = \frac{1}{2}(x + \sqrt{(d^2 - x^2)} - AB)$, which by the question is a maximum, or $x - \sqrt{(d^2 - x^2)}$ a maximum. This in fluxions and reduced gives $x = \sqrt{(d^2 - x^2)}$, and $x = d\sqrt{\frac{1}{2}} = AC = BC$. Hence the area is $\frac{1}{4}d^2 = 400$.

IX. QUESTION 1202, by Mr. W. Putsey, of Pickering.

A pendulum rod, 36 inches long, is suspended 4 inches below its upper end, with a weight of 50lb. connected to that end; what weight must be fixed to the lower end, so that in vibrating, it may have, with the most ease, the greatest momentum possible?

Answered by Mr. J. T. M'Doneld.

Putting $m = 32$ inches, $n = 4$, $w = 50$ lb. and $x =$ the weight to be attached, we have by art. 242, Dr. Hutton's Course, $\frac{m^2x + n^2w}{mx - nw}$ for the length of the pendulum, and $\sqrt{\frac{m^2x + n^2w}{mx - nw}}$ as the time of vibration, or as the velocity of x ; therefore $x \sqrt{\frac{m^2x + n^2w}{mx - nw}}$ is as the momentum, which by the question must be a maximum. From the fluxional equation there results $x^2 - \frac{3mnw - n^2w}{2m^2} x = \frac{n^2w^2}{m^3}$; where x is $= 9.4984$ lb. the weight required.

X. QUESTION 1203, by Mr. Wm. Saint, R. M. A.

What two numbers are those whose sum being increased or dimi-

nished by their difference or the difference of their squares, the sums and differences shall be all squares?

Answered by Mr. J. Cunliffe, jun. Bolton-le-Moors.

If the sum of two numbers be 1, then their difference = the difference of their squares. Therefore let x denote the less number, then $1 - x$ will be the greater, and $1 - 2x$ will denote both their difference and the difference of their squares. Hence by the question $1 + (1 - 2x) = 2 - 2x$ and $1 - (1 - 2x) = 2x$ are both to be squares. Put $2x = 4n^2$, or $x = 2n^2$; then $2 - 2x = 2 - 4n^2 =$ a square. Put $n = s - \frac{1}{2}$; then $2 - 4n^2 = 1 + 4s - 4s^2$ must be a square, suppose =

$$(2rs - 1)^2 = 4r^2s^2 - 4rs + 1; \text{ hence } s = \frac{1+r}{1+r^2}, \text{ and then } n =$$

$$s - \frac{1}{2} = \frac{1+r}{1+r^2} - \frac{1}{2}, \text{ where } r \text{ may be taken at pleasure. Take } r =$$

2; then $n = \frac{1}{10}$, and $x = 2n^2 = \frac{1}{50}$, and $1 - x = \frac{49}{50}$, which are two numbers that will answer. Take $r = 5$; then $n = \frac{7}{25}$, and $x = 2n^2 = \frac{49}{3125}$, and $1 - x = \frac{3125}{3125}$, two other numbers that will answer.

The same by Mr. Geo. Harvey, jun. Plymouth.

Let $x + y$ and $x - y$ be the two numbers; then is $2y$ their difference and $4xy$ the difference of their squares. Suppose $2y = 4xy$; then $x = \frac{1}{2}$, hence $2 + 2y$ and $1 - 2y$ must be two squares. Let $2y = r^2 + 2r$, then $1 + 2y = r^2 + 2r + 1 = (r + 1)^2$ a square; then also must $1 - 2y = 1 - 2r - r^2$ be a square, suppose = $(1 +$

$$nr)^2 = 1 + 2nr + n^2r^2; \text{ hence } n^2r + r = 2n - 2, \text{ and } r = \frac{2n - 2}{n^2 + 1}.$$

Take $n = 2$; then $r = \frac{2}{5}$, and $y = \frac{1}{10}$; hence the two numbers are $\frac{11}{20}$ and $\frac{9}{20}$.

The same answered by Mr. John Ryley, jun. Leeds.

Let x and y be the required numbers; then by the question $x + y + x - y = 2x =$ a square, and $x + y - x + y = 2y =$ a square, or x and y are each half a square, and consequently $x + y$ must be a square. Next, $x + y - x^2 + y^2$ and $x + y + x^2 - y^2$ must both be squares, which will also be squares when divided by $x + y$, which has been shown to be a square; that is, $1 - x + y$ and $1 + x - y$ are squares. Assume $1 + x - y = s^2$, then $x = s^2 + y - 1$, which substituted in $1 - x + y$, gives $2 - s^2$ a square, and it is so when $s = 1$. Therefore take $s = r + 1$, then $2 - s^2 = 1 - 2r - r^2 =$

x square = suppose $(1 - rn)^2 = 1 - 2rn + r^2n^2$; hence $r = \frac{2n-2}{n^2+1}$

and $s^2 = \left(\frac{n^2 + 2n - 1}{n^2 + 1}\right)^2$, also $2 - s^2 = \left(\frac{n^2 - 2n - 1}{n^2 + 1}\right)^2$. Take

$n = 3$, then the two formulas give $\frac{4}{25}$ and $\frac{1}{25}$. Therefore if $x + y = 1$, we have $x = \frac{4}{25}$, and $y = \frac{1}{25}$, which are perhaps the least two numbers that will answer the question.

XI. QUESTION 1204, by Mr. J. H. Harding.

In the expression $(aa + bb - 2ab \times \cos. z)^n = A \pm B \times \cos. z + C \times \cos. 2z \pm D \times \cos. 3z$, &c. ad infinitum, which is useful in physical astronomy, the values of A and B, on which C, D, &c. depend, have been computed by means of series thus: viz.

$$A = a^{2n} \times (1 + a^2 \cdot \frac{b^2}{a^2} + c^2 \cdot \frac{b^4}{a^4} + \gamma^2 \cdot \frac{b^6}{a^6} + \&c. \text{ ad infin.});$$

$$B = 2a^{2n} \times (a \cdot \frac{b}{a} + ac \cdot \frac{b^3}{a^3} + c\gamma \cdot \frac{b^5}{a^5} + \&c. \text{ ad infin.});$$

in which series a is $= n$, $c = n \cdot \frac{n-1}{2}$, $\gamma = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, &c.

Now Mr. Woodhouse (fellow of Caius College in Cambridge, has asserted in the Philosophical Transactions of R. S. for the year 1804, page 273, that when the index $n = \frac{1}{2}$, "the series for A and B, with respect to its numeral coefficients, decreases the fastest." It is required to prove the truth or the falsehood of this assertion.

Answered by Senex.

To prove the falsehood of Mr. Woodhouse's assertion, nothing more is requisite than to substitute $\frac{1}{2}$ and $\frac{3}{2}$ for n , and to compare the two results with each other. Thus, when $n = \frac{1}{2}$, we have $a = \frac{1}{2}$, $c = \frac{-1.1}{2.4}$, $\gamma = \frac{1.1.3}{2.4.6}$, $\delta = \frac{-1.1.3.5}{2.4.6.8}$, &c. ad infin: the 10th term

of which series, here denoted by χ , is $\frac{-1.1.3.5.7.9.11.13.15.17}{2.4.6.8.10.12.14.16.18.20}$

When $n = \frac{3}{2}$, we have $a = \frac{3}{2}$, $c = \frac{3.1}{2.4}$, $\gamma = \frac{-3.1.1}{2.4.6}$, $\delta =$

$\frac{3.1.1.3}{2.4.6.8}$, &c. ad infin. the 10th term of which series is $\chi =$

$\frac{3.1.1.3.5.7.9.11.13.15}{2.4.6.8.10.12.14.16.18.20}$

Here it is evident by inspection, that the 2d series, has an advantage over the first, in point of convergency, which is the greater the farther the terms are from the beginning. For instance, χ^2 , which occurs in computing the value of Δ , (see the last year's Diary,) is to χ^1 , as 17^1 is to 3^1 , that is, as 32 to 1 nearly. Such is the advantage of the 2d series over the 1st at the 10th term; and were we to compare the 20th terms, the ratio would be found as 37^2 to 9, or as 152 to 1 nearly. And that the series for computing the value of π obtains nearly the same increase of convergency, by changing the index n from $\frac{1}{2}$ to $\frac{3}{2}$, is so very obvious from what is here done, that I need not specify it. Thus the falsehood of Mr. Woodhouse's assertion might be made to appear by innumerable other values of n . And I may farther add, that this is not the only error which may be found in Mr. Woodhouse's writings.

XII. QUESTION 1205, by Mr. P. Barlow, R. M. A.

If a be any prime number; then is it a divisor of the sum of all the consecutive squares $1^2 + 2^2 + 3^2 \dots \left(\frac{a-1}{2}\right)^2$. And if the product of all those squares be divided by a , the remainder will be always ± 1 . Required the demonstration?

Answered by Mr. William Burdon, Acaster Malbis.

The sum of the series $1^2 + 2^2 + 3^2 \dots \left(\frac{a-1}{2}\right)^2$, is $\frac{a^3 - a}{4 \cdot 6} = \frac{(a+1) \times (a-1) \times a}{2 \times 2 \times 6}$. By Dr. Hutton's Dictionary, art. cube, the difference between the cube of any number and the number itself, is always divisible by 6; and every prime being an odd number, both $a+1$ and $a-1$ can be divided by 2 without a remainder. Therefore the sum of the series is a multiple of a . Hence, if the square of each of the prime numbers 5, 7, &c. be divided by 24, the remainder will always be 1.

The same by Mr. John Ryley, Leeds.

First, it is well known that the sum of the series $1^2 + 2^2 + 3^2 + \&c.$ to a terms is, $\frac{a \times (a+1) \times (2a+1)}{2 \times 3}$; which, if $\frac{a-1}{2}$ be substituted for a , becomes $\frac{a \times (a+1) \times (a-1)}{2 \times 3 \times 4}$, which is evidently divisible by a .

Again, if a be a prime in the series of odd numbers 1, 3, 5, 7, &c. then will $\left(\frac{a-1}{2}\right)^2$ or $\frac{a^2-2a+1}{4}$ be a term in the series of squares $1^2, 2^2, 3^2, 4^2, \&c.$; and by changing the value of a , all the terms in this series may be exhibited from 1 to any number whatever. Now if the above formula be divided by a , it is evident that 1 will remain; and as this happens in a single factor, it will happen in any number of them, and also in their products; the negative sign taking place when the number of terms is even, and the affirmative when odd. But if a is not a prime, the continual product of the squares will divide by a without any remainder.

XIII. QUESTION 1206, by Mr. Wm. Marrat, of Boston:

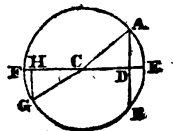
A circular plate of metal, of uniform thickness, is at rest on its centre, with its upper surface parallel to the horizon: now if two weights, one of 7 and the other of 5lb. be suspended from its outer edge, at the distance of 60° from each other; how many degrees, along the arc from the less weight, must another of 3lb. be placed, that the whole may be in equilibrio?

Answered.

In answering this question most of the correspondents remark on some small impropriety in the proposition, as the small weight of 3lb. will no where in the circumference, balance the two larger weights, when the plate is supported on the central point only; for in this case, any two weights, and their distance being given, both the 3d weight and its situation may thence be ascertained, and neither of these ought to be given in the question, for if the weight be given, its place may happen to fall out of the circumference. Some of the contributors, therefore, make the weights 7 and 3 change places; while others compute where the weight 3 will fall, out of the circle; and some make the plate rest not simply on the central point only, but on a line or edge passing through the centre, which is the method of the author, as follows:

1. By the Author, Mr. Wm. Marrat, of Boston.

In forming this question I made a small mistake; but if the 3 and 5lb. weights be suspended at the distance of 60° from each other, on the outer edge, the place of the 7lb. weight, when in equilibrium, may be found thus: Let B be the place of the 3lb, A that of the 5, and G that of the 7lb; join AB , and through the centre C draw FE bisecting AB .—Now A, B, G , denoting the three weights, by mechanics, the difference of the momenta of A and B ,



$\frac{3}{8}ab\sqrt{\frac{1}{3}} \div \frac{3}{8}ab = \frac{2}{3}\sqrt{\frac{1}{3}}$ is the ratio of ABC to EDF. Hence all the series of parabolas inscribed within each other will form a series of geometrical proportionals, in the constant ratio of $\frac{2}{3}\sqrt{\frac{1}{3}} = r$, the first term being $\frac{3}{8}ab = c$. Hence the sum of any number n of those terms

will be $\frac{1-r^n}{1-r} c = \frac{1-(\frac{2}{3}\sqrt{\frac{1}{3}})^n}{1-\frac{2}{3}\sqrt{\frac{1}{3}}} \times \frac{3}{8}ab$; and the sum of the infinite series of them $\frac{1}{1-\frac{2}{3}\sqrt{\frac{1}{3}}} \times \frac{3}{8}ab = \frac{4ab}{3-2\sqrt{\frac{1}{3}}}$.

Answered by Mr. P. Barlow, R. M. A.

The greatest inscribed parabola, is to that in which it is inscribed, as the greatest inscribed rectangle is to the circumscribing one; which is easily shown to be as $\frac{2}{3}\sqrt{\frac{1}{3}}$ to 1. Therefore calling the first parabola $= a$, and $\frac{2}{3}\sqrt{\frac{1}{3}} = m$, we have only to sum the series $a(1+m+m^2+m^3$

&c.) to n terms, which is known to be $\frac{1-m^n}{1-m} a$, or only $\frac{a}{1-\frac{2}{3}\sqrt{\frac{1}{3}}}$ when n is infinite.

The same by Mr. J. Collins, Kensington.

The inscribed parabola EDF will be a maximum, when the product DE . EF is a max. which is known to be the case when $DE = 2EG = \frac{2}{3}BD$. Hence, calling $BD = a$, and $AC = 2b$, then is $DE = \frac{2}{3}a$, $EG = b\sqrt{\frac{1}{3}}$, and the area of EDF $= \frac{3}{8}ab\sqrt{\frac{1}{3}}$, which is to the circumscribing parabola in the constant ratio of $\frac{2}{3}\sqrt{\frac{1}{3}}$ to 1, and the same for all the successive parabolas, thus forming a geometrical series of the constant ratio

$\frac{2}{3}\sqrt{\frac{1}{3}}$: hence the sum of n terms is $\frac{1-(\frac{2}{3}\sqrt{\frac{1}{3}})^n}{1-\frac{2}{3}\sqrt{\frac{1}{3}}} \times m$, and only

$\frac{m}{1-\frac{2}{3}\sqrt{\frac{1}{3}}}$ when the series is infinite, where m is the area of the first parabola.

The same by Mr. J. T. M'Doneld.

It being pretty generally known that the dimensions of the greatest rectangle or parabola, inscribed in a given parabola, are $\frac{2}{3}$ the height, $\sqrt{\frac{1}{3}}$ the base or double ordinate, it is obvious that $\frac{2}{3}\sqrt{\frac{1}{3}}$ will express the ratio of decrease. Therefore, calling this r , and the area of the parabola first inscribed a , the series to any number n of maximum pa-

rabolas will be $a(1+r+r^2+r^3+\dots+r^{n-1})$, its sum $= \frac{1-r^n}{1-r} a$,

or only $\frac{a}{1-r}$ for the infinite number.

The same by Mr. Wm. Saint, R. M. A.

It is well known that the area of any parabola = $\frac{2}{3}$ of its circumscribing parallelogram, and that the greatest parabola $\mathbf{B}$, which can be inscribed in any parabola $\mathbf{A}$, is that which is circumscribed by the greatest inscribed rectangle of $\mathbf{A}$. Hence, if b = base, and a = altitude of parabola $\mathbf{A}$, its area will be $\frac{2}{3}ab$; and if the base of its inscribed

rectangle = x , and height = y , then $b^2 : x^2 :: a : \frac{ax^2}{b^2}$, therefore,

$$y = \frac{b^2 - x^2}{b^2} \times a; \text{ consequently } xy = \frac{b^2x - x^3}{b^2} \times a = \text{a maxi-}$$

imum, from which $x = b\sqrt{\frac{1}{2}}$, therefore $y = \frac{1}{2}a$; hence the area of the inscribed rectangle = $xy = \frac{2}{3}ab\sqrt{\frac{1}{2}}$, that is = $\sqrt{\frac{1}{2}} \times$ parabola $\mathbf{A}$. Therefore parabola $\mathbf{B} = \frac{2}{3}\sqrt{\frac{1}{2}} \times$ parabola $\mathbf{A}$. In like manner, parabola $\mathbf{C} = \frac{2}{3}\sqrt{\frac{1}{2}} \times$ parabola $\mathbf{B}$. Hence we have only to find the sum of a geometrical progression whose first terms is $\frac{2}{3}ab$, and ratio $\frac{2}{3}\sqrt{\frac{1}{2}}$,

and number of terms n , which sum is $\frac{1 - (\frac{2}{3}\sqrt{\frac{1}{2}})^n}{1 - \frac{2}{3}\sqrt{\frac{1}{2}}} \times \frac{2}{3}ab$, and only

$$\frac{\frac{2}{3}ab}{1 - \frac{2}{3}\sqrt{\frac{1}{2}}} \text{ when } n \text{ is infinite.}$$

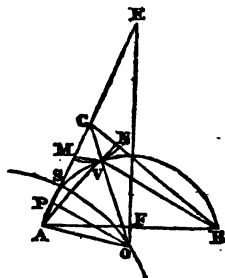
PRIZE QUESTION, *by the Rev. J. Furnass, Ponteland.*

A gentleman dying left a plot of ground, in the form of a triangle, whose three sides are 60, 84, and 100 chains, lying towards the north-west, north-east, and south respectively, to his four children. One part was to be a sector of a circle described from the north or vertical angle with a radius of 10 chains, the arc cutting both sides, and passing through a fountain somewhere in the field, which was to be in common; another part was to be a triangle, formed by the longest or south side, and two lines connecting its extremities with the fountain; the remaining spaces the other two parts. Quere each share and area, when the vertical angle at the fountain of the second part is a maximum.

Answered by Mr. J. A. Dotchen, Chelsea.

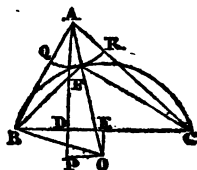
If a circle be described to pass through the extremes $\mathbf{AB}$ of the south side of the field $\mathbf{ABC}$, and touch the arc $\mathbf{MN}$, whose radius is given = 10, and $\mathbf{AV}$ and $\mathbf{BV}$ be drawn to the point of contact $\mathbf{v}$, it is pretty obvious that the $\angle \mathbf{AVB}$ will then be a maximum, as required, and $\mathbf{v}$ the place of the fountain. From $\mathbf{o}$, the centre of the cir-

cle passing through AVB, draw OFE perpendicular to AB, meeting AC produced in E; also make OF perpendicular to AC, and join AO, CO. Then the triangle ABC being wholly given, and AB bisected in F, AE and FE are easily found = 91.687042 and 76.853846 respectively; and since AO, CO have a given difference CV, O will be situated in the arc of an hyperbola whose foci are AC, and transverse diameter CV = 10, and conjugate $\sqrt{(AC^2 - CV^2)} = 10\sqrt{35}$. Call the abscissa, SF, x , then, FE = $x + 66.687042$, and ordinate FO = $\sqrt{(350x + 35x^2)}$, and by similar triangles, FO : FE :: AF : FE, or $\sqrt{(350x + 35x^2)} : x + 66.687042 :: 50 : 76.853846$, and equating the products of extremes and means gives $x = 4.267331 = SF$, and FO = $\sqrt{(350x^2 + 35x)} = 46.1610887 = \text{tang. } \angle ACO \text{ to rad. } PC = 39.267331$; hence $\angle ACO = 49^\circ 36' 50'' \frac{2}{3}$, which taken from $\angle ACB = 86^\circ 16' 6'' \cdot 9$, leaves $\angle BCO = 36^\circ 39' 16'' \cdot 3$; then the area of $\triangle ACV = AC \cdot CV \cdot \frac{1}{2} \sin. \angle ACO$ to rad. $1 = 228.50916$ square chains, and the sector $MCV = 43.29643$, their difference is 185.21273 or 18 acres, 2 roods, 3.40368 perches, the area of the share AMV. In like manner BNV is found = 21 acres, 3 roods, 19.96963 perches, the whole sector MCN = 7 acres, 2 roods, 4.53664 perches, and lastly the whole triangle ABC minus the sum of the three shares just found, leaves AVB = 203 acres, 2 roods, 6.60358 perches. Mr. D. gave also a geometrical construction.



The same by Mr. J. H. Hearing, Pall Mall.

Let ABC represent the triangular field, and AQR the circular sector, or first division. Through the given points A and C describe the circle OBFC to touch the given circle QR in F, the place of the fountain: then draw AF, BF, CF, so shall BFC, BFQ, CFR be the other three portions of the field. From the centres A, O, draw ADP and OE perpendicular to BC the former meeting OF = and parallel to DE in F; and put BE = $\frac{1}{2}BC = 50 = a$, DE = FO = $17.28 = b$, AD = $50.29316 = c$, the radius AF = $10 = r$, and the perp. OE = x . Then AF = $c + x$; AO = $\sqrt{[(c + x)^2 + b^2]}$, and OF = OB = $\sqrt{[(c + x)^2 + b^2]} - r$; but OB is also = $\sqrt{(a^2 + x^2)}$, therefore $\sqrt{(a^2 + x^2)} = \sqrt{[(c + x)^2 + b^2]} - r$, hence $x^2 + 4.720028x = 97.5565$, and $x = 7.795094 = OE$. Hence BO = OF = 50.603986 , AO = 60.603986 , $\angle BAF = 49^\circ 36' 51''$, $\angle BAC = 86^\circ 16' 7''$, and $\angle CAF = 36^\circ 39' 16''$; theref. area of triangle BAC = $BA \times AC \times \frac{1}{2} \text{nat. sin. } 86^\circ 16' 7'' = 2514.657852$, the triangle BAF = 228.50958 , and triangle CAF = 250.734708 square chains.



Now the sector AQR , to the radius 10, and arc $QR = 86^\circ 16' 7''$, is $= 75.2836$, or 7 acres, 2 roods, 4.5 perches, the area of the first part, and if the sum of the two triangles BAF , CAF , be taken from the triangle BAC , the remainder 2035.413564, or 203 acres, 2 roods, 6.6 perches is the area of the triangle BFC , or the second part; also, the areas of the sectors AQF , ARF , viz. 43.29652 and 31.98704, being taken from the two triangles BAF , ACF , leave 185.21306 and 218.747668, or 18 acres, 2 roods, 3.4 perches, and 21 acres, 3 roods, 20 perches for the two remaining parts.

Questions proposed in 1810, and answered in 1811.

I. QUESTION 1209, by Mr. Tho. Kirton, of Cranford.

An old gentleman bequeathed 100000l. to his four sons, in such proportion, that every son at the time of their successively coming of age, will possess the same sum as his fortune; what legacy did the father give to each, supposing their ages were 12, 13, 14, 15 years, and allowing compound interest at 5 per cent. per annum?

Answered by Mr. Wm. Burdon, Acaster Malbis.

Put $r = 1.05$ the amount of 1l. for one year, $m = 100000$ l.; and let the four shares be denoted by x, rx, r^2x, r^3x : then $x + rx + r^2x + r^3x = m$, hence $x = m \div (1 + r + r^2 + r^3) = 23201.183$ l. the first share; then the other three shares are 24361.242, and 25579.304, and 26858.271. For, by the table at art. Compound Interest, Dr. Hutton's Dictionary, the amount of each of these sums will be 35992.65l.

The same answered by Mr. Tho. Ford, Bighton.

Put $p = 100000$, and $r = 1.05$; then will the ratios of the shares be denoted by 1, r, r^2, r^3 , the sum of which is $4.310125 = a$. Then as

$a : r^3 :: p : 26858$ l.	5s. 4d. 2.98206q.	the eldest son's share.
$a : r^2 :: p : 25579$	6 1 0.36224	the second ditto.
$a : r :: p : 24361$	4 10 0.72608	the third ditto.
$a : 1 :: p : 23201$	3 7 3.92962	the fourth ditto.

The same by Mr. W. Putsey, Pickering Academy.

The present worths of 1l. forborn during their respective minorities, are $\cdot 74622$, and $\cdot 71068$, and $\cdot 67684$, and $\cdot 64462$ l. the sum of which is $2\cdot 77836$ l.; consequently $2\cdot 77836 : 100000 :: \cdot 74622 : 26858$ l. 5s. 9 $\frac{1}{2}$ d. the sum left to the eldest son. In like manner the sums bequeathed to the others, are found to be 25579 l. 2s. 7 $\frac{1}{2}$ d., and 24361 l. 2s. 7 $\frac{1}{2}$ d., and 23201 l. 9s.

II. QUESTION 1210, by Mr. John Bains, jun. *Horbury Bridge.*

Suppose a person on the top of the Peak of Teneriffe, whose height is 15379 feet, could just see the top of a mountain over the horizon, at the distance of 300 miles. Required the height of that mountain, supposing the earth's radius 3978 $\frac{1}{2}$ miles, or circumference 25000 miles, and the height of the spectator's eye 5 feet?

Answered by the Proposer, Mr. John Bains, jun.

Let τ represent the Peak of Teneriffe, and m the mountain. Then $15379 + 5 = 15384$ feet = $2\frac{29}{32}$ miles = $21\frac{9}{16}$ nearly = CT : hence $3978\frac{1}{2} + 2\frac{29}{32} = 3981\frac{141}{160} = AT$. Now, as 25000 miles : $360^\circ :: 300 : 4^\circ 19' 12'' = \angle MAT$; and as $AT : \text{radius} :: AB : \cos. \angle BAT = 2^\circ 11' 15''$; therefore $\angle BAM = 2^\circ 7' 57''$; hence as $\cos. \angle BAM : AB :: \text{radius} : AM = 3981\cdot 642$; therefore $DM = 14610$ feet = $2\cdot 767$ miles the answer.



The same answered by Mr. Rob. Rogers, of Delapre Abbey.

By right-angled triangles $BT = \sqrt{(AT^2 - AB^2)} = 814129\cdot 94269$ feet; hence $BM = 300 \times 5280 - 814129\cdot 94269 = 779870\cdot 05731$ feet; then $AM = \sqrt{(AB^2 + BM^2)} = 21022930\cdot 071$ feet; and consequently $DM = 14470\cdot 071$ feet, the height of the mountain:

The same by Miss Mary Weston, Gainsborough.

Put the semidiameter of the earth AB or $AC = 21008460$ feet = r ; the height of the observer's eye from the base of the mountain $CT = 15384$ feet = a ; the tang. $MT = 300$ miles = 1584000 feet.

Then $AT^2 - AB^2 = (a + r)^2 - r^2 = BT^2$; hence $BT = \sqrt{(a^2 + 2ar)} = 804129\cdot 942$ feet.

And $MT - BT = MB = 1584000 - 804129\cdot 942 = 779870\cdot 058$ ft.

But $AB^2 + BM^2 = AM^2$; hence $AM = 21022930\cdot 0735$ feet.

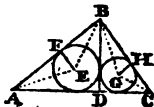
Therefore $AM - AD = DM = 14470\cdot 0735$ feet, the mountain's height required.

III. QUESTION 1211, by Mr. S. Titlow, of Harleston.

If, in a right-angled triangle, a perpendicular be drawn from the right angle upon the hypotenuse, and circles inscribed, within the two smaller triangles on each side of it, their diameters will be to each other as the subtending legs of the right-angled triangle. Required the demonstration?

Answered by Mr. John Davey, of St. Just, in Penwith.

Draw the lines as in the question, the dotted lines each bisecting its respective angle, and meeting each other in their correspondent centres E and G ; from which, to the points of contact, F , H , draw the radii EF , GH . Then, since the angle ABC is a right one, the triangles ABD , BCD , are similar, and consequently the triangles ABE , BCG , are similar also; and, therefore as $AB : BC :: EF : GH :: 2EF : 2GH$.



The same by Mr. A. Nesbit, of Farnley, near Leeds.

By Dr. Hutton's Course, vol. 1, p. 336, theor. 87, the three triangles ABC , ABD , BCD , are all similar; and it is well known, that if there be drawn similar lines within similar triangles, those lines will have to each other the proportion as the like sides of the respective triangles. Hence, the diameters of the two inscribed circles will not only be in proportion to each other as the sides AB and BC , but also as any other like sides of the circumscribing triangles. Moreover, if a circle be inscribed within the large $\triangle ABC$, its diameter will be in proportion to the diameters of the two circles inscribed in the two smaller triangles, as the like sides of the three triangles. Also, the circumferences of the inscribed circles, or any like chords drawn within them, will still be in proportion to each other as the like sides of the triangles circumscribing the circles.

IV. QUESTION 1212, by Mr. W. Burdon, Acaster Malbis.

- The perpendicular from the centre of gravity of a cone, to the slant side is the greatest possible to have the cone's solidity 231 inches. Required its height, and the diameter of its base?

Answered by Mr. D. Anderson, R. M. A.

Let ACD be the cone, AB its altitude, and or the perpendicular from

its centre of gravity o , on the side AD .—Put $AB = x$; then $OB = \frac{1}{2}x$, and $AO = \frac{1}{2}x$. Let $231 = a$, and $.7854 = c$. Now $\sqrt{(3a \div 4c)} = BD$, and $\sqrt{(x^2 + 3a \div 4c)} = AD$; then

by similar trian^g. $AD : BD :: AO : OP = \frac{3x\sqrt{3a}}{4\sqrt{(3a + 4cx^2)}}$

which must be a maximum, or $\frac{x^2}{3a + 4cx^2}$ a maximum;

the fluxion of which made $= 0$, gives $x = \sqrt[3]{(3a \div 2c)} = 7.6127$ the altitude. Hence $2BD = CD = \sqrt{(3a \div cx)} = 10.766$ the diameter of the base.



The same by Mr. Thomas Clifton, Nottingham Academy.

If y, z, x denote the height, slant side, and semi-base of the cone. Then $3.1416x^2 \times \frac{1}{3}y = 231 = a$; hence $x^2 = a \div \frac{1}{3}$

$1.0472y$ or by . But $\sqrt{(x^2 + y^2)} = z$, and $z : y :: x : AC = \sqrt{\frac{ay}{b} \div \sqrt{\frac{a + by^2}{by^3}}} = \sqrt{\frac{ay^2}{a + by^3}}$, which must be a max.

because parallel to the max. line drawn from the centre of gravity. Its fluxion put $= 0$, and reduced, gives $2a \div b = y^2$; hence $y = 7.61268$, and $2x = 2\sqrt{(a \div by)} = 10.76595$.

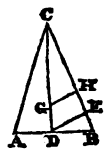


The same by Mr. Jas. Cunliffe, jun.

Let ABC represent a section of the cone through its axis CD ; and let DE be perpendicular to the slant side BC . In the axis CD take $CG = \frac{1}{3}CD$, then G will be the centre of gravity of the cone, as is well known. Draw GH perpendicular to CB ; then is $GH = \frac{1}{3}DE$, consequently when DE is a maximum GH will be so too. Put $CD = y$, $s = 231$, and $a = 3.1416$. Then, by mensuration, $\frac{1}{3}axy^2 = s$, or $xy^2 = 3s \div a = b$, and $y^2 = b \div x$.

Again, $DE = \frac{AD \times CD}{AC} = \frac{xy}{\sqrt{(x^2 + y^2)}}$

$\frac{x\sqrt{b}}{\sqrt{(b + x^2)}}$; the fluxion of which put $= 0$, and reduced gives $x^3 = 2b = 6s \div a$; whence $x = 7.61268$, the height, and $y = 5.38298$, the radius of the base.



V. QUESTION 1213, by Mr. Wm. Burdon.

The solidity of a cone is 282 inches, which is cut into two parts by a plane perpendicular to its slant side, and passing through the

centre of gravity and the middle of the radius of the base. Required the solidity of each part.

Answered by Mr. Henry Atkinson, Newcastle.

Let ADF represent the cone, and c its centre of gravity. Put $x =$ BD its altitude, $a = 282$ its solidity, and $\cdot 785398 = b$: then will $AB = \frac{1}{2} \sqrt{(3a \div bx)}$, and $BC = \frac{1}{4} x$; and in the sim. triangles ABD, GBC, it is $BD : AB :: BG$ or $\frac{1}{2} AB : BC$, hence $\frac{1}{4} x^2 = 3a \div 8bx$, and $x = \sqrt[3]{(3a \div 2b)} = 8 \cdot 136107 = BD$ the alt. hence $AF = 2AB = \sqrt{(3a \div bx)} = 11 \cdot 50620$, $EH = 5 \cdot 75310$, $BI = \frac{1}{2} BD = 4 \cdot 0680535$, $re = 8 \cdot 62965$, and the elliptic hoof GEF (by prob. 27, pt. 3, sect. 1, of Dr. Hutton's Mensuration), is $135 \cdot 2866$; hence the remaining part ADEG is $146 \cdot 7134$.



The same by Mr. John Davey, of St. Just.

Suppose ABF to be a similar cone, whose base AF is $= 2$, and $n = \cdot 261799$. Then by similar triangles as $GB = \frac{1}{2} : BC = \frac{1}{4} BD :: BD : AB = 1$; hence $BD = \sqrt{2}$, and the solidity $4n \sqrt{2}$. Again, $AD = \sqrt{(AB^2 + BD^2)} = \sqrt{3}$; and as $AD : AB :: GF : EF = \frac{1}{2} \sqrt{2}$; also, as $GF : EF :: EF : KF = \frac{1}{2}$; and as $DF : EF :: BD : BE = \frac{1}{2} \sqrt{2}$; also $HE = AF - 2KF = 1$. Hence will be found, by Mensuration, the content of the part GEF, which call c ; then by similar solids, as $4n \sqrt{2} : c :: 282 : 135 \cdot 2865$ the solidity of the lower part; and $282 - 135 \cdot 2865 = 146 \cdot 7135$ that of the upper part.

The same answered by Mr. J. A. Dotchen, of Chelsea.

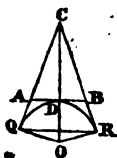
Suppose ADF to be the cone, GE the cutting plane, &c. then by the question the triangles ABD, GBC are sim. and BD or $4BC : AB$ or $2GB :: GB : BC$; hence $GB = BC \sqrt{2}$, and $GC = \sqrt{(GB^2 + BC^2)} = BC \sqrt{3}$, that is, BC, BG, GC , or AB, BD, AD , are as $\sqrt{1}, \sqrt{2}, \sqrt{3}$. Further, because GBC and CDE are similar, $\sqrt{3} : \sqrt{1} :: CD$ or $3BC : CE = BC \sqrt{3} = GC$, from above: therefore $CI = CB$. Now call $AB = x$, then by Mensuration, &c. $\frac{1}{2} \times \cdot 7854 x^2 \times x \sqrt{2} = 282$; hence $x = 5 \cdot 753095$, and $BD = x \sqrt{2} = 8 \cdot 136102$, also $BI = \frac{1}{2} BD = 4 \cdot 068051$, $HE = AB = 5 \cdot 753095$, $GF = \frac{1}{2} AB = 2 \cdot 8765475$; whence, by the rule at p. 171, Dr. Hutton's Mensuration, 3d edition, the content of the elliptic ungula GEF is $135 \cdot 286176$, which taken from 282 leaves $ADEG = 146 \cdot 743824$.

VI. QUESTION 1214, by Mr. Dan. T. Sheridan, *Wolverhampton.*

Given the perimeter and vertical angle, to construct the plane triangle, when the area is a maximum.

Answered by Mr. J. H. Harding, Pall Mall.

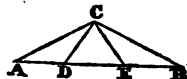
Constr. With the lines cq , cr , each = the given semiperim. make the $\angle qcr =$ the given vertical angle; draw the two perps. qo , ro , intersecting in o , the centre of the circle touching the base and continuation of the sides; draw co , and with radius oq or or , and centre o , describe the circle qdr ; draw qr , and parallel to it through the point d , draw adb ; so will abc be the triangle required.



Demonst. Because the semiperim. drawn into the radius of the inscribed circle is = the area, and the area is to be a maximum, it is evident that the inscribed circle itself must be a maximum, the semiperim. as well as the vertical angle, or position of the lines, ec , or qr being given, and consequently the triangle must be isosceles, or ab parallel to qr , as in the construction.

The same answered by Mr. W. G. Horner, Classical Seminary, Bath.

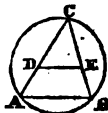
Make $ab =$ the given perimeter, and the angles a and b each = $\frac{1}{2}$ the supplement of the given one, by lines meeting in c . Draw cd , ce , making the angles acd , bce each = a or b ; then cde is the triangle required. For, the vertical angle being given, the area is as the rectangle of the including sides, and a maximum when they are equal.



But, by the construction, they are each = ad or eb ; and the $\angle dce =$ supplement (sum of $cde + ced = dac + acd + ebc + ecb =$) $4a$ or $4b$, as was assumed.

The same answered by Mr. Edw. Riddle, Whitburn, Sunderland.

About any triangle abc let a circle be described: then if the $\angle c$ be given, its opposite side ab is also given; and, by theor. 6, Simpson on the Maxima and Minima, the area of the triangle is a max. when $ac = cb$. Hence, describe any isosceles triangle acb , having the $\angle c =$ the given one, and on ac , or ac produced, take cd , so that $cd : ca ::$ as the given perim. : $ac + ab + bc$. Through d draw de parallel to ab , and cde will be the triangle required,



The same answered by the Proposer, Mr. D. T. Sheridan.

On EF = the given perimeter describe a segment of a circle to contain the given vertical angle $+ \frac{1}{2}$ its supplement: bisect EF in D , on which erect the perp. CD ; draw CE , CF ; and make the angles ECA , FCB , each = the $\angle E$ or F ; then is ABC the required triangle.—For, because the $\angle BAC = 2\angle ACE$ by construction $= 2\angle BCF$, the $\angle ACE + BCF = BAC = \frac{1}{2}$ supplement of vert. $\angle$; therefore $\angle ACB$ = the given vert. $\angle$, because $AC = AF$, and $BC = BF$, by construction $AB + AC + BC = EF$, and ABC is a maximum by theor. 6, Simpson on the Maxima and Minia.



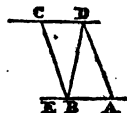
Several contributors remark that this is the same as the Diary question 480, in Hutton's Diarian Miscellany. And the proposer adds, "I was not aware, when I proposed this question, that it was an old one: but my method of construction is so different from any yet given to it, that I trust it will secure me from the imputation of being a wilful plagiarist".

VII. QUESTION 1215, by Mr. Wm. Cole, of Colchester.

Travelling on a road that tended directly west, at the rate of six miles an hour, I observed that the wind appeared to strike me from the north-west; but having occasion to stop, I then found that it actually came from a point 10° more to the north. Required the velocity of the wind?

Answered by Mr. W. Bean, of Ridgmont.

Let AB = the road = 6, AD or BC the apparent direction of the wind when travelling, and BD the true direction when stopped at B : then $\angle A = 45^\circ$, $\angle ADB = 10^\circ$, $\angle DBA = 125^\circ$. Hence as $s. \angle ADB$: $AB :: s. \angle A$: $BD = 24.43$ miles, the velocity required.



The same, answered by Mr. Geo. Harvey, jun. Plymouth.

Let AB be the road, tending directly west; the $\angle A$ the apparent, and the $\angle DBE$, the real direction of the wind; $AB = 6$, the distance travelled in one hour. Then, in the triangle ABD , as $s. \angle D$ (10°): AB (6) :: $s. \angle A$ (45°): $BD = 24.432$ miles, the velocity required.

The same answered by Mr. S. Parlour.

Let AB express the direction and velocity of the traveller; ca the apparent, and DB the real velocity and direction of the wind:

(see Dr. Hutton's Dictionary, art. Aberration). Hence, the $\angle D$ being given $= 10^\circ$, and the $\angle A = 45^\circ$, and $AB = 6$, we have $s.10^\circ : AB = 6 :: s.45^\circ : BD = 24.4326$ miles, the velocity of the wind per hour.

VIII. QUESTION 1216, by Mr. John Ryley, jun. Leeds.

Having a desire to know the depth of a coal-pit, I dropped a stone into it, and between the stone leaving my hand, and the sound it made by dashing against the bottom reached my ear, I counted 8 pulsations, of which 75 make a minute. Required the depth of the pit?

Answered by the Rev. J. Furness, of Ponteland.

First, as $75 : 60 :: 8 : 6\frac{2}{3}$ seconds $= t$ the whole time : also put $a = 1142$, $b = 16\frac{1}{12}$, and $x =$ time of the sound's ascending from bottom to top ; then $t - x =$ the time of the stone's descent. Then, (by Dr. Hutton's Conics, p. 149) $ax =$ the depth of the pit ; and, by theor. 5, p. 169, of the same, $b(t - x)^2 =$ the depth also ; therefore $b(t - x)^2 = ax$; the root of which gives $x = .49164$ seconds ; hence $ax = 561.4529$ feet, or 93.5755 fathoms nearly, the depth required.

The same by the Rev. Mr. T. Scurr, Hexham.

Put $a = 16\frac{1}{12}$ feet, $b = 1142$, $m = 6\frac{2}{3}$ sec. the whole time, and $x =$ the depth of the pit. Then as $\sqrt{a} : 1'' :: \sqrt{x} : \sqrt{(x \div a)}$ the time of the stone's descent ; and $b : 1'' :: x : (x \div b)$ the time of the sound's ascent ; therefore $\frac{x}{b} + \sqrt{\frac{x}{a}} = m$; which equation reduced gives $x = 561.4677$ feet, the depth required.

The same answered by Mr. Charles Stamp, of Boston.

Let the time the stone was falling be denoted by x ; also put $a = 1142$, $t = 6\frac{2}{3}$ sec. the time of 8 pulses, and $g = 16\frac{1}{12}$. Then $t - x$ is the time of the sound's ascending : Now, by falling bodies, $gx^2 =$ the depth, and by the uniform motion of sound $ta - ax =$ the depth also ; therefore $gx^2 = ta - ax$, the root of which equation is $x = 5.90834$ sec. therefore the depth is 561.4756 feet.

IX. QUESTION 1217, by Mr. Wm. Saint, R. M. A.

To find three square numbers in arithmetical progression.

Answered by Mr. John Surtees, Houghton-le-spring.

Let $(a-n)^2$, a^2+n^2 , and $(a+n)^2$ represent the three square numbers; then it only remains to make the second a square; which, from the property of a right-angled triangle, will be so, if $a = r^2-1$, and $n = 2r$; and then the three numbers will be $(r^2-2r-1)^2$, $(r^2+1)^2$, and $(r^2+2r-1)^2$.

If $r = 2$,	the numbers are	1, 25, 49,
$r = 3$,	ditto	4, 100, 196,
$r = 4$,	ditto	49, 289, 529,
&c.		

The same by Mr. S. Titlow, at Bungay School.

Let x^2 denote the first number, and $2rx + r^2$ the common difference. Then $x^2 + 2rx + r^2$ is the second number, a square, and $x^2 + 4rx + 2r^2$ is the third to be made a square, which put $= (x+s)^2$
 $= x^2 + 2sx + s^2$. Hence $x = \frac{s^2 - 2r^2}{4r - 2s}$. Where, if r be taken $= 2$, and $s = 3$, we have $x = \frac{1}{2}$ and the numbers are $\frac{1}{4}$, $\frac{25}{4}$, $\frac{49}{4}$, or, by rejecting the common denominator 1, 25, 49.

The same answered by Mr. James Wood, of Dudley.

Put x = the middle term, and $x + 1$ the common difference, then $x^2 - 2x - 1$ and $x^2 + 2x + 1$ are the two extremes; of which it remains only to make the first $x^2 - 2x - 1$ a square, which put $= (x-n)^2 = x^2 - 2nx + n^2$; hence $x = \frac{n^2 + 1}{2n - 2}$, where n may be any number greater than 1, make it $= 2$; then $x = \frac{5}{2}$ and the three numbers $\frac{1}{4}$, $\frac{25}{4}$, $\frac{49}{4}$; or, for whole numbers, rejecting the denominators, they are 1, 25, 49.

X. QUESTION 1218, by Mr. Wm. Saint, R. M. A.

To find the least possible integer square number that terminates with the greatest number of equal digits.

Answered by Mr. John Davey, of St. Just.

1st. Suppose the number to consist of three digits, and the last two equal. Then if r denote the first, and s each of the two last, we

have $100r + 10s + s =$ a square; which put $= (a+b)^2$; then by equating the several terms, $s = b^2$; and $10s = 10b^2 = 2ab$, and $5b = a$; also $a^2 = 25b^2 = 100r$; and $b^2 = 4r$. Now if r be taken $= 1$, we have $b^2 = s = 4$, and the number is 144, being the least square number, having two equal digits.

2dly. If the number be supposed to consist of 4 digits, the first denoted by r , and the other three by s ; we then have $1000r + 111s =$ a square; which I find to be the case also when $r = 1$, and $s = 4$; hence the number is 1444, being the least square integer number ending with three equal digits.

The same answered by Mr. Wm. Putsey, Master of the Pickering Academy.

By the extensive table of squares, cubes, &c. in the 1st vol. of Dr. Hutton's Course, the square of 38 is 1444, ending with three equal digits, being one more than any number in that table, which is, therefore, the number sought. And in like manner it appears that no square number whatever can terminate with more than three equal digits.

The same answered by Mr. John Ryley, of Leeds.

It does not appear that this question will admit of a general solution. It will, therefore, be the readiest method of finding an answer, to consult a table of roots and squares, such as that in Dr. Hutton's Dictionary; where it will be seen that 12 and 38 are the least numbers whose squares terminate with the greatest number of equal digits, viz. 144 and 1444.

The same answered by Mr. Wm. Saint, Woolwich.

The required square number must necessarily end in 1, 4, 5, 6, or 9; and if it could end in any two of these, as 2 ones, 2 fours, &c. it would be expressed in one of the five following forms, viz. $100z + 11$, or $100z + 44$, or $100z + 55$, or $100z + 66$, or $100z + 99$. Now every square number is known to be of the form $4n$, or $4n + 1$. But the 1st, 3d, and 5th, of the above forms are of the form $4n + 3$, and the 4th of the form $4n + 2$; so that none of these can be squares. But the 2d, being of the form $4n$, may be a square. The required number, therefore, must terminate with 4. It cannot, however, terminate with more than three 4's; for, if it could, it might be expressed by $z \cdot 10^4 + 4 \cdot 10^3 + 4 \cdot 10^2 + 4$; and, being a square number, it would also be such when divided by 4; that is, $25z \cdot 10^4 + 10^3 + 10^2 + 10 + 1$, or its equal $(25z + 10) \cdot 10^4 + 10^3 + 10 + 1$, or $(25z + 11) \cdot 10^4 + 10 + 1$, would be a square. Now if

is evident, that whatever value may be given to z , this expression would terminate with two 1's, which we have shown above to be impossible. We have now, therefore, only to find the least square number that ends in three 4's: and such a number must be of the form $(10n \pm 2)^2 = 100n^2 \pm 40n + 4$; which, when $n = 3$, gives 1444 for the number sought.

The same answered by Mr. Thomas Smith, of Hethersett.

By inspecting the tables of squares, &c. in Dr. Hutton's Course, vol. 1, p. 90, and in his Mathematical and Philosophical Dictionary, the least square integer terminating with the greatest number of equal digits, is easily found, and is $1444 = 38^2$.

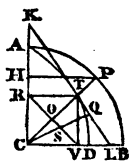
We are sorry that our limits will not allow us to insert the very ingenious and elaborate solution given to this question, by Mr. Henry Atkinson, of Newcastle; who, besides shewing that the number sought must be $1444 = 38^2$, proves also that no integer square number can end with more than three equal digits; and that, when any square number terminates with either two or three equal digits, these digits will always be 4's.

II. QUESTION 1219, by Mr. Wm. Watkins, Haydon Bridge.

A gentleman having a piece of ground in form of Dinostratus's quadratrix, the radius of the generating circle being ten chains, ordered his gardener to trench a rectangle in it of the greatest possible area; but the poor fellow being as ignorant of numbers as the man in the moon, humbly solicits your assistance to tell him the dimensions and area.

Answered by Mr. J. Cunliffe, jun.

Let CATD denote the quadratrix; CAB the generating circle; CRTV the greatest rectangle. Put $CA = CB = r$; $AR = x$; $RT = y$; $z =$ a circular arc, radius 1, tangent t , centre c , intercepted between CA and CT ; also q the quadrantal arc of the same circle, or to the rad. 1. Then the arc $AP = rz = qx$; hence $AR = x = rz \div q$, and $CR = r - rz \div q = (r \div q)(q - z)$. Again, $1 : t :: CR : RT = t$. $CR = (rt \div q)(q - z) = y$; hence the area $RTVC = CR \cdot RT = (r^2 t^2 \div q^2)(q - z)^2$, which is to be a maximum, therefore $t(q - z)^2$ will be a maximum. The flux-



ion of this made $= 0$ gives $t(q - z) = 2tz = \frac{2tt}{1+t^2}$, because $z =$

$\frac{t}{1+t^2}$; hence $q - z = \frac{2t}{1+t^2}$. So that the question is reduced to

finding such an arc of a circle, radius 1, that the length of its complement may be equal to double its tangent, divided by the square of its secant. By a few approximations by the method of Trial and Error, which may readily be performed by the help of Dr. Hutton's Mathematical Tables, it is found that z is an arc of $35^{\circ}.698088$, and therefore its length is $.62304917$; also $t = .7185223$; and $x = rz \div q = .3966454$ $r = AR$; hence $CR = .6033546$ $r = 6.033546$ chains, $RT = y = t. CR = 4.3352372$ chains; therefore the area $CR. RT = 26.15684$ square chains $= 2.615684$ acres $= 2$ acres, 2 roods, $18\frac{1}{2}$ poles, nearly.

The same by Mr. J. A. Dotchen, of Chelsea.

Put the abs. $cv = x$, ord. $tv = y$, and the base $cb = 1$. Then by the method at art. 218, Lyons' Fluxions, x is found $= 1 - \frac{1}{2}y' - \frac{1}{24}y'^2 - \frac{1}{960}y'^3 - \frac{1}{40320}y'^4 - \&c.$ and the rectangle $cv. tv$ or $xy = y - \frac{1}{2}y'^2 - \frac{1}{24}y'^3 - \frac{1}{960}y'^4 - \&c.$ which per question is to be a maximum; hence, taking the fluxions, $\&c. 1 - y' - \frac{1}{2}y'' - \frac{1}{80}y'^3 - \frac{1}{5120}y'^4 - \&c. = 0$; and by approx. y is found $= .947744$, and $x = (1 - \frac{1}{2}y' - \frac{1}{24}y'^2 - \&c.) = .680981$. Now, by the nature of the curve, the rad. cb is a mean proportional between the quad. ab and the base cb ; hence when $cb = 1$, as above supposed, $cb = 1.5708$; but since cb is given $= 10$, therefore we must increase the values of x and y , above found, in the ratio of 1.5708 to 10 , and then the sides of the required rectangle will be 4.33526 and 6.03352 , and its area 26.15689 square chains, or 2 acres, 2 roods, 18.5104 poles.

The same by Mr. J. H. Hearing, Pall Mall.

Draw the tangent tl ; and put ac or $cb = 10 = a$, the quadrant $ab = b$, the arc $ap = y$, $AR = x$, $CT = z$, the sine $PH = s$, and the cos. $CH = c$; then $CR = a - x$, $CD = ax \div b$, $CL = bzx \div aa$, and $x = ay \div b$; hence, by the known property of the greatest inscribed rectangle, $cv = vl = RT = bzx \div 2aa$. But PH , or s , is known to

$$be = y - \frac{y^3}{2.3a^2} + \frac{y^5}{2.3.4.5a^4} \&c. \text{ and } CH \text{ or } c = a - \frac{y^2}{2a} + \frac{y^4}{2.3.4a^3} - \frac{y^6}{2 \dots 6a^5} \&c.; \text{ then by similar triangles } CH : HP ::$$

$CR : RT$, or $c :: a - x : (s \div c) (a - x) = RT = vl = bzx \div 2aa$ as above; also $CH^2 : CP^2 :: CR^2 : CT^2$, that is $c^2 : a^2 :: (a - x)^2 : (a^2 \div c^2) (a - x)^2 = CT^2 = z^2$; consequently $bzx \div 2aa = RT = vl$ is $= (b \div 2c^2) (a - x)^2$; hence $(s \div c) (a - x) = (b \div 2c^2) (a - x)^2$, or, multiplying all by $c^2 \div (a - x)$, and writing $ay \div b$ instead of x , it gives $sc = (ab \div 2) (1 - y \div b)$, or $y = 6.230498$; consequently $x = 3.966459$ chains. Therefore the length CR of the rectangle is 6.043541 , its breadth $cv = 4.335239$, and its area 26.156842 square chains, or 2 acres, 2 roods, 18.51 poles.

The same, by Mr. D. T. Sheridan, Wolverhampton.

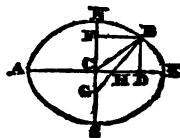
The lines in the foregoing figure being still employed, the tangent $TL = TK = TC$. Draw CQ perpendicular to the tangent KL . Then since CT bisects KL , and KL bisects CK , their intersection O is the centre of gravity of the triangle CKL ; consequently $CO = 2 OT$, and $CL = SR = SC$; therefore $\angle TCL = TLC$, and $\angle SCL = SLC$; therefore by subtraction $\angle OCQ = QLS$; but $\angle OSC = QSL$, therefore $\angle O = \angle Q$, a right angle. But $CT, \frac{1}{2} CT (OT) = \frac{1}{2} CT' = RT'$, and $CT, \frac{2}{3} CT (CO) = \frac{2}{3} CT' = CR'$, therefore $CR : RT :: \sqrt{\frac{2}{3}} : \sqrt{\frac{1}{3}} :: \sqrt{2} : 1 ::$ radius : tangent $\angle ACP = 35^\circ 16'$; and $90^\circ : \angle ACP :: AC : AR = 3.9185185$; then $CR = 6.08148148$, and $RT = 4.30025$, therefore the content required = 26.15189 chains = 2 acres, 2 roods, 18.43 poles.

XII. QUESTION 1220, by Mr. Wm. Wright, Ostringham.

To find in what latitude a degree of the meridian is equal in length to 2 degrees of longitude in the same latitude; the earth being considered as an oblate spheroid, the axes of which are as 229 to 230?

Answered by Mr. Henry Atkinson, Newcastle.

Let ANES represent the meridian, the poles of which are N and S; and the point B be the required latitude and BF the radius of its parallel. Put AE = a = 230, NS = c = 229, $b^2 = a^2 - c^2$, CD or BF = x , and BD or CF = y . Then, by Simpson's Fluxions, art. 71, the radius of curvature at the point B =
$$\frac{\sqrt{[a^2c^2 + b^2(a^2 - 4x^2)]^2}}{2a^2c}$$
. Now the length of a



degree of latitude is as the radius of curvature, and the length of a degree of longitude as the radius of its parallel of latitude, therof. $\frac{[a^2c^2 + b^2(a^2 - 4x^2)]^{\frac{3}{2}}}{2a^2c} = 2x$, or $[a^2c^2 + b^2(a^2 - 4x^2)]^3 = 16a^2c^2x^2$; this put into numbers gives $x = 57.56814$, and hence $y = 99.120707$. Again, by Emerson's Conics, prop. 15, or Hutton's Conics, prop. 15, $c^2 : b^2 :: y : CG = 867573$, the line CG being drawn perpendicular to the curve; so that $FG = 99.98828$. Now $FG : FB :: \text{radius} : \text{cotangent } FBG$ or $BHE = 60^\circ 4' 8''$ the latitude required.

The same by Mr. John Surtees, Houghton-le-Spring.

Let $a = 230, b = 229, m = 6.2832$ the circumference of a circle

to radius 1, $c^2 = a^2 - b^2$, $s = \text{sine}$, and $z = \text{cosine}$ of latitude to radius 1. Then, by cor. 3, prob. 1, p. 206, vol. 2, Robertson's Navig.

the length of a degree of the meridian is $= \frac{amb^3}{360(a^2 - c^2s^2) \times \sqrt{(a^2 - c^2s^2)}}$;

and by prob. 2, ib. $\frac{2ma^2 \sqrt{(1 - s^2)}}{360 \sqrt{(a^2 - c^2s^2)}}$ is the length of 2 degrees of the corresponding parallel of latitude; these two equated and reduced, give $2a(a^2 - c^2s^2) \times \sqrt{(1 - s^2)} = b^3$, or $2ac^2z^3 + 2ab^2z = b^3$: hence $z = .496752$ nearly, $= \text{cosine of } 60^\circ 13'$, the latitude required.

XIII. QUESTION 1221, by Mr. Wm. Marrat, Boston.

A cone of marble, whose length is 16, and base diameter 6 feet, stands inverted on its vertex, its axis making an angle of 75° with the plane of the horizon; what must be the direction and quantity of the least power, applied to the centre of its base that will just sustain the cone in that position?

Answered by Mr. D. Anderson, R. M. A.

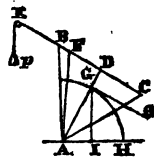
Let ADE be the cone, and DE its axis; also take $nc = \frac{1}{4}BD$, then will c be the centre of gravity of the cone; let DM be parallel to the horizon, and CM perpendicular to it. Now there are given $BD = 16$, $BE = 3$, and $\angle BDM = 75^\circ$; hence $AE^2 \times \frac{1}{4}BD \times .7854 = 150.7968$ the content of the cone; and 1 foot : 150.7968 :: 2700 ounces : 25447lb. the weight of the same, by prop. 61, Dr. Hutton's 2d. vol. This weight may be conceived as acting at c, in direction cm ; and the line BD as a lever moveable about the point D; then if a force be applied at B to support the cone, BD will be a lever of the 2d kind, and in the direction EA the necessary sustaining force will be less than in any other direction: hence the weight of the cone at c acts by the lever DM , while the required sustaining force acts at B by the lever BD ; therefore, in case of an equilibrium, $BD : DM :: \text{weight } 25447\text{lb.} : w$ the weight required, by Dr. Hutton's prop. 31, vol. 2. Now in the right-angled triangle CDM , it is radius : CD (12) :: $\text{sine } \angle c$ (15°) : $DM = 3.105828$; so then the former propor. is $16 : 3.105828 :: 25447 : 4939\frac{1}{4}\text{lb.} = 44\frac{1}{4}\text{cwt.}$ the force required.



The same by Mr. Wm. Marrat, Boston.

Let ABC be a section of the cone, and G its centre of gravity. It

is evident that, as the cone revolves about A, the centre of gravity G will describe the arc CH; it is also obvious that the least sustaining power will be parallel to the tangent GO, or perpendicular to AD. Let AF, perpendicular to AH, denote the weight of the cone; then, by the resolution of forces, FD will denote the pressure at A, in the direction GO parallel to FD; hence, by similar triangles the weight w of the cone : f the force at G in the direction GO :: FA : FD :: radius AG : AI = s the sine of $\angle AGI$; hence $f = sw \div \text{radius} = 227.2044 \times 2.58819 = 588.048 \text{ cwt.} = \text{force at G} = \text{suppose } w$. Again, by the property of the lever, $w : p :: AD : AG :: 16 : 12$ or $4 : 3$, therefore $p = \frac{3}{4}w = 441.036 \text{ cwt.} = \text{the power at D in the direction DE to keep the cone in the required position.}$

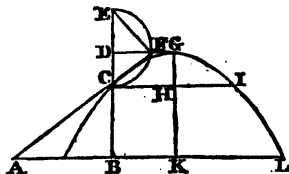


XIV. QUESTION 1222, *by Mr. Wm. Moore, R. M. A.*

Given the base of an inclined plane, to find its inclination so that a body projected directly up it, with a given velocity, shall, after passing the top of it, fall at the greatest possible distance beyond the base in the same horizontal line?

Answered by Mr. Edward Riddle, Whitburn.

Put $a = AB$ the given base, $x = BC$ the height of the plane AC , $b = BE$ the altitude due to the given velocity. Now the velocity at c , where the body leaves the plane AC , and begins to describe the parabola $cghl$, will be the same, whether the body be projected up AC or BC . Hence ce , the altitude due to the velocity at c , is $b - x = y$. On CE describe a semicircle, and draw the lines as in the figure.



Then by similar triangles $AC = \sqrt{(a^2 + x^2)} = z : BC = x :: CE = y : CF = xy \div z :: CF : CD = GH = x^2 y \div z^2$. And $AC : AB :: 2CF : 2DF :: DG = CH = HI = 2axy \div z^2$. Again, by the parabola, $\sqrt{GH} : \sqrt{GK} = \sqrt{(DC + CB)} :: HI : KL$, or $\sqrt{GH} : \sqrt{GH} + \sqrt{GK} :: HI : HI +$

$$KL = BL = \frac{z}{x\sqrt{y}} \times \left(\frac{x\sqrt{y}}{z} + \frac{\sqrt{(x^2y + xz^2)}}{z} \right) \times \frac{2axy}{z^3} =$$

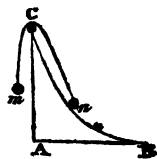
$$\frac{xy + \sqrt{(x^2y^2 + xyz^2)}}{z^2} \times 2a = (\text{by restoring the values of } y \text{ and } z)$$

$$\frac{x - x^2 + \sqrt{(a^2bx + (b^2 - a^2)x^2 - bx^3)}}{a^2 + x^2} \times 2a, \text{ a max. Its flux. put =}$$

0, and reduced, gives $bx^2 - 2a^2x - ba^2 + \frac{a^2b + (b^2 - a^2) \cdot 2x - 3bx^2}{2\sqrt{(a^2bx + (b^2 - a^2)x^2 - bx^3)}} \times (a^2 + x^2) - 2x\sqrt{(a^2bx + (b^2 - a^2)x^2 - bx^3)} = 0$. Hence, if $a = 3$, and $b = 10$, then $x = 2.23186$. And $AB : BC :: \text{radius} : \tan. A = 36^\circ 38' 51''$, the angle of the plane's inclination.

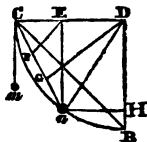
PRIZE QUESTION, by Mr. B. R.

bnc is a quadrantal arc, and AB , AC two equal lines of 10 feet each, parallel and perpendicular to the horizon; m and n are two weights, of 2lb. and 1lb. connected by a thread passing freely over the point c ; required the time in which the less weight n will be drawn up, from b to c , along the arc bc , by the greater weight m descending perpendicularly; and what will the last velocity be?



Answered by Mr. B. R. the Proposer.

Let cb be the first position of the string, or when the body m is at c , and n at b ; also let ncm be any other position of the thread and the two bodies. Join bn , and draw ne perpendicular to cd , dg perpendicular to cn , and ef parallel to dn .



Put $r = CB = BD = 10$. $BC = r\sqrt{2}$, $cn = x$, $v = \text{velocity of } m \text{ in direction } cm$, or of n in nc , which is the same, t the time in motion, $g = 32\frac{1}{2}$ ft. the force of gravity, $u = \text{the velocity of } n \text{ in the curve at } n$, and $y = BH$ the height ascended by n , while m descends the space cm ; then, by a well known formula, $mv\dot{v} + nu\dot{u} = gm\dot{x} - gn\dot{y}$, the fluents of which give $mv^2 + nu^2 = 2g(mx - ny) = \text{by correct. } 2g\sqrt{(m \times cm - ny)}$. But $n : v :: dn : dg :: r : \frac{1}{2}\sqrt{(4r^2 - x^2)}$,

therefore $u = \frac{2rv}{\sqrt{(4r^2 - x^2)}}$, and $cm = CB - cn = r\sqrt{2} - x$,

also BH or $y = DB - En = r - \frac{x}{2r}\sqrt{(4r^2 - x^2)}$; therefore $mv^2 +$

$\frac{4nr^2v^2}{4r^2 - x^2} = 2g[m(r\sqrt{2} - x) - nr + \frac{nx}{2r}\sqrt{(4r^2 - x^2)}]$; hence $v =$

$\sqrt{[2g(4r^2 - x^2) \times \frac{m(r\sqrt{2} - x) - nr + \frac{nx}{2r}\sqrt{(4r^2 - x^2)}}{4nr^2 + m(4r^2 - x^2)}]}$ the general expression for the velocity of the descending body m , which

when n arrives at c , or $x = 0$, or $cm = r\sqrt{2}$, becomes $\sqrt{(2gr \times \frac{m\sqrt{2}-n}{m+n})}$, the final velocity required.

$$\text{Then } i = \frac{-\dot{x}}{v} = -\dot{x} \times$$

$$\sqrt{\frac{4nr^2 + m(4r^2 - x^2)}{2g(4r^2 - x^2) \times [m(r\sqrt{2} - x) - nr + \frac{nx}{2r}\sqrt{(4r^2 - x^2)}]}}$$

the fluent of which will give the time as required.



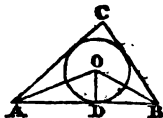
Questions proposed in 1811, and answered in 1812.

I. QUESTION 1224, by Mr. D. T. Sheridan, *Wolverhampton*.

Given the vertical angle, the radius of the inscribed circle, and the sum of the lines drawn from its centre to the angles at the base, to construct the plane triangle.

Answered by Mr. D. T. Sheridan, the Proposer.

Analysis. Suppose the thing done; and that ABC is the required triangle, o the centre of the inscribed circle, and od its radius. Then, because the $\angle c$ is given, the $\angle AOB$ is also given $= \angle c +$ half its supplement. Therefore in the triangle ABO , are given the perpendicular od , the vertical $\angle AOB$, and the sum of the sides, to construct it: which is problem 80, Appendix to Simpson's Algebra.



Mr. Edward Riddle, after his solution, referring to Mr. Simpson's problem, adds, "I take this opportunity to observe, that the method of solving this problem numerically, as given by Simpson, is incorrect; for tangent $\frac{1}{2}q$, ought to be substituted $\text{tang. } \frac{1}{2} \text{ comp. } q$."

II. QUESTION 1225, by Mr. A. Nesbit, *Farnley, Leeds*.

The triangular field ABC has its base $AC = 50$ chains, the vertical angle at $B = 100^\circ 17'$, and the ratio of the sides AB to BC as 7 to 6;

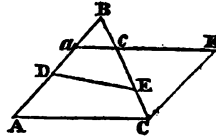
now it is required to divide the triangle equally between two persons, so that the division fence, passing from the side AB to the side BC , may be the shortest possible?

Answered.

To this question have been received solutions in a great variety of ways; of which, the following specimens are as many as room can be found for.

1. *By Mr. Rd. Forster, Newcastle-upon-Tyne.*

ABC being the proposed triangle, and DE the dividing line: then by plane trigonometry the sides AB and BC are found $= 35$ and 30 ; thence the area $\frac{1}{2}AB \times BC \times \sin. \angle B = 516.566925$. Now (by the Max. and Min. Simpson's Geom. or by p. 171, Hutton's Course, vol. 3) the line DE will be shortest when $BD = BE$; then $\frac{1}{2}BD^2 \times \sin. B = 258.2834625$ the area of BDE ; hence BD or BE is found $= \sqrt{525} = 22.9128785$; and $\sin. D : BE :: \sin. B : DE = 35.1773$.



2. *By Mr. Joseph Hine, Plymouth.*

Upon BA , BC containing the given $\angle B = 100^\circ 17'$, take $ba = 7$, $bc = 6$; join ac , and produce it till $acr = 50$; draw rc parallel to BA , and ac to ac .

Calcul. The angles A , c , equal to a , c , are easily found by trigonometry also $BA = 35$, and $BC = 30$; hence the area of $BAC = 516.5685$; and because the shortest line DE cuts off an isosceles triangle BDE , it is $516.5685 \div \sin. B = BD \times BE = BD^2$, or BE^2 , then $BD = BE = 22.913$; hence $DE = 35.2$ nearly.

3. *By Mr. Edw. Riddle, Whitburn.*

It is easily found that the side $AB = 35$, and $BC = 30$; and it is very well known that the line DE , which cuts off any given part from the triangle ABC , is a minimum when $BD = BE$. Hence BD or $BE = \sqrt{\frac{1}{2}AB \times BC} = \sqrt{525} = 22.9128785$.

4. *By Mr. John Smith, Alton Park.*

Assume $ba = 7$, and $bc = 6$: then, the angle B being given $= 100^\circ 17'$, ac will be found $= 10$ nearly. But AC is given $= 50$; therefore $10 : 50 :: 7 : 35 = AB :: 6 : 30 = BC$. Now the triangle ABC being double the triangle BDE , and the angle B common, there-

fore $2 : 1 :: AB \times BC : DE \times BE$. But DE being the shortest possible, DB and BE are equal; hence BD or $BE = \sqrt{\frac{1}{2} AB \times BC} = \sqrt{525} = 5\sqrt{21} = 22.9129$.

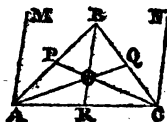
III. QUESTION 1226, by Mr. Thomas Smith, Hethersett.

In the triangle ABC given the three sides, viz. $AB = 6$, $BC = 5$, $AC = 7$; now if this triangle be divided into three equal parts, by the straight lines AO , BO , CO . Required the length of these three lines?

Answered by Mr. J. A. Dotchen, of Chelsea.

Produce AO , BO , CO , to meet the three sides of the triangle in P , R , and draw AM , EN parallel to BO .

Since the triangles BOA , BOC are equal by the suppos. and stand on the same base BO , they must be between equidistant parallels, that is, $AR = EC$; therefore trian. $AOR = \frac{1}{2} AOC = \frac{1}{2} AOB$, and $OR = \frac{1}{2} OB = \frac{1}{3} BR$. In like manner $OQ = \frac{1}{3} AQ$, and $OP = \frac{1}{3} CP$. Then, by theorem 38, Dr. Hutton's Geomet. $2BR^2 + 2AR^2 = AB^2 + BC^2$; hence $BR = \sqrt{(\frac{1}{2} AB^2 + \frac{1}{2} BC^2 - AR^2)} = \sqrt{(18 + 12\frac{1}{2} - 12\frac{1}{2})} = \frac{2}{3}\sqrt{73}$; and $BO = \frac{2}{3} BR = \frac{2}{3}\sqrt{73} = 2.848001$. In like manner $AO = \frac{2}{3}\sqrt{145} = 4.013865$, and $CO = \frac{2}{3}\sqrt{112} = \frac{2}{3}\sqrt{7} = 3.527668$.



Rem. From what has been above proved, it appears that o is the centre of gravity of the triangle ABC .

The same by Mr. Wm. Dunn, Broughton, Lincolnshire.

With the given sides make the triangle ABC ; then it is well known that the dividing lines AO , BO , CO , will meet in the centre of gravity o of the triangle; which continued will bisect the sides of the triangle. Then, by Hutton's Course, vol. 1, theor. 38, $BR = \sqrt{\frac{1}{2}(BA^2 + BC^2 - 2AR^2)} = \sqrt{\frac{1}{2}(36 + 25 - 24\frac{1}{2})} = 4.272$, two-thirds of which is $2.848 = BO$. In like manner $\frac{2}{3}\sqrt{\frac{1}{2}(49 + 25 - 18)} = 3.527668 = CO$, and $\frac{2}{3}\sqrt{\frac{1}{2}(49 + 36 - 12\frac{1}{2})} = 4.01386 = AO$.

IV. QUESTION 1227, by Mr. S. Titlow, Bungay.

To find two cube numbers, whose difference shall be a square number.

Answered by Mr. John Collins, Hatton Garden.

Put x and $x + 1$ for the required numbers: then the difference of their cubes will be $3x^2 + 3x + 1$, which must be a square number; let

$rx - 1 =$ its root; then $3x^3 + 3x + 1 = r^3x^3 - 2rx + 1$, or $3x^3 + 3x = r^3x^3 - 2rx$, which gives $x = \frac{3 + 2r}{r^3 - 3}$, where r may be any number greater than 1. If $r = 2$, then $x = 7$, and $x + 1 = 8$, and the cubes will be 343 and 512, the least that can be in whole numbers.

The same by Mr. S. Parlour, East India Co's Institution.

Let $x^3 + 3x^2 + 3x + 1$ and x^3 be the two cubes: then $3x^2 + 3x + 1$ is to be a square. Let $2ax - 1$ be assumed for its side; then $3x^2 + 3x + 1 = 4a^2x^2 - 4ax + 1$; hence $x = (4a + 3) \div (4a^2 - 3)$; and by taking $a = 1$, we have $x = 7$, and the two cubes will be 343 and 512.

Mr. Henry Atkinson and Mr. John Bruce, after a general analytical solution, add, "By inspecting the extensive table of squares and cubes in Dr. Hutton's Mathematical Dictionary, it will be found that the former two numbers are the least whole numbers that will answer the conditions of the question. It will also be found that 14^3 and 7^3 will answer." And some other gentlemen refer to the same table of squares and cubes, in the first vol. of the Course, for the same purpose.

V. QUESTION 1228, by Mr. Thomas Ford, of Beighton.

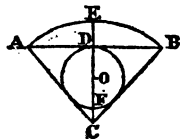
Given the radius of a sector, CA , or $CB = 10$, to determine the chord AB such, that the area of a circle inscribed in the plane triangle ABC , shall be equal to the area of the segment, whose arc is subtended by the said chord AB .

Answered by Mr. John Bruce, Newcastle.

Let AC or $BC = a$, $AB = x$. Then $DE = a - \sqrt{(a^2 - \frac{1}{4}x^2)}$ the versed sine of the half arc. Now putting $n =$ tabular number answering to the versed sine divided by the diameter, then $400n =$ area of the segment. Again, by prob. 24, Simp. Select Exercises,

$$OD^2 = \frac{a - \frac{1}{2}x}{a + \frac{1}{2}x} \times \frac{1}{4}x^2, \text{ hence } \frac{a - \frac{1}{2}x}{a + \frac{1}{2}x} \times x^2 \times .785398$$

will be the area of the circle; which being put $= 400n$ the segment, and applying Dr. Hutton's method of approximating by double position, x is found $= 13.95345$.



The same by Mr. John Craggs, Hilton, near Sunderland.

After equating the analytical expression for the area of the in-

scribed circle, and that of the segment derived from approximation in the mensuration, Mr. Craggs then adds, "The area will be found by Dr. Hutton's large table of Segments, and Trial and Error. Thus,

If the versed sine $v = 2.83$, then circle's area $= 27.2409$
 and the segment's area $\dots\dots = 27.1504$
 their diff. (circle too much) $= 0.0905$

Again, if $v = 2.835$, then the circle $\dots\dots = 27.2343$
 and the segment's area $\dots\dots = 27.2204$
 their diff. (circle too much) $= 0.0139$

Then $.0766 : .005 :: .0139 : .0009$; hence $v = 2.8359$ or 2.836 , and the chord $AB = 13.954$."

The same by Mr. John Davey, teacher of mathematics at St. Just, in Penwith, Cornwall.

In the sector $AEBCA$, let $z =$ the length of the arc AEB to radius 1, and $n = .7854$; then is $\frac{1}{2}z - AD \times CD =$ the area of the segment AEB ; and because $OD = \frac{AD \times CD}{AD + AC}$, we have $\frac{AD^2 \times CD^2}{(AD + AC)^2} \times 4n =$ the

circle's area, therefore $\frac{1}{2}z - AD \times CD = \frac{AD^2 \times CD^2}{(AD + AC)^2} \times 4n$. Now,

by means of Dr. Hutton's Tables of Natural Sines, &c. and his method of double position, I find $z \doteq 15.4429032$ to the radius 10, and the sine $AD = 6.976724$; hence $AB = 2AD = 13.953448$.

The same by Mr. J. A. Dotchen, of Chelsea.

Put $x = AD =$ half the chord AB , $z =$ arc AE or EB , $a =$ the radius AC or CB , $r =$ inscribed radius OD or OF , and $n = 3.14159$, &c. Then $CD = \sqrt{(a^2 - x^2)}$; also $a + x = \frac{1}{2}$ perimeter of the triangle ABC , and $r(a + x) =$ its area; but the same area is also $= AD \times CD = x\sqrt{(a^2 - x^2)}$; therefore $r(a + x) = x\sqrt{(a^2 - x^2)}$, hence $r = x\sqrt{((a - x) \div (a + x))}$ consequently $nr^2 = nx^2 \times (a - x) \div (a + x) =$ area of the circle DF ; this, per question is $=$ the segment AEB , $=$ sector AEB $-$ triangle $ABC = az - x\sqrt{(a^2 - x^2)}$; or taking at present $a = 1$, our equation will be $z - x\sqrt{(1 - x^2)} = nx^2 \cdot (1 - x) \div (1 + x)$, that is $z - \text{sine } z \cdot \text{cosine } z = n \cdot \text{sine }^2 z \cdot (1 - \text{sine } z) \div (1 + \text{sine } z)$. The value of z in this equation may now be easily determined by the method of trial and error, assisted by the table of the lengths of circular arcs given in Dr. Hutton's Mathematical Tables, and the table of sines. Hence, by a few approximations, z is found to be an arc of $44^\circ 14' 21''\frac{1}{2}$, and its sine x , to rad. 1, is $.6976570$; consequently the whole AB to rad. 10, is 13.95314 chains, the length required.

VI. QUESTION 1229, by Mr. Thomas Kirton, Cranford.

If the length of a piece of round timber be 60 feet, the girt at the greater end 12 feet, and that at the less end 2 feet; required the true content, and the value at 4 shillings per foot, of the piece that may be cut off at the less end, so that the remaining part may measure exactly to the same quantity as the whole did before it was cut, supposing the whole and the said remaining part to be measured by the common or quarter-girt method of measuring?

Answered by Mr. Henry Atkinson, Newcastle-upon-Tyne.

There is a general and very elegant solution to questions of this kind given in Dr. Hutton's Mensuration, near the end. Now, if we here employ the same notation as is there used, we shall have $s = 14$, $d = 10$, $L = 60$, and $z =$ the length required, which by the place referred to is $(L \div 2d) \times [\sqrt{((4s + d)d) - 2s + d}] = 23.071395$ feet, and the circumference at the section $= \frac{1}{2}\sqrt{((4s + d)d) - \frac{1}{2}s} = 5.8452326$; whence is obtained the true solidity of the part cut off $= 30.512$ feet; which at four shillings per foot is worth 6l. 2s. $0\frac{1}{2}d$.

In the same manner is the solution given by all the contributors, several of whom make remarks on the erroneousness of the quarter-girt method of measuring, and in particular Mr. Chas. Holt remarks, that, if a person had 100 such trees, he may cut off in all 3050 feet, which at 4s. each, amounts to £610.

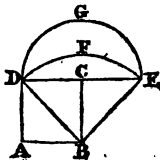
VII. QUESTION 1230, by Mr. John Baines, jun. Horbury-bridge.

If ABCD be a square, and circles be described from the points B and C, with the radii BD and CD, the area of the lune DGEF will be equal to the area of the square ABCD. Required the demonstration?

Answered by Mr. Wm. Burdon, Acaster Malbis.

Join BD, BE; then it is proved at the 62d geometrical problem in Dr. Hutton's Recreations, that the lune DGEF is equal the triangle BDE, or = the square ABCD, since the triangle BDC = BCE.

N. B. Several other properties of lunes may be seen in the same work.



The same by Mr. Jere. Craven, Bamburg Castle.

This lune is called the lune of Hippocrates; and it is shown at p. 58, Dr. Hutton's Dictionary, that the quadrant $BDFE$ is = the semicircle $DGEC$; then, taking away the segment $DFEC$ from each, there remains the lune $DGEF$ = the triangle BDE = the square AC .

The same by Mr. Wm. Gregory, Islington.

$DB^2 = 2BC^2$, therefore the quadrant $BDFE$ = the semicircle $DGEC$; take away the common segment $DFEC$, and there remains the lune $DGEF$ = triangle DEB = the square $DCBA$.

The same by Mr. Jos. Jones Habershon, Rotherham.

The right-angled triangles BAD , BCD , BCE , are equal in all respects; therefore the triangle BDE = the square AC , and the angle DBE = ABC a right angle; consequently the arc DFE is a quadrant. Therefore, by Hutton's Mensuration, problem 12, pt. 2, sect. 1, the lune $DGEF$ = the triangle DBE , and therefore = the square AC .

VIII. QUESTION 1231, by Mr. Joseph Walker, near Cheadle,

To find two square numbers, consisting of the same digits, but in an inverted order, and having their roots, consisting of two digits, x and y , in an inverted order; also that the sum of the digits in each square, shall be the square of the sum of the digits in each root; and lastly, that the sum of the digits in each root shall be the square of their difference.

Answered by Mr. W. Putsey, Pickering Academy.

Put $10x + n$ and $10n + x$ for the roots of the two squares: then $x + n$ will be the sum of the digits in each root, and the sum of the digits in each square $x^2 + 2nx + n^2$, which is $= (x+n)^2$, as per question. Again, $x + n = (x - n)^2$ by the question, or by reduction $x = \frac{1}{2}\sqrt{(8n + 1)} + n + \frac{1}{2}$. Here, if $n = 1$, then $x = 3$; and the numbers are 961 and 169.

The same, by Mr. Thomas Smith, of Hethersett.

Since, by the last condition of the question, $x + y = (x - y)^2$, $x + y$ must necessarily be a rational square integer, and $x - y$ its root: but it can be no other than 4, and 2 its root, to answer the

other conditions. Hence $x = 3$, and $y = 1$, and the numbers are 31 and 13, the squares of which are 961 and 169, also $9 + 6 + 1 = (3 + 1)^2$; so that all the conditions are answered,

The same by Miss Mary Weston, Gainsborough.

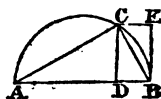
As the roots of the required numbers consist of two digits, their squares, or the numbers themselves, cannot exceed four digits. Now, as the sum of any four digits will not exceed 30, of which the nearest square root is 5, it is plain that the sum of the two digits cannot exceed that number. But, by the question, their sum is to be a square number: now 1 and 3 are the only two numbers under 5 whose sum will make a square number. Therefore they are the required digits, and 13 and 31 the roots of the required numbers, which are 169 and 961, answering the conditions of the question.

IX. QUESTION 1232, by Mr. John Ryley, of Leeds.

In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle, upon the hypotenuse; to construct the triangle.

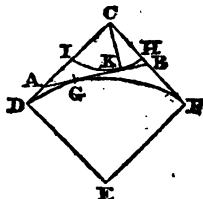
Answered by Mr. Thomas Ford, of Beighton.

It is well known, that, in every right-angled triangle, as the sum of the perimeter and perpendicular is to the perimeter so is half the perimeter to the hypotenuse or base; the base therefore is given. Hence this easy construction: On the given base AB describe the semicircle ACB , and at B erect the perpendicular $BE =$ to the given one; from E , and parallel to AB , draw EC to cut the semicircle in C . Join AC and BC , then will ACB be the triangle required. Which is too obvious to need demonstration.



The same by the Rev. George Harker, of Sebergham.

Const. Take $cd =$ half the perimeter, on which describe a square $cdef$; on centre E , with ED or EF as radius, describe the arc dgr ; also on c as a centre, and radius $=$ the given perpendicular describe an arc ikh ; draw AB to touch the two circles, and ABC is the required triangle.



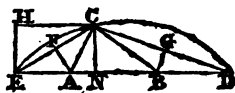
Dem. The angle c , and the perpendicular cx (this being drawn to the point of contact) are of the given magnitudes by construction,

and because $CD + CF =$ the given perimeter; and that, by the property of the tangents, $AG + GB (= AB) = AD + BF$; therefore $AB + AC + BC =$ the given perimeter.

Nor will the mode of constructing be at all different whatever be the vertical angle: it is only, instead of describing the square $CDEF$, drawing DE , EF perpendicular to CD , CF , and on E the point of intersection as a centre, &c.

The same, by Mr. Richard Taylor, Schoolmaster, Carlisle.

Const. On $ED =$ the given perimeter constitute a segment of a circle to contain an angle of 135° ; take EH perpendicular to ED , and $=$ the given perpendicular, parallel to ED draw HC to the circle; then join CE , CD , which bisect by the perps. FA , GB , so shall ABC be the triangle.



Dem. By construction $CA = AE$, and $CB = BD$, therefore $CA + CB + BA = ED$. Again, $\angle E = \angle ECA$, and $\angle D = \angle DCB$, therefore $\angle ECA + \angle DCB = \frac{1}{2}$ a right angle, consequently $\angle ACB =$ a right angle. Also the perpendicular $CN = HE$.

The same, by Mr. W. Truman, Schoolmaster, Exeter.

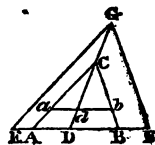
Make $ED =$ the given perimeter, on which describe the segment ECD to contain an angle $= 135^\circ =$ a right angle and a half, and let CN be the given perpendicular. Also join CD , CE , and draw CA , CB , making the $\angle ACE = \angle E$, and $\angle BCD = \angle D$, and it is done. The demonstration as above.

X. QUESTION 1233, by Mr. Wm. Cole, Colchester.

Given the angles of a plane triangle, with the length of a line drawn from the vertical angle, to cut the base in a given ratio; to construct the triangle.

Answered by Mr. Charles Holt, Heath-Charnock.

On any line, take DE , DF in the given ratio of the segments; draw EG and FG making E and F the given angles at the base. Join DG , in which take $PC =$ the given line; then draw Ac and Bc parallel to EG and FG , and ABC will be the triangle required. Which is too evident to need demonstration.



The same, by Mr. John Johnson, Birmingham.

Construct any triangle abc , having the given angles; and divide ab at d in the given ratio of the segments; on cd produced take $cn =$ the given line, then draw an parallel to adb , and $\triangle acn$ will evidently be the required triangle.

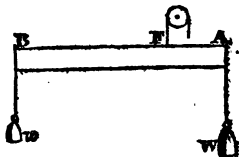
XI. QUESTION 1234, *by Mr. Wm. Burdon, Acaster Malbis.*

A cylindrical piece of oak timber, 6 feet in length, being suspended at the distance of 61 inches from one of the ends, required a weight of $5\frac{1}{2}$ cwt. to be attached to the other end to keep it parallel to the horizon: what is the diameter of its end?

Answered by Mr. William Burdon, the Proposer.

Let AB be the lever, of which F is the fulcrum, and W, w two weights fixed to the ends keeping it in equilibrio. Put $BF = 61$ inches $= a$, $FA = 11$ inches $= b$, $W = 9856$ oz. and $x =$ the weight of an inch in length of the cylinder. Then by mechanics $wa + \frac{1}{2}xa^2 = Wb + \frac{1}{2}xb^2$, or, since $w = 0$, $\frac{1}{2}xa^2 = Wb + \frac{1}{2}xb^2$, and $x = 2Wb \div (a^2 - b^2) = 60.2311$ oz. or 112.5182 cubic inches, in one inch length of the timber:

consequently $\sqrt{\frac{112.5182}{.7854}} = 11.969$ inches the diameter of the cylinder



The same, by Mr. John Ryley, Leeds.

As the distance between the point of suspension and middle of the beam is given $= 25$ inches, and from the same point to where the weight is attached $= 11$ inches; therefore if $w =$ the weight of the cylinder, by mechanics it is $25w = 11 \times 112 \times 5\frac{1}{2}$, hence $w = 11 \times 112 \times 5\frac{1}{2} \div 25 = 271.04$ lbs. Now, by Dr. Hutton's Tables of Specific Gravity $925 : 1728 :: 271.04 \times 16 : 8101.3123$ cubic inches contained in the cylinder. Therefore if $x =$ its diameter $l =$ its length, and $c = .7854$, the content will be expressed by $clx^2 = 8101.3123$; hence $x = \sqrt{(8101.3123 \div cl)} = 11.96922$ inches, the diameter required.

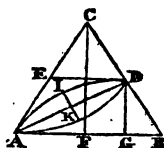
XII. QUESTION 1235, *by Mr. Wm. Burdon.*

A right cone being cut into two parts, by a plane passing through the extremity of the base diameter, has the transverse axe of the

elliptic section 6, and its conjugate 5; also the solidity of the part next the vertex, is to that of the lower part, as 2 to 1. Required the dimensions of the cone?

Answered by Mr. Edward Riddle, of Whitburn.

Let $AB = D$, then $ED = 25 \div D = d$; and by Dr. Hutton's Mensuration, cor. 4, pa. 173, 1st edition, and the data in the question it is $D^3 : d^3 :: 9 : 4$, hence $D = 5.72357076$, and $ED = d = 4.3679026$, then $AG = \frac{1}{2} AB + \frac{1}{2} D = 5.0457367$, $GB = .67783406$, and $GD = \sqrt{(AD^2 - AG^2)} = 3.2466199$; lastly, $GB : GD :: FB : \frac{1}{2} AB : CF = 13.7070854$.



The same, by Mr. John Smith, Alton Park.

Let ABC be the cone, and $AIDK$ the elliptic section. Put $AB = x$, then (by Dr. Hutton's Mensuration, pa. 170, 3d edition) $\sqrt[3]{3^3} : \sqrt[3]{2^3} :: x : \frac{1}{2} x \sqrt[3]{12} = .7631427 x = DE$. By conics, $AB \times DE = IK^2$, that is, $x \times .7631427 x = 5^2$; hence $x = \sqrt{(25 \div .7631427)} = 5.72357155 = AB$ the base diameter, and $.7631427 x = 4.367902 = DE$. Then $EG = \frac{1}{2} (AB - DE) = .6778347$, and $AG = AB - EG = 5.045737$, consequently $DG = \sqrt{(AD^2 - AG^2)} = 3.24662$; therefore by similar triangles $GB : GD :: FB (\frac{1}{2} AB) : FC = 13.707073$ the altitude.

XIII. QUESTION 1236, *by Mr. Jos. Hine, Plymouth.*

There is a hemispherical bowl, of 16 inches diameter, into which is thrown a globe of 3 inches diameter, which vibrates in it in such sort, that a great circle of the globe always coincides with the same great circle of the bowl. Required the length of the path that the highest point in the globe describes in one revolution?

Answered by Mr. J. Cunliffe, jun. Bolton-le-Moors.

If the globe be supposed to roll, without sliding in its motion in the bowl, the point on its surface will describe a curve called an epicycloid, as appears from the generation of that curve. Let b denote the radius of the bowl, and a the radius of the revolving globe. Then the whole length of the curve described in a complete revolution by a point on the revolving great circle of the globe, will be expressed by $(b - a) \times 8a \div b$. (See Dr. Hutton's Mathematical

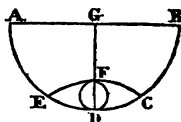
Dictionary, Epicycl. inferior.) When $b = 8$, and $a = \frac{1}{2}$, as in this case, that expression gives $9\frac{1}{4}$ inches, for the required length of the curve.

The same, by the Rev. J. Furnass, of Ponteland.

It is plain that the path of the point in question is the inner epicycloid ; and double the versed sine of the half arc in the rule given in Dr. Hutton's Dictionary, vol. i. p. 433, being equal to the diameter 3 of the globe, that rule becomes, as $8 : 16 - 8 :: 6 : 9\frac{1}{4}$ the length of the path.

The same, by Mr. A. Nesbit, Farnley, near Leeds.

Let ADB be a section of the hemisphere, and DF of the globe. Then, by the question $BA = 16$, and $DF = 3$. Now, by Newton's Princip. vol. 1, pr. 49, or Dr. Hutton's Dictionary, the curve EFC described by the point F , is that of an epicycloid ; hence, as $DG : 2DG - DF :: 2DF : EFC$; that is as $8 : 13 :: 6 : 9\frac{1}{4}$ the length of the whole curve.



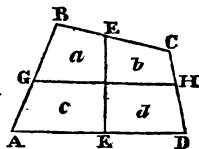
XIV. QUESTION 1237, by Mr. R. Rogers.

In Dr. Hutton's translation of Montucla's Mathematical Recreations, vol. 1, p. 422, prob. 19, is the following problem, the solution of which by the learned contributors to the Diary would be obliging : viz. "To cut a given trapezium into four equal parts, by lines intersecting each other at right angles."

Answered.

By prob. 67, Simpson's Algebra, or by prob. 4, p. 168, vol. 3, of the Acad. Course, just published, divide the given quadrilateral $ABCD$ into two equal parts by the right line EF ; and again, by the same, into two equal parts by the line GH .

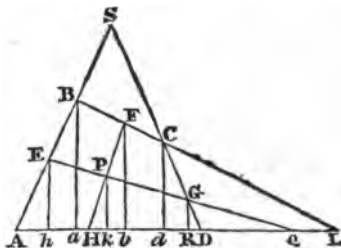
In this way the construction was given geometrically by Messrs. Craggs, Dewhurst, Furnass, Rutherford, Scurr, and Surtees ; and algebraically by several other gentlemen. But it will soon appear that this method is not sufficient : for though the two parts $a + b$ and $c + d$ be equal, and also the two parts $a + c$ and $b + d$, yet the four parts a, b, c, d , are not necessarily equal : for though $a + b = a + c$, each of these being the half of the quadrilateral ; then taking away the part a from both sides, there remains



$b = c$, and in like manner is easily obtained $a = d$. Yet, notwithstanding all these equalities there is no proof that a is equal to b or to c , or that d is equal to b or to c ; in short, that any one of these four parts is just the quarter or $\frac{1}{4}$ part of the whole. In fact the problem is not to be performed by plane geometry. As to those solutions by drawing the two lines through the centre of gravity of the trapezium, they are quite from the purpose, as the parts in that situation weigh both according to their quantity, and to the various distances of their centres of gravity.

An Algebraical Solution by Amicus.

Let $abcd$ be the given trapezium, and eg , fh the lines required intersecting perpendicularly at p , and dividing the trapezium into 4 equal parts. Produce ad , bc to meet in L , and ab , dc to meet in s , then the triangles dcl and bcs will be given in magnitude, and if half the trapezium be added to each, the triangles hfl and egs will be given in magnitude; therefore because the $\angle s$ is given, the rectangle $es \cdot sg$ will be given. Let perpendiculars be drawn upon al from the points e , b , f , c , g , and produce eg to meet al in e .



—Put $a^2 = 2 \text{ tri. } hfl$, $b^2 = es \times sg$, $A^2 = \text{the trapez. } abcd$, $c = al$, $d = ad$, $e = ab$, $h = cd$, $m = ba$, $n = As$, $k = cd$, $p = al$, $r = As$, $s = ds$, $x = lh$, $y = aq$, and $z = es$. Then $z \times sg = b^2$, or $sg = \frac{b^2}{z}$, and $gd = s - \frac{b^2}{z} = \frac{sz - b^2}{z}$; but $cd : cd :: gd : gr = \frac{k}{h} \times \frac{sz - b^2}{z}$;

theref. $2 \triangle dgq = dq \times \frac{k}{h} \times \frac{sz - b^2}{z} = (y - d) \times (sz - b^2) \frac{k}{hz}$, to which if

A^2 , or $2\triangle egd$, be added, we have $2\triangle eq = (y - d) \times (sz - b^2) \frac{k}{hz} + A^2$;

which is also $= aq \times eh$. But $ba : ba :: ae : eh = \frac{m}{c}(r - z)$, theref.

$\frac{my}{c}(r - z) = (y - d) \times (sz - b^2) \frac{k}{hz} + A^2$, which call equation (1).

This equation expresses the relation between y and z when eg bisects $abcd$. For simplicity, put $v^2 = (y - d) \times (sz - b^2) \frac{k}{hz} + A^2 = 2\triangle eq$;

then $eh = \frac{v^2}{y}$, and by sim. trian. $BA : \Lambda A :: eh : \Lambda h = \frac{nv^2}{my}$; therefore

$eh = y - \frac{nv^2}{my} = \frac{my^2 - nv^2}{my}$. Again $HL \times Fb = a^2$, or $Fb = \frac{a^2}{x}$; and

by sim. tri. $BA : \Lambda L :: Fb : bL = \frac{pa^2}{mx}$; therefore $hb = x - \frac{pa^2}{mx} =$

$\frac{mx^2 - pa^2}{mx}$. But the tri. ehq and hfb are evidently similar, hence

$eh : hq :: hb : fb$, that is $\frac{v^2}{y} : \frac{my^2 - nv^2}{my} :: \frac{mx^2 - pa^2}{mx} : \frac{a^2}{x}$, or $(my^2 -$

$nv^2) \times (mx^2 - pa^2) = a^2 v^2 m^2$, which call equation (II.); which is an equation for the relation between x and y , as v is always determinable in terms of y from equation (I.), when EG , FH are perpendicular to each other, and bisect $ABCD$: that is, if the values of x , y , z are such as to satisfy equations (I.) and (II.) then ΔEPH will be equal to $PFCG$, and $EBFP = HPGD$.

There is yet, however, another condition to be fulfilled, viz. that each of these parts shall be just a 4th part of the whole, and this will evidently be the case when any one of them is equal to that fourth part. Now to get an equation that will express this condition, draw rk perpendicular to AD ; then by similar triangles $eh : qb :: rk : qk$

$= rk \times (\frac{my^2 - nv^2}{mv^2})$, and $rh : hb :: rk : hk = rk \times (\frac{mx^2 - pa^2}{ma^2})$, then by

add. $HQ = qk + hk = rk \times \left\{ \frac{(my^2 - nv^2)a^2 + (mx^2 - pa^2)v^2}{ma^2v^2} \right\}$. But HQ

$= \Lambda Q - \Lambda H = \Lambda Q - (c - x) = y + x - c$, therefore $rk =$

$\frac{(y + x - c) ma^2v^2}{(my^2 - nv^2)a^2 + (mx^2 - pa^2)v^2}$, and the area of $HPQ (= \frac{1}{2} HQ \times rk)$

$= \frac{(y + x - c)^2 ma^2v^2}{2(my^2 - nv^2)a^2 + 2(mx^2 - pa^2)v^2}$. Theref. subtr. GDQ or $\frac{v^2 - \Lambda^2}{2}$,

we have $HPGD = \frac{(y + x - c)^2 ma^2v^2}{2(my^2 - nv^2)a^2 + 2(mx^2 - pa^2)v^2} - \frac{v^2 - \Lambda^2}{2} =$

$\frac{\Lambda^2}{4}$, equat. (III.) We have now three independent equations (I, II, III),

involving the three unknowns x , y , z , any two of which may be exterminated by the known rules, and there results a final equation containing only one unknown. The resulting equation, however, is too complex to be set down here.

When the given figure is a triangle, instead of a trapezium, the solution will be much simpler, and will terminate in an equation of the 8th degree.

Several other algebraical solutions were attempted, but they were not perfect.

PRIZE QUESTION, by Col. Mudge.

If a cylinder, whose internal diameter is 20 inches, and altitude 6 feet, be close at the top, and having a round hole in the bottom of half an inch diameter, have five feet deep of water in it, and the other foot occupied with air condensed eight times; required the time of complete exhaustion, after the whole is suddenly opened, and without allowing for any contraction of the stream at the hole?

Answered by Mr. H. Atkinson, Newcastle.

Let AB be the cylinder's height, AC that of the condensed air at beginning, or CF the surface of the water at first, and DE the same at any other variable time. Put $AC = a$, $AB = b$, $AD = x$, diameter DE of cylinder $= c$, and diameter of hole $= d$. Then will $8a \div x$ be the density of included air at D , and $8ah \div x$ the height of a column of water of equal pressure as that air, putting $h = 34$ feet, the height equal to one atmosphere; therefore $DB + 8ah \div x = b - x + 8ah \div x$ is the whole internal pressure on the bottom; from this deducting the pressure of one atmosphere (h) upwards against the hole on the outside, leaves $8ah \div x + b - x - h$ the pressure on the column of water, or $= 8ah \div x - e - x$, putting $e = h - b = 28$. But the velocity in the hole due to this whole height is $2\sqrt{g} \times \sqrt{(8ah \div x - e - x)}$, putting $g = 16\frac{1}{2}$, and hence the velocity of the descending surface DE will be $(2d^2 \div c^2) \sqrt{g} \times \sqrt{(8ah \div x - e - x)}$. Therefore the flux. of



the time $t = \frac{\dot{x}}{\text{veloc.}} = \frac{\dot{x}}{\frac{2d^2}{c^2} \sqrt{g} \times \sqrt{(\frac{8ah}{x} - e - x)}} = \frac{c^2}{2d\sqrt{g}} \times \frac{\dot{x} \sqrt{x}}{\sqrt{(l - ex - x^2)}}$, putting $l = 8ah = 272$; which being expanded in a series, and the correct fluents taken when $x = 6$, it gives $t = 156.29$ sec. $= 2$ min. $36\frac{1}{4}$ sec. the time required.

Nearly in the same manner was the solution given by I. T. McDonald, who brought out the time $= 155\frac{1}{2}$ seconds $= 2$ minutes $35\frac{1}{2}$ seconds.—Senex also brought out the time $= 156.359$ sec. $= 2$ min. $36\frac{1}{4}$ sec. and in his solution, by a very ingenious contrivance, brings out a series by means similar to those explained in Dr. Hutton's Mensuration, pa. 206 of the 3d edition, which series converges so fast that five or six terms are sufficient to give the sum true to as many places of figures. With much regret, for want of room, these solutions are obliged to be omitted, as well as those by Mr. Cunliffe, Mr. Moore, Colonel Mudge, and Mr. Watkins. The other solutions received to this question were either incomplete or incorrect.

Questions proposed in 1812, and answered in 1813.

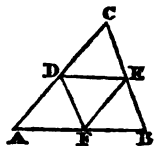
I. QUESTION 1239, by Mr. Joseph Hine, Plymouth.

It is required to divide any triangular field into four equal parts, by three lines, terminating in the three sides, so that one of the parts may be inclosed by the required three lines?

Answered by Mr. John Collins, London.

Let ABC be the proposed triangle. Bisect the base AB in F : draw FE parallel to AC , and FD parallel to BC ; then join DE and the thing is done.

For, by the nature of parallel lines, the triangles ADF , BEF , ACB , are all equiangular or similar. Again, F being the middle of AB , and FD , FE , being parallel to the opposite sides of the triangle; therefore D is the middle of AC , and E the middle of BC ; consequently DE is parallel to AB , and all the four triangles are mutually similar and equal.



The same by Mr. Joseph Hine, of Plymouth.

Bisect the sides of the given triangle in D, E, F ; then join these points, and the thing is done. For because AB and BC are bisected, EF is parallel and equal to AD or DC . In like manner, DE is parallel and equal to AF or FB ; also DF is equal and parallel to BE or EC . Therefore AE , BD , CF , are equal parallelograms, and consequently their halves, or the four triangles, all equal.

The same by Mr. A. Nesbit, Farnley, Leeds.

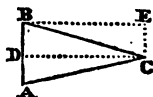
Let ABC be the triangle to be divided. Bisect the sides, and join the bisecting points by the lines DE , DF , EF , and the thing is done. For, by Dr. Hutton's Course, theor. 82, Geom. the lines of division are respectively parallel to the sides of the given triangle; therefore $CD = EF$, and $CE = DF$, consequently the triangle $CDE = DEF$; and in the same manner it may be proved that all the triangles are equal.

II. QUESTION 1240, by Mr. D. T. Sheridan, *Wolverhampton*.

If the circular sides of a circular bellows, 12 feet in diameter, form an angle of 30° at the nose, which is a circular orifice of three inches diameter, be compressed in one second of time; what is the velocity of the issuing fluid?

Answered by Mr. Henry Atkinson, Newcastle-upon-Tyne.

Let ABC represent the bellows open, and CD a section through the middle. Now, if we suppose the under part ADC placed upon the other, as represented by BEC, they will form a cylinder, the base of which will be an ellipsis, having CD for its conjugate, and BC or 12 for its transverse diameter, also BD its height: hence its solidity is $12 \times DC \times DB \times .7854 = 12^3 \times \cos.$



$15^\circ \times \sin. 15^\circ \times .7854 = 12^3 \times \frac{1}{2} \sin. 30^\circ \times .7854$ (because $\sin. \times \cos. = \frac{1}{2} \sin.$ of the double arc.) $= \frac{1}{4} \times 12^3 \times .7854$, because $\sin. 30^\circ = \frac{1}{2}$. But the area of the orifice is $\frac{1}{4} \times \frac{1}{4} \times .7854 = \frac{1}{16} \times .7854$.

Therefore $\frac{\frac{1}{4} \times 12^3 \times .7854}{\frac{1}{16} \times .7854} = 4 \times 12^3 = 6912$ feet, is the velocity

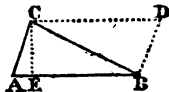
per second if the bellows be single. But, if they be double, as they commonly are, the velocity will be $32 \times 12^3 \times \cos. 7^\circ \frac{1}{2} \times \sin. 7^\circ \frac{1}{2} = 16 \times 12^3 \times \sin. 15^\circ = 7155.83$ feet per second.

The same by Mr. George Duckett, Liverpool.

It is evident that the bellows will contain a quantity of air equal to a cylinder whose base is 12, and altitude 3 feet. Hence $12 \times 12 \times 3 \div \frac{3^2}{144} = 6912$ feet per second, is the velocity of the air.

The same by Mr. D. T. Sheridan, the Proposer.

Let ABC be a central vertical section of the bellows: complete the parallelogram ABDC; then it is evident this represents an oblique cylinder, the half content of which is equal to the bellows. By trigonometry the perpendicular CE = 6, and the solidity is $AB^3 \times \frac{1}{2} CE \times .7854 = 339.2928$ feet. Also 3 inches = $\frac{1}{4}$ foot, and $\frac{1}{4} \times \frac{1}{4} \times .7854 = .0490875$ feet = area of the nose. Then $339.2928 \div .0490875 = 6912$ feet the velocity per second.

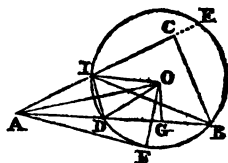


III. QUESTION 1241, *by the Rev. J. Furnass, of Ponteland.*

A triangular field, of which the base AB is 40, and the other two sides AC , BC , are 30 and 20 chains, is to be divided into three parts by two straight lines, drawn from the base, one from A , and the other from B , which is 15 chains distant from A , meeting each other in a point I in the side AC , and there forming between them an angle of 60° . It is required to find the areas of the three parts or triangles thus constituted.

Answered by Mr. John Rutherford, Trinity-house School, Newcastle.

In the given triangle ABC , make AD of the given length, and on DB describe the segment of a circle to contain the given angle 60° ; complete the circle, and join BI and DI ; the intersection at I on AC being the point required, and O the centre of the circle. Draw the tangent AF , and OG perpendicular to AB , also the other lines as in the figure. Then by geometry the $\angle DOG = \angle DIB$ (60°), and $DG = \frac{1}{2} DB$, hence by trigonometry $OG = 7.21687875$, and $DO = 14.4337575$: but (by Dr. Hutton's Geometry, theor. 61, cor. 1) $AD \times AB = AF^2 = 600$, and $AF = 24.4948974$; also in the triangle AOD , AO (28.431204): $\sin \angle ADO$ (150°) :: DO : $\sin. OAD = 14^\circ 42' 16'' 45'''$. But the angle $OAI = 14^\circ 15' 1'' 20'''$, and the two sides IO and AO being given, the side AI is found $= 14.939523$; then the areas of the three triangles AID , IDB , IBC , are 54.244586 and 90.407644 , and 145.939523 chains respectively.

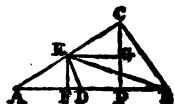


The same by Mr. John Whitley, of Masbro', near Rotherham.

After the construction as above, &c. the angles A and C are found from the data $= 28^\circ 57'$ and $104^\circ 29'$ respectively. In the right-angled triangle DOG are given $\angle DOG = DIB = 60^\circ$, and $DG = \frac{1}{2} DB = 12.5$, to find $DO = 10 = 14.43375$, and $OG = \frac{1}{2} OD = 7.216875$. Also in the right-angled triangle AGO are given $AG = 27.5$, and $OG = 7.216875$, to find $AO = 28.4312$, and $\angle OAG = 14^\circ 42'$. And in the triangle AOI are given $AO = 28.4312$, $OI = 14.43375$, and $\angle OAI = (\angle A - OAG =) 14^\circ 15'$, to find $AI = 14.94335$: hence $IC = 15.05665$: then $AI \times AD \times \frac{1}{2} \sin. \angle A = \text{area of triangle } AID = 54.256$; and $AC \times IC \times \frac{1}{2} \sin. \angle C = \text{area of triangle } AIC = 415.779$: then $AD : DB :: 3 : 5 :: \triangle AID : \triangle BDI = 90.427$ chains; all the three areas.

The same by Miss Susannah Jackson, Mile End.

The content of the triangular field ABC is = 290.47 chains. Let fall the perpendicular CP; then $\frac{AC^2 - BC^2}{2AB} + \frac{AB}{2} = AP = 26.25$; and the $\angle GEB$ being = FED extremely near; also the $\angle ACP = AEF = 61^\circ 2' 42''$, leaves $\angle A = 28^\circ 57' 18''$; therefore $\angle AED = 76^\circ 2' 42''$, and $\angle ADE = 75^\circ$; now we have given the angles and side AD; then $\sin. AED : AD :: \sin. A : DE = 7.5$; and $\sin. F : DE :: \sin. E : FD = 1.94$; also $\sqrt{(DE^2 - FD^2)} = EF = 7.244$. Then the areas are, $\triangle ADE = 54.33$, $PEB = 90.55$, and $BCE = 145.592$.

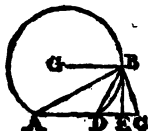


IV. QUESTION 1242, by Mr. John Dewhurst, Blackburn.

Given the difference of the angles at the base, and the difference of the segments of the base, made by a perpendicular from the vertex; to construct the triangle when the rectangle under the sum and difference of the sides is a maximum.

Answered by Mr. Wm. Burdon, Acaster Malbis.

Constr. Take AD = the given difference of the segments of the base, on which describe (by problem 14, Geometry, Dr. Hutton's Course,) a segment of a circle to contain the given difference of the angles at the base; from G the centre of the circle, draw GB parallel to AD, cutting the circle in B; join BA, DB, and to AD produced apply BC = BD, so will ABC be the triangle required.



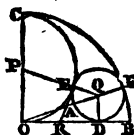
Demonst. Demit the perpendicular BE. Then $\angle ABD = BDC - A = C - A$ the given difference of the angles at the base. And $(AB + BC) \times (AB - BC) = (AE + EC) \times (AE - EC) = (AD + 2DE) \times AD$, which, as AD is known, will be a maximum, when DE is the greatest, that is, when BE touches the circle.

The same by Mr. Edw. Riddle, Whitburn, Sunderland.

Suppose the thing done, and that ABC is the given triangle, AE the

Answered by Mr. I. T. M'Donald.

Draw FEQ joining the centres of the two circles, and ER their common tangent ($= OR = DR$), then are the points F, Q, R , in a circle; and, by the circle, $FO \cdot OA' = OD' = 4ER' = 2FE \cdot 2EQ$: therefore $OA = 2EQ = AF = \frac{1}{2} OF$.



The same by Mr. John Whitley, Masbro', near Rotherham.

The figure being described as in the question, draw the radius OAF to the point of contact F , which will pass through the centre of $DAEF$ (by the 43 Dr. Hutton's Geom.), and (by cor. 1, th. 61) the rectangle $OF \times OA = OD'^2$ (by question 1061, Ladies' Diary, for 1800) the rectangle $OC \times AF$: hence, since $OF = OC$, we have $OA = AF$, or $2AF = OF = OC$. Then, since circles are to each other as the squares of their diameters, the area of the circle of which OC is the diameter, is quadruple the area of the circle DEF : and consequently the area of the semicircle OEC is equal to double the area of the circle DEF .

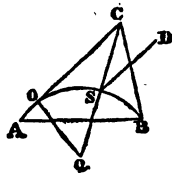
Corol. The three whole circles have their diameters as the progressional numbers, 1, 2, 4, and consequently their areas as the squares 1, 4, 16.

VII. QUESTION 1245, by Mr. John Mark, Sebergham.

Given the sum of the sides, the radius of the inscribed circle, and the line drawn to bisect the vertical angle, and terminate in the periphery of the circumscribing circle: to construct the triangle.

Answered by Mr. W. Burdon, Acaster Malbis.

Constr. Take co = half the given sum of the sides, and oq perpendicular to it, also cq equal the line bisecting the vertical angle, and terminating in the circumference of the circumscribing circle; make the $\angle qcb = qco$; draw ps parallel to co , at a distance = the given radius of the inscribed circle, cutting cq in s ; then, with centre q , and radius qs , describe an arc, cutting ca and cb in A and B ; join AB , so shall ABC be the triangle required. For, it

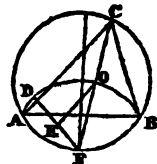


is well known that $oc =$ half the sum, and $ao =$ half the difference of the sides; and that the locus of s , the centre of the inscribed circle, is a circle of which the centre is q .

The same by Mr. J. Collins, Schoolmaster, Hatton Garden.

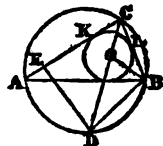
Analysis. Let ABC be the triangle required, and $ACBF$ the circumscribing circle; draw CF the bisecting line, and FD perpendicular to AC ; then it is known that $CD =$ half the sum of the sides: hence this

Constr. With CF the given line, and CD the given half sum, form the right-angled triangle CDF ; on DF set off $DE =$ radius of the inscribed circle, and draw EO parallel to AC , cutting CF in O , the centre of that circle; with centre F , and radius FO , describe the arch AOB ; draw CB making the angle $FCB = FCD$, cutting the arch in A and B ; join AB , then ABC is the triangle, as is evident.



The same answered by Mr. J. A. Dotchen, of Chelsea.

Conceive ABC to be the triangle, having a circle inscribed and circumscribed, the centre of the former being o ; let D be the middle point of the arc ADB , demit DX perpendicular to AC , and join CD , DB , BO . Then, it is well known that CX is equal to half the sum of the sides, and is therefore given; CD , the bisecting line is also given, therefore the right-angled triangle DCX is wholly given, and the angle ECD , or its double ACB , is known; whence, because the radius of the inscribed circle is given, its centre o is given, as also the distances oc and od . Further, since the angle $DOB = (OCB + OBC = ACD + OBA = ABD + OBA =) OBD$, the triangle DOB is isosceles, and $DO = DB$, therefore DB is given, and the triangle ABC may be thus determined.



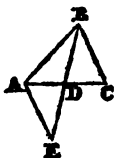
Construct a right-angled triangle CED , having its base $CE =$ half the given sum of the sides, and its hypotenuse $CD =$ the given bisecting line; make the $\angle DCB = DCE$, and within the angle ECB with the given radius describe the tangential circle KL . From D set off $DB = DO$, meeting CB in B ; draw the tangent AB meeting CE produced in A , and ABC is the triangle required, as is obvious from the analysis.

VIII. QUESTION 1246, by Mr. James Cunliffe, jun. Bolton-le-Moors.

Given the vertical angle, difference of the including sides, and the line from the vertical angle to the middle of the base: to construct the triangle.

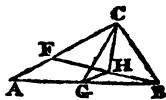
Answered by Mr. Charles Thompson, Edinburgh.

Let ABC be the required triangle, BD the line bisecting the base. Draw AE parallel to BC , meeting BD produced in E . Then $BD = DE$, and $AE = BC$; but BD is given, therefore BE is given; also $\angle BAE = \text{sup. } \angle ABC$ is given, as well as $AB - AE = AB - BC$. Therefore in the triangle ABE , are given the base BE , the $\angle BAE$, and the difference of the sides AB , AE , whence it may be constructed by prob. 2, Append. Simpson's Algebra: and then, by drawing ADC to bisect BE , and BC parallel to AE , the triangle ABC is completed.



The same answered by Mr. W. Wiseman, Hull.

Take $CH = \text{half difference of sides}$; through H draw FB , making $\angle FHC = \text{comp. half given vertical angle}$; erect HC perpendicular to FB ; apply $GC = \text{the given bisecting line}$; make HCB and UCF each $= \text{half the given vertical angle}$; draw BGA cutting CF produced in A ; then ABC will evidently be the triangle required.



The same by Mr. Thomas York, Northallerton.

Let ABC be the triangle. Take $CF = CB$, and draw FB , also let AB and BF be bisected in G and H , and join GH : then CG and GH are given, CG being the given bisecting line, and $GH = \frac{1}{2} AF$ the given difference of the sides. Now the $\angle FHC = CFH = \text{half the sup. of the given vertical angle}$, consequently the triangle CGH is given.

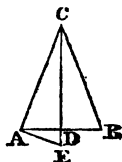
Constr. From the foregoing analysis constitute the triangle CGH ; draw the indefinite line BF perpendicular to CH ; make the angles FCH and HCB each $= \text{half the vertical angle}$; draw BGA to meet CF produced in A : then ABC is evidently the triangle required.

IX. QUESTION 1247, by Mr. Jesse Winward, Plymouth.

A cone, whose altitude is 20 feet, revolves about its axis with a velocity of 10 feet per second at the circumference of the base, but in the time of one such revolution a body would descend down its slant side when the cone is at rest. Required the base?

Answered by Mr. D. T. Sheridan.

Put $cd = a = 20$ feet, $10 = b$, $3.1416 = c$, $16\frac{1}{2} = d$, and $x =$
 ad . Then $2cx =$ circumference of the base; $b : 1''$
 $:: 2cx : 2cx \div b =$ the time of one revolution. Again,
 a body descends down ca in the same time as down ce ,
 and $a : x :: x : x^2 \div a = de$, therefore $a + x^2 \div a$



$= ce$, and $d : 1'' :: \frac{a^2 + x^2}{a} : \frac{a^2 + x^2}{ad} =$ square of

the time down ce or ca ; hence $\frac{a^2 + x^2}{ad} = \frac{4c^2 x^2}{b^2}$, and

$x = \sqrt{\frac{a^2 b^2}{4adc^2 - b^2}} = 1.7818 = AD$ the radius of the base.

The same by Mr. John Smith, Allon Park.

Let abc be the cone. Put $a = cd = 20$ feet the altitude, $x = AD$ or
 rs the radius of the base, $b = 10$ the velocity at the circumference, $g =$
 $16\frac{1}{2}$ the space freely descended in 1 second, and $n = 3.1416$. Then,
 by the laws of gravity, $\sqrt{a \div g}$ is the time of falling through cd ; but,
 by prop. 26, vol. ii, Dr. Hutton's Course, the time in cp is to the time

in ca , as cd to ca , that is, $a : \sqrt{a^2 + x^2} :: \sqrt{\frac{a}{g}} : \sqrt{\frac{a^2 + x^2}{ag}}$

the time down ca . But $\frac{2\pi x}{b}$ is the time of 1 revolution; therefore,

$\sqrt{\frac{a^2 + x^2}{ag}} = \frac{2\pi x}{b}$, and hence $x = \sqrt{\frac{a^2 b^2}{4agn^2 - b^2}} = 1.78182$

feet, the radius of the base.

The same by Miss Mary Weston, Gainsborough.

Put $x =$ the diameter of the cone's base.

Then $6.2832x =$ its circumference,

And $.62832x =$ time of the cone's revolution round the axis.

Now $\sqrt{20 \div 16\frac{1}{2}} = 1.115'' =$ time of descending down the axis,

And $(1.115 \div 20) \sqrt{400 + x^2} =$ the time along the slant side;

Therefore $(1.115 \div 20) \sqrt{400 + x^2} = .62832x$; hence $x =$
 1.7818 feet, and the diameter of the base is 3.5636 feet.

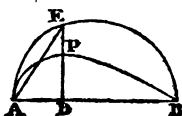
X. QUESTION 1248, *by Mr. J. A. Dotchen, Chelsea.*

If AB be the diameter of a given semicircle, DE a perpendicular or-
 dinate, and AE a corresponding chord; also, if DF be taken on the or-
 dinate DE always equal to the difference between the chord AE and the

abscissa AD ; it is required to determine the length of the curve which is the locus of P , without the use of series? See the Diary for 1806, question 1157.

Answered by Mr. J. Brierley, and Mr. Hacking, near Blackburn.

Put $AB = a$ the circle's diameter, $AD = x$, $DP = y$; then $AE = \sqrt{ax}$, and $y = \sqrt{ax} - x$, hence $\dot{y} = \frac{a}{2\sqrt{ax}} \dot{x} - \dot{x}$, $\dot{x}$, and $\dot{z} = \sqrt{(\dot{x}^2 + \dot{y}^2)} = \dot{x}\sqrt{1 + \left(\frac{a}{2\sqrt{ax}} - 1\right)^2}$. To find the fluent, let $a = b^2$, and $x = v^2$, then will $\dot{z} = 2b\sqrt{2} \times v \sqrt{\left(\frac{v^2}{b^2} + \frac{v}{2b} - \frac{1}{2}\right)}$. Now put $s = \frac{v}{b} - \frac{1}{2}$, then this substituted, and the correct fluent taken when $x=a$ or b^2 , it will be $a \times \left(\frac{3\sqrt{5} + 1}{8} + \frac{\sqrt{2}}{16} \times \text{hyp. log. } \frac{3 + \sqrt{10}}{\sqrt{2} - 1}\right) = a \times 1.20216$ nearly, the length of the curve required.



The same by Mr. Wm. Burdon, Acaster Malbis.

Put $AB = a$, $AD = x$, and $DP = y$. Then, by the circle, $AE = \sqrt{ax}$, and by the question, $y = \sqrt{ax} - x$, or $y + x = \sqrt{ax}$, this squared gives $y^2 + 2yx + x^2 = ax$, an equation to the parabola. By making the fluxion of $\sqrt{ax} - x = 0$, it gives $x = \frac{1}{2}a = AD$, and then the greatest value of $DP = \frac{1}{2}a$ also. So that taking P for the vertex of each of the parabolic arcs PA , PB , their lengths will be found by prob. 4, page 271, Dr. Hutton's Mensuration, 4th edition, and the sum of both will be the length of the curve APB .

The same answered by Mr. W. G. Horner, of Bath.

Make $AB = d$, $AD = x$, $DP = y$, $AE = x + y = z$. By the circle $z^2 = dx = dz - dy$. Hence $y = \frac{dz - z^2}{d}$, and $\dot{y} = \frac{d - 2z}{a} \dot{z}$, and $\dot{z} = \dot{z} - \dot{y} = \frac{2z\dot{z}}{d}$. From these data the length of the curve between A and P is $\int \sqrt{(\dot{x}^2 + \dot{y}^2)} = \int \frac{\dot{z}\sqrt{(d^2 - 4dz + 8z^2)}}{d}$. To transform the radical trinomial into a binomial, make $z = v + \frac{1}{4}d$, then the last expression becomes $\int \frac{\sqrt{(16v^2 + d^2)}}{d\sqrt{2}} v$; the value of which, by methods pointed out in the second and third vols. of

Dr. Hutton's Course, is found to be $\frac{1}{d\sqrt{2}} \times [2v\sqrt{(16v^2 + d^2)} + \frac{d^2}{2} \times \text{hyp. log. } (4v + \sqrt{(16v^2 + d^2)})]$. But, as the origin of the curve, x, y, z , are each = 0, therefore $v = \frac{1}{4}d$, and the correction for the length above found is $\frac{1}{4}d - \frac{1}{4}d\sqrt{2} + \text{hyp. log. } d(\sqrt{2} - 1)$. Hence the length of the complete curve (when $y = 0, x = z = d$, and therefore, $v = \frac{1}{4}d$) is $d \times \left(\frac{3\sqrt{5}+1}{2} + \frac{\sqrt{2}}{4} \times \text{hyp. log. } \frac{\sqrt{10}+3}{\sqrt{2}-1} \right)$.

Scholium 1. Since $\dot{y} = \frac{d-2x}{d} \dot{z}$, y is a maximum, when $z = \frac{1}{2}d$, at which time $x = y = \frac{1}{4}d$, and the length of the curve is $d \times (1 + \frac{\sqrt{2}}{4} \times \text{hyp. log. } \frac{\sqrt{2}+1}{\sqrt{2}-1})$.

2. The area is $fyx = f \frac{2dx^2 - 2x^2\dot{z}}{d^2} = f \frac{2x^2\dot{z}}{d} - f \frac{2x^2\dot{z}}{d^2} = \frac{2x^3}{3d} - \frac{x^3}{2d^2}$, which requires no correction, and gives the whole area = $\frac{1}{6}d^3$, and the area when y is arrived at its maximum $\frac{5}{24}d^3$.

3. Hence the whole area is the same as that of a parabola of the same base and height.

The same answered by Mr. John Whitley, Masbro'.

Let $AB = d, AD = x, DP = y, AE = z$; then by the circle $x = \frac{z^2}{d}$, and $y = z - \frac{z^2}{d}$, by the question, hence the fluxion of the curve $\sqrt{(x^2 + y^2)} = \frac{z}{d} \sqrt{(d^2 - 4dz + 8z^2)}$. To find the fluent, put $z = \frac{1}{2}(d + v)$; then $\dot{z} = \frac{1}{2}\dot{v}$, and the fluxion of the curve is $\frac{v}{4d} \sqrt{(\frac{1}{2}d^2 + \frac{1}{2}v^2)} = \frac{\sqrt{2}}{8d} \times \dot{v} \sqrt{(d^2 + v^2)}$, whose fluent is $= \frac{\sqrt{2}}{8d} \times [\frac{1}{2}v \sqrt{(d^2 + v^2)} + \frac{1}{2}d^2 \times \text{hyp. log. } (v + \sqrt{(d^2 + v^2)})] + c$. Now when $z = 0$, the curve $AP = 0, v = -d$: hence $c = \frac{1}{8}d - \frac{\sqrt{2}}{16}d \times \text{hyp. log. } (d\sqrt{2}-d)$: therefore the correct fluent (when $z = d$, in which case $v = 3d$) is $= [(3\sqrt{5}+1) + \frac{\sqrt{2}}{2} \times \text{hyp. log. } (3 + \sqrt{10}) \times (1 + \sqrt{2})] \times \frac{1}{8}d = 1.20215 \times d = \text{length of the curve } APB$.

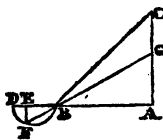
Note. The curve APB is a portion of the common parabola whose ordinate is $= x + y$, abscissa $= x\sqrt{2}$, and parameter $= a\sqrt{\frac{1}{2}}$. See solution of question 134, vol. 2, *Leybourn's Repository*, New Series.

XI. QUESTION 1249, by Mr. John Rutherford, *Newcastle*.

If the hypotenuse BC , of a right-angled triangle ABC move through the point a , while the end c moves along the side AC till it arrives at the right angle A ; it is required to find the length and area of the curve described by the point a ?

Answered by Mr. John Whitley, of Masbro'.

It is evident that the curve BRD described by the point a , is the modated conchoid, whose equation, by Example 3, pa. 207, vol. 3, Dr. Hutton's Course, is $y = \frac{x-b}{x} \sqrt{(h^2 - x^2)}$, where $AB = b$, $BC = AD = h$, $AE = x$, and $EF = y$: hence the flux. of the area DEF is $y\dot{x} = \frac{b\dot{x}}{x} \sqrt{(h^2 - x^2)} - \dot{x} \sqrt{(h^2 - x^2)} =$



$\frac{b h^2 \dot{x}}{x \sqrt{(h^2 - x^2)}} - \frac{b x \dot{x}}{\sqrt{(h^2 - x^2)}} - \dot{x} \sqrt{(h^2 - x^2)}$: whose fluent is $\frac{1}{2} b \dot{h} \times \text{hyp. log.} \frac{h - \sqrt{(h^2 - x^2)}}{h + \sqrt{(h^2 - x^2)}} + b \sqrt{(h^2 - x^2)} + \text{area of the circ. seg.}$ whose radius $= h$, and versed sine $= hx$; and the whole area of the semi-nodus DRB (putting $AC = p$) is $= \frac{1}{2} b \dot{h} \times \text{hyp. log.} \frac{h-p}{h+p} + \dot{h} p + \text{cir. seg. rad.} = h$, and vers. $= h - b$.

Again: we have $\dot{y} = \frac{b h^2 \dot{x} - x^2 \dot{x}}{x^2 \sqrt{(h^2 - x^2)}}$, and the fluxion of the curve is $\frac{h \dot{x}}{x^2} \sqrt{\frac{b^2 h^2 - 2 b x^2 + x^4}{h^2 - x^2}}$, which being expanded in a series, and the correct fluents taken, will give the length of the curve.

XII. QUESTION 1250, by the Rev. George Harker, *Sebergham*.

In an horizontal dial, the angle included between two given hour-lines was observed to be a maximum. To find the lat. independently of fluxions or algebra.

Answered by the Rev. George Harker, the Proposer.

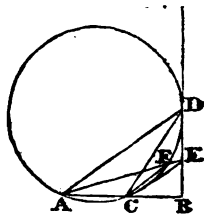
Put A, a , for the given hour arcs respectively; then by the principles of dialling,

as $\text{rad.} : s. \text{ lat.} :: t. A : t. H. \text{ angle,}$

and $s. \text{ lat.} : \text{rad.} :: t. h. \angle : t. a.$

theref. $t. A : t. a. :: t. H. \angle : t. h. \angle$, a given ratio.

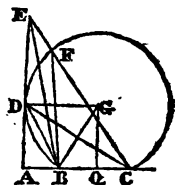
Therefore take AB to BC in the given ratio of the tangents of the hour angles; erect BD perpendicular to AB ; through A, C , draw a circle to touch BD in D , and join CD ; then the $\angle ADC$ (included between the two given hour lines) is a max. For, draw any other right line AE , and join EC ; then the $\angle AFC$, or its equal ADC , is manifestly greater than $\angle FEC$. Now, because $BD = \sqrt{(AB \cdot BC)}$ is given, therefore the hour $\angle ADB$ or CDB becomes known; and consequently $\sin. \text{ lat.} (= \frac{\text{rad.} \times t. H. \angle}{t. A})$ is known.



The same by Mr. Wm. Burdon.

Take AB to AC in the ratio of the tangents of the two given hour angles; bisect BC in Q with the perpendicular QG ; apply $BG = AQ$, and draw GD parallel to BQ , meeting the perpendicular AE in D ; join BD, CD , the two hour lines when the included $\angle BDC$ is a maximum.

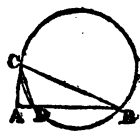
For, with the centre G , and radius GD or GB , describe a circle, and take any point E in the perp. AD ; join BE, CE, BF ; then the $\angle BFC = BDC$ is greater than the $\angle BEC$.



Example. Suppose the lat. be required where the angle between the hour lines of 3 and 4 o'clock is the greatest. Here, $AB = 1$, $AC = \sqrt{3}$, $AQ = BG = \frac{1}{2} \sqrt{3} + \frac{1}{2}$, $BQ = \frac{1}{2} \sqrt{3} - \frac{1}{2}$, and $QG = AD = \sqrt{(BG^2 - BQ^2)} = \sqrt{3} = 1.316074$; also by trig. $AD : \text{rad.} :: AB : \tan. \angle ADB = 37^\circ 13' 43'' \cdot 9$. Consequently by the principles of dialling, $\tan. 45^\circ : \text{rad.} :: \tan. \angle ADB : \sin. \text{ lat.} = 49^\circ 27'$.

The same by Mr. D. T. Sheridan.

Take AD, AB the respective tangents of the given hour angles, and draw AC perpendicular to AB and = a mean proportional between AB and AD ; through the points B, D, C , describe a circle, then join CB, CD , and the thing is done.



For, AB, AD = AC by the construction therefore AC is a tangent to the circle at c; and the $\angle BCD$ is evidently a maximum.

Calcul. In the right-angled triangle ACD, are given the sides, to find the $\angle ACD$; then, as tang. arc AD : tang. hour angle ACD :: radius : sine of the latitude. ?

XIII. QUESTION 1251, by Mr. John Ryley, Leeds.

Required a solution of the 53d question of Dr. Hutton's Course, vol. ii. pa. 282, 6th edit. which is as follows: If a cannon ball of 11lb. weight be fired against a pendulous block of wood, and, striking the centre of oscillation, cause it to vibrate an arc whose chord is 30 inches; the radius of that arc, or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the centre of oscillation of the pendulum, at the lowest point of the arc; the whole weight of the pendulum being 500lb. ?

Answered by Mr. Henry Atkinson, Newcastle-upon-Tyne.

As no data are here given, from which the form of the block of wood may be determined, I shall suppose the figure of the whole to be a prism, or the centre of gravity to be in the middle point of its length, viz. at 59 inches, the whole length given being 118; in which case the solution will be as follows.

By art. 293, vol. 2, Dr. Hutton's Course, 6th edit. the distance of the centre of oscillation, below the point of suspension, is found to be $88\frac{1}{3}$ inches; and, using the notation in the general solution at art. 298, we have $p = 500$, $b = 1$, $g = 59$, $o = i = 88\frac{1}{3}$, $v =$ velocity of the ball, $u =$ velocity of the centre of oscillation, and $a = 16\frac{1}{2}$ feet = 193 inches. Then, as $118 : 30 :: 88\frac{1}{3} : 22.38083 = c$; hence $c\sqrt{(2a \div o)} = 46.8653$ inches = 3.90544 feet, the first or greatest velocity of the centre of oscillation. Then $\frac{bii + gop}{bii} \times c \times \sqrt{\frac{2a}{o}} = \frac{bi + gp}{bi} \times c \sqrt{\frac{2a}{i}} = 15715.59$ inches = 1309.63 feet, the velocity of the ball.

The same by Mr. M'Donald.

By the pendulum, as $1' : 39\frac{1}{2} :: (\frac{49}{16})'$ or $\frac{2}{3} : 88\frac{1}{2}$, the distance of the centre of oscillation, (which call o , its velocity u , that of the ball v); also by sim. triangles or sectors, as $118 : 30 :: 88\frac{1}{2} : 22.380826 = c$, the chord of the arc described by the centre of oscillation, being that point at which, if the whole mass or centre of gravity, of the pendulum were placed, it would receive the same angular velocity. Hence, as $500 + 1 : 1 :: v : u$, or $v = 501 u$. But the velocity acquired by the point of oscillation, in descending through the arc whose chord is c , is $c \sqrt{(2a \div o)}$ where $a = 16\frac{1}{2}$ feet) $= 3.90543$ feet $= u$. Therefore, $501 u = 1956.6204$ feet, the velocity of the ball.

The same by Mr. Ed. Riddle, Whitburn, near Sunderland.

This question is only a particular case of the general problem solved prob. 58, Dr. Hutton's Course, vol. 2, 6th edit.—Assuming here the notation there used, we have $i = o = 88\frac{1}{2}$ inches, $g = 59$, and $118 : 30 :: 88\frac{1}{2} : 22\frac{719}{1888}$ inches $= c$. And the velocity of the centre of oscillation is by the above problem $\frac{ci}{o} \sqrt{\frac{2a}{o}} =$ in this case $c \times$

$\sqrt{\frac{2a}{o}} = 3.90544$ feet; also the ball's velocity is by the same prob. $=$

$\frac{bi^2 + gop}{bio} \times c \sqrt{\frac{2a}{o}} = 1312.651003$ feet; which differs from the

answer in the course, this latter having been produced by making i , o , and g , all equal, whereas only i and o are equal.*

* Most of the gentlemen who have solved this question have remarked on the want of a certain datum in this question, viz. the place of the centre of gravity, which certainly ought to be given, though the omission was not observed when the question was proposed. We may now remark, however, that if, in supplying that defect, the distance of the centre of gravity, below the point of suspension, be supposed 88 inches $= g$, then the velocity of the ball will come out 1778.457 feet.

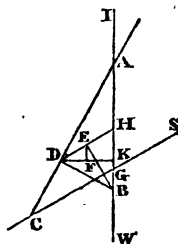
XIV. QUESTION 1252, by Philo-Nauticus,

How near must a ship sail to the wind, and what angle must her sails make with her course, so that she may make the most way to windward? Or, what angle must a ship's course make with the direc-

tion of the wind, and what angle must her sails make with her course, so that her motion in a direction opposite to the wind, shall be the greatest?

Answered by Philo-Nauticus, the Proposer.

Let w be the direction of the wind, sc the course of the ship, ca the position of the sail, ba the force of the wind. Then construct as in the figure. Hence bd will be the effective part of ba acting on the sail; de the effect of bd estimated in the direction cs , or that in which the ship moves: and ef the ship's motion in the direction iw . Hence ef is to be a maximum. Now



$$BA : BD :: 1 : \sin. BAC$$

$$BD : DE :: 1 : \sin. ACG$$

$$DE : EF :: 1 : \cos. CGB$$

therefore $AB : EF :: 1 : s. BAC \times s. ACG \times \cos CGB$,
and $EF = BA \times \sin. BAC \times \sin. BAG \times \cos. CGB$, which must be a max.
or $\sin. BAC \times \sin. ACG \times \cos. CGB$ a max. since BA is constant,
or $\sin. BAC \times \sin. ACG \times \cos. (BAC + ACG)$ a max.
or $\sin. A \times \cos. A \times \sin. C \times \cos. c - \sin.^2 A \times \sin.^2 c$ a max.

Let x be the sine A , and y the sine c , to radius 1.

Then $x\sqrt{(1-x^2)} \cdot y\sqrt{(1-y^2)} - x^2 y^2$ is a max.

$$\text{therefore } \frac{y\sqrt{(1-y^2)} \cdot (1-2x^2)x}{\sqrt{(1-x^2)}} - 2xy^2x = 0,$$

$$\text{and } \frac{x\sqrt{(1-x^2)} \cdot (1-2y^2)y}{\sqrt{(1-y^2)}} - 2x^2 y y = 0.$$

Which equations, being reduced, gives $x = y$, and $x = \frac{1}{2}$:

Hence the $\angle ACG$, which the sail makes with the ship's course, is 30° , and the $\angle CGB$, or the angle which the ship's course makes with the wind, is 60° .

Note. This result agrees very well with experience, since it is found that those vessels make most way to windward, which sail about $5\frac{1}{2}$ points $61\frac{1}{2}^\circ$ from the wind.

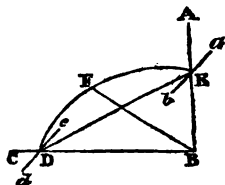
PRIZE QUESTION, by Llerien.

AB and CB are two equal lines perpendicular to each other, and DE equal to either, is inserted between them. Two equal and perfectly elastic balls fall from A and C , towards a force at B , which varies as the distance from B ; and at E and D are reflected by planes, ab , cd , inclined at an angle of 45° to AB and CB . What will then be-

come of the balls? Will they ever meet, and, if so, where? And, after such meeting, if it should happen, what will become of them then?

Answered by Llerien, the Proposer.

With EB , BD , as semiaxes, describe the elliptic quadrant DPE , and bisect the area DPE by the line BF : then the balls will meet in F . For it is evident that each of the balls will describe some ellipse, (Newton's Prin. pr. 10, cor. 1). Also to revolve round the centre of an ellipse, a body must have the velocity acquired in falling to the periphery, through a distance from the centre equal to the line that joins the extremities of the two axes. Each of the balls therefore will describe the ellipse DPE , and therefore (ib. pr. 1) will meet in the point F .

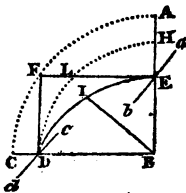


* * It is manifest, that this solution of our correspondent Llerien, supposes the bodies to be reflected from D and E , in thence commencing their motions in the curve, at the same instant, as he bisects the area by the line BF , which, indeed, he had a right to suppose, as no particular time was mentioned in the question. But all our other correspondents have taken it for granted, that the two bodies commence motion at the same instant at A and C , thence falling through unequal spaces AE , CD , they begin their motions in the curve at D and E , at different times; in which case the point of meeting F , will be different from the extremity of a line BF bisecting the area. We can therefore only find room for the sentiments of a very few of our correspondents, touching the situation of the point of collision, as follows.

Mr. Henry Atkinson, after reasoning as above, infers.

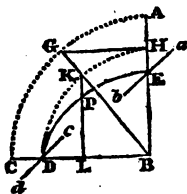
Hence, the ball which falls from A , will, after being reflected from the point E , always pass by the elliptic arc through the point D . And, in like manner, the ball which falls from C would always pass by the same elliptic arc, through the point E , if not prevented by some obstacle: therefore the balls must meet somewhere in the elliptic arc ED .

Describe the quadrants CFA , DLH , with the radii ac and BD . Now, if different balls were projected from the points C and D , with velocities such as would make them describe the curves CFA , DLH , DIE , it is a well known property of this law of attraction, that those quadrants



would all be described in the same time, and that time would be the same as any of them would be in falling to B . Also, a ball, falling from c , would be the same time in reaching D , as the ball which describes the quadrant cFA would be in describing CF , the versed sine of which is CD . Hence, if the whole quadrantal arch cFA represent the time of falling to B , the arches CF and FA will represent the times of falling to D and E (see vol. 2, Hutton's Course, prob. 12, pa. 351, edit. 6); for F is the termination of the perp. raised from D , and also of that raised from E . And, if BI be drawn to bisect the right angle ABC , then I will be the point where the two balls will meet: also, the time from the first beginning to fall till the two balls meet, will be equal to the time in which either of them would fall to B . Hence, the answers to the question proposed are manifest, viz. That the balls will be reflected from the two planes, perpendicular to the directions in which they were falling; after which they will each proceed forward in the elliptic orbit DIB , till they meet in the line which bisects the $\angle ABC$; then, being perfectly elastic, and meeting directly with equal velocities, they will each return again by the same path in which they advanced, and with the same velocity too, till the one strikes the plane ab , and the other the plane cd , which will reflect them in the directions EA and DC , in which directions they will continue, till the one arrives at A , and the very instant the other arrives at c : the whole force which the two balls had acquired, being now destroyed, they will describe the very same track over again, thus continuing to go backward and forward for ever.

Mr. M'Donald also says, "Describe ED an ellipse to the semiaxes DB , BE , and it will be the path in which the balls will move after impinging on the planes ab , cd : and, if the circular quadrants AC , DH , be described, and HE be drawn parallel to BC , and AG be joined; then a perpendicular from K , where the circle is intersected, demitted on EC , will cut the elliptic arc in I , the point where the balls meet.



These constructions Mr. M. then demonstrates at great length, from Newton's Principia and Dr. Hutton's Course; which, with great reluctance, are obliged to be omitted for want of room, as well as other ingenious solutions by our learned correspondents.

Mr. M'Donald adds, in the conclusion of his solution, as follows: "And what will become of them then?" Why, by the laws of collision, they will recede from each other with the same velocity with which they met, and will therefore move in the same trajectory till they collide at the other extremity of EB completed (supposing the two planes ab , cd , removed out of the way); whence they will return to the same point E , and will continue thus to meet and repel each other everlastingly.

Questions proposed in 1813, and answered in 1814.

1. QUESTION 1254, by Mr. A. Nesbit, *Farnley, Leeds.*

Having to survey an elliptical field, and to divide it into two parts, by a line parallel to the conjugate axis; I found the transverse axis measured just 10 chains, also the divisional line 480 links, and the parameter, or double ordinate through the focus, 360 links. Required the areas of the two parts?

Answered by Mr. John Davey, of St. Just, in Penwith.

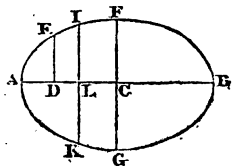
By the question and the nature of the ellipse, we have $\sqrt{(3 \cdot 6 \times 10)} = 6 =$ the conjugate axis; also as $6 : 10 :: \sqrt{(3^2 - 2 \cdot 4^2)} : 3 =$ the distance between the ordinate and centre: hence $5 + 3 = 8$, and $5 - 3 = 2$, the greater and less abscisses respectively. Then, by Dr. Hutton's Mensuration, p.244, 1st edit. the area of the greater segment is $= 4$ ac. 0 r. 6. p. and that of the less $= 2$ r. 27 p.

The same by Mr. Alfred Fox, of Falmouth.

By the properties of the ellipse, the conjugate axis $= \sqrt{(\text{param.} \times \text{trans.})} = 600$ links; and (Dr. Hutton's Course, ellipse, th. 3, cor. 1,) as conjugate 600 : transv. 1000 :: divisional line 480 : corresponding chord of circumscribing circle $= 800$; hence the areas of the segments of the ellipse (by Dr. Hutton's large Treatise on Mensuration, prop. 6, ellipse,) $= 4 \cdot 04145$, and $\cdot 67094$ acres.

The same answered by Miss Susannah Jackson, Mile End,

By the property of the ellipse, $AD \times DB = AC \times DE = CF^2 = 9$; consequently $CF = 3$, and $AD = 1$; also $FG : AB :: \sqrt{(FC^2 - IL^2)} : CL = 3$. Then by par. 42, 6th edition, Dr. Hutton's Course, the areas are $AIK = 6 \cdot 709428$, and $BK = 40 \cdot 414572$.



II. QUESTION 1255, by Mr. Joseph Williams, Bristol.

A circular area of marshy land is to be improved, by making a trench or drain of the outer part, 3 feet deep, and of a regular breadth continuing to increase the breadth till so much earth is displaced, as will raise the remaining inclosed circle 2 feet higher. Required the diameter of such remaining circle?

Answered by Mr. Wm. Putsey, Pickering Academy.

Let the diameter of the given circle be assumed $= d$, and that of the inclosed area $= x$; then will their areas be as d^2 and x^2 , and that of the ditch as $d^2 - x^2$. But by the question $d^2 - x^2 : x^2 :: 2 : 3$, or $d^2 : x^2 :: 5 : 3$, and $d : x :: \sqrt{5} : \sqrt{3}$, or $x = d\sqrt{\frac{3}{5}}$.

The same by Mr. Thomas Smith, Hethersett, Norfolk.

Let d and x be the two diameters of the circles, and $n = .7854$; then $3n(d^2 - x^2)$ is the solid content of the ditch, and $2nx^2$ that of the earth laid upon the inner circle; hence, by equality, $5x^2 = 3d^2$, and $x = d\sqrt{\frac{3}{5}}$.

A Geometrical Solution by Mr. Geo. Ducket, Northwich.

On the given radius AC , describe a semicircle, and take $AD : DC :: 2 : 3$, and erect the perpendicular DE cutting the semicircle in E , join CE , which will be the radius of the circle required.—For, since it is evident that the area of the trench is to that of the remaining circle as 2 is to 3, and by sim. triangles $AC : CE :: CE : CD$, therefore $AC : CD :: AC^2 : CE^2$, and by division $AD : CD :: AC^2 - CE^2 : CE^2 :: 2 : 3$ by the construction.



III. QUESTION 1256, by Mr. Joseph Williams.

A stone descending freely, by the force of gravity, from the top of a precipice, was found to fall one-third of the way in the last second of time. Required the height of the precipice, and the whole time of descent?

Answered by Mr. J. Brooke, Schoolmaster, Wortley.

Let t = the whole time of the stone's descent, then will $t - 1$ = the time of descent through $\frac{1}{3}$ of the whole height; and as the spaces descended are as the squares of the times, we have $t^2 : (t - 1)^2 :: 3 : 2$; hence $t^2 - 6t = -3$, and $t = 3 + \sqrt{6} = 5.4494$: then 477.61 feet is the height of the precipice.

The same answered by Mr. H. Hubert, jun. Rock Insur. Office.

Put x = time of descent, and $a = 16\frac{1}{2}$; then ax^2 = the whole height, and $a(x - 1)^2 = \frac{1}{3}$ of the same; therefore $2ax^2 = 3a(x - 1)^2$, hence $x^2 - 6x + 3 = 0$, which, by completing the square, &c. gives $x = 3 + \sqrt{6} = 5.4494 = 5'' 27'''$ nearly the whole time of descent, and $ax^2 = 477.6$ the height of the precipice.

The same by Mr. John Whitley, Rotherham.

Since the times are as the square roots of the spaces, we have, as $\sqrt{3} : \sqrt{2} ::$ the whole time : to the time in falling through $\frac{1}{3}$ of the height; hence, as $\sqrt{3} - \sqrt{2} : \sqrt{3} :: 1'' : \sqrt{3} \div (\sqrt{3} - \sqrt{2}) = 3 + \sqrt{6} = 5.449489$ seconds, the whole time = t ; then $16\frac{1}{2}t^2 = 477.6257$ feet, the height of the precipice.

IV. QUESTION 1257, by Mr. Chas. Pritty, Thornham.

A ball, whose elasticity is to perfect elasticity, as 1 to n , falls from a given height, suppose 100 feet, on a hard horizontal plane, and rebounds continually till its whole motion is lost. Quere the space passed over, and the same in numbers when the elasticity is the half of perfect, that is, rising from the plane always with the half of the velocity with which it strikes it?

Answered by Mr. John Davey, of St. Just, Cornwall.

By Dr. Hutton's Course, vol. ii. p. 140, the spaces described by falling bodies are as the squares of their velocities: therefore, if s denote the space fallen through at first, we soon find that the ball,

by ascending and descending between the first and second impact, will pass through the space $\frac{2s}{n^2}$, between the second and third, a space $= \frac{2s}{n^4}$, &c.; and by continuing thus till its motion is lost, will constitute a series of the form $s + \frac{2s}{n^2} + \frac{2s}{n^4} + \frac{2s}{n^6} + \&c. \dots 0$, the sum of which is $\frac{n^2 + 1}{n^2 - 1}s$; which, when $n = 2$, and $s = 100$, gives $166\frac{2}{3}$ for the whole space passed over.

The same answered by Mr. J. A. Dotchen, of Chelsea.

Since the ball, after falling through the first 100 feet, rebounds with only an n th part of the velocity it has acquired, by the laws of gravity it will only ascend to the n^2 part of 100 feet, or $\frac{100}{n^2}$; it then returns through the same space, and meeting the plane again, rises with an n th part of the last velocity, and consequently will ascend only to the n th part of the height through which it last fell, that is, to $\frac{100}{n^4}$ feet; in like manner the next height risen to will be $\frac{100}{n^6}$: hence it appears, that the whole space passed over is equal to the series $100 + \frac{200}{n^2} + \frac{200}{n^4} + \frac{200}{n^6} + \&c.$ ad. inf. which, exclusive of the first term, being a geometrical progression, its sum is easily found $= 100 + \frac{200}{n^2 - 1} = 100 \times \frac{n^2 + 1}{n^2 - 1}$. So that, taking $n = 2$, the entire space passed over is $100 \times \frac{5}{3} = 166\frac{2}{3}$ feet.

The same answered by Mr. John Smith, Alton Park.

The heights to which the ball successively rises, are as the squares of the velocities with which it last rebounds from the plane; the velocities with which it rebounds as the elasticity of the ball; and the velocities with which it strikes the plane equal to those with which it last rebounded. Now put $a = 200$ double the first height, the spaces described will form an infinitely decreasing geometrical progression, the first term of which is a , and the second $\frac{a}{n^2}$; therefore $a \div (a - \frac{a}{n^2})$ is the sum of all the terms, and con-

sequently $a^2 \div (a - \frac{a}{n}) - \frac{1}{2} a$ the space passed over by the ball. If

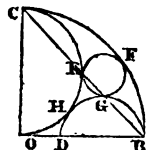
$a = 200$ feet, and $n = 2$, then $\frac{40000}{200 - 50} - 100 = 166\frac{2}{3}$ feet is the space required.

V. QUESTION 1258, by Mr. John Dawes, Birmingham.

If obc be a quadrant of a circle, and oec a semicircle on the radius oc as a diameter; and on the other radius ob , if the semicircle dhb be described touching the other curves in h and b ; its area will be equal to double the circle efg inscribed in the space $cfbghec$. Required the demonstration?

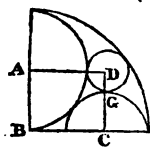
Answered by Mr. John Davey.

Having drawn the lines as in the question, draw also the chord cb , which it is evident cuts off from each of the four circles, an arc of 90° ; which arcs being similar, their chords are to each other as their respective diameters: therefore, putting $cb = 6$, (by question 1197, Ladies Diary for 1810), we have $ce = 3$, $gb = 2$, and consequently $eg = 1$. Now the areas of circles being as the squares of their similar chords, the circle on the diameter db is = four times the circle efg , or the semicircle dgb = double the circle efg .



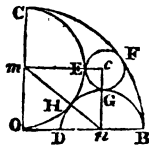
The same by Mr. Jos. Hall, Hurdsfield, near Macclesfield.

Join the centres of the circles as in the figure. Then it easily appears that $ABCD$ is a right-angled parallelogram. Now $cg = \frac{2}{3} AB$, by Ladies Diary, 1810, question 1197; consequently dg must be $\frac{1}{3} AB$, and cg double of dg .



The same by Mr. R. Nunn, Academy, Eyc, Suffolk.

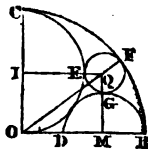
If lines be drawn from m and n , through the points of contact g and e , they will meet in the centre c . In a former Diary it was proved that ob and db are as 3 to 2. It is evident that nc is equal and parallel to mo , hence the radius gc or $ec = \frac{1}{2} gn$; consequently the area of the semicircle dgb is double the whole circle efg .



Corol. The triangle *mnc* is right-angled, and the sides are 3, 4, and 5.

The same by Miss Mary Weston.

It is demonstrated in the answer to the 4th question, 1197, Ladies Diary, 1810, that $DB = \frac{2}{3} OC$. On the centres *i* and *m* erect the perpendiculars *io*, *mq*, and from the centre *o* of the quadrant, draw the radius of through the intersection *q*.



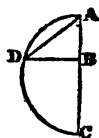
Then $EQ = OM - IE = \frac{2}{3} OC - \frac{1}{2} OC = \frac{1}{6} OC$,
 $GQ = OI - MG = \frac{1}{2} OC - \frac{1}{3} OC = \frac{1}{6} OC$,
 $FQ = OF - OQ = OC - OQ = OC - \frac{5}{6} OC = \frac{1}{6} OC$,
 (for $OQ = \sqrt{(OM^2 + MQ^2)} = \sqrt{[(\frac{2}{3} OC)^2 + (\frac{1}{3} OC)^2]} = \frac{5}{6} OC$.
 Consequently as the points of contact *E*, *F*, *G*, are equidistant from the common centre *Q*, they are in the circumference of a circle, the diam. of which is $\frac{1}{3} OC$ or $\frac{1}{2} DB$.—Hence the area of the circle *EFG* is half that of the semicircle *DGB*.

VI. QUESTION 1259, by Mr. John Collins, London.

Through what chord of a circle must a body fall, to acquire half the velocity gained by falling through the diameter.

Answered by Mr. Thomas Exley, Kingsdown, Bristol.

Draw the semicircle *ADC*, take $AB = \frac{1}{2} AC$, draw *BD* perpendicular *AC*, and join *AD*; then the velocity acquired in falling through *AD* is the same as that in *AB* (Dr. H.'s Course, Motion, prop. 27) $= \frac{1}{2}$ the velocity in falling through *AC* (prop. 18), theref. *AD* is the chord required.



The same by Mr. Alfred Fox, of Falmouth.

Let *AC* be the diameter of the given circle. Take $AB = \frac{1}{2} AC$, draw *BD* perpendicular to *AC*, and join *AD*, and *AD* will be the chord required, Dr. Hutton's Course, pr. 27, and pr. 6, cor. 2, on motion, &c.

Note. The chord *AD* is equal to the radius.

The same by Mr. T. Harleston.

Since the velocities acquired in falling down the chords of a circle, vary as their lengths, it must be that chord which is equal to half the diameter, or the chord of 60 degrees.

bola, when $os = sv = vi$. But by the circle $ns \times sg = as^2 = so \times si = so \times 2so$; theref. $ns = 2os$, or $os = \frac{1}{2}ns = \frac{1}{2}on$; theref. $sg = \frac{1}{2}on$, and $vg = \frac{1}{2}on$. Hence, by the parabola, $vs : vg = 2vs :: as^2 = \frac{1}{2}on^2 : fg^2 = \frac{1}{2}on^2$; and the content of the paraboloid $= \frac{2}{3} \frac{1}{2}on^3 \times 2618 = 59574$.

VIII. QUESTION 1261, by Mr. Wm. Burdon, *Acaster Malbis*.

A carpenter has a tree, the diameters of which are 5 and 2 feet, and its length 9 feet. He requests a theorem by which the tree may be cut, without completing the cone, so that it may make the greatest square beam possible.

Answered by Mr. James Beckerleg, Penzance.

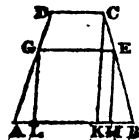
Let x denote the length to be cut off the smaller end, and $y + 2$ the diameter at that section: then as $9 : 3$ (the diff. of the diameters) $:: x : y = \frac{1}{3}x$; hence $(\frac{1}{3}x + 2)^2 \times (9 - x)$ is a maximum, or $8x - \frac{1}{3}x^3$ a max.; in fluxions, &c. this gives $x + 2x = 24$, whence $x = 4$ the distance required, or $9 - 4 = 5$, the length of the greatest square beam.

The same by Mr. Jos. Brewer, of Preston.

Put $a = 5$ the greater diameter, $b = 2$ the less, $d = a - b = 3$ their diff. and $l = 9$ the length of the tree, also $x =$ the length from the greater end where it must be cut. Then, by sim. figures, as $l : d :: x : dx \div l$ the diff. between the diameter at the greater end and the section; consequently $a - dx \div l =$ the diameter at the section, and $(a - dx \div l)^2 x$ a max. which in fluxions, &c. gives $x = al \div 3d$ (the theorem) $= 5 \times 9 \div 3 \times 3 = 5$ feet, the length required.

Answered by Mr. W. Burdon, the Proposer.

Let ABCD represent the tree, and LMEN the greatest parallelopipedon that can be cut from it. By sim. tri. $KB : KC :: HB : HE = CK . BH \div KB = CK . (AB - LH) \div 2KB$. Hence the content of the beam is $LH^2 . HE = LH^2 . (AB - LH) . CK \div 2KB =$ a maximum by the question or $LH^3 . (AB - LH)$ a maximum which gives $LH = \frac{1}{3}AB = 3\frac{1}{3}$ feet, each side of the ends of the beam, and $HE = \frac{CK . AB}{6KB} = \frac{CK . AB}{3(AB - DC)} = 5$ feet, the length of the beam.

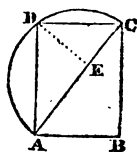


IX. QUESTION 1262, by Mr. William Bardon.

A block of marble, in the form of a square parallelopipedon, is cut diagonally into two equal parts, and the length of the plane of the section is 5 feet. Required the length of the piece, its solidity being the greatest possible?

Answered by Mr. D. T. Sheridan.

Let ABCD represent the marble block, AC the length of the diagonal section $= 5$, AB the length of the block, and AD the breadth and depth of the square end; then $AB \times BC^2$ is a maximum by the question, hence $BC^2 = 2AB^2$, by the theory of max. and min. consequently $AB = 5\sqrt{\frac{1}{3}} = 2.886$, and $BC = 5\sqrt{\frac{2}{3}} = 4.0825$.



Constr. Divide the given line AC, at E, in the ratio of 2 to 1; on E, erect the perpendicular ED, meeting a semicircle on AC in D; draw AD, CD, and complete the parallelogram ABCD, one side of the block, as is evident.

The same by Mr. Wm. Wiseman, of Hull.

Put x = the height, then $5^2 - x^2$ = the area of the end, and $25x - x^3$ = the content a max. therefore $25 - 3x^2 = 0$, and hence $x = 5\sqrt{\frac{1}{3}} = 2.886$, the length or height; also 4.082 the side of the square end.

X. QUESTION 1263, by Mr. J. Baines, jun. Horbury-bridge:

A given cone is immersed in water, with its vertex downwards; what part of the axis will be immersed, if the specific gravity of the fluid, be to that of the cone, as 8 to 5?

Answered by Mr. Wm. Birkin, of Derby.

Put the axis of the cone $= a$, and the immersed part of it $= x$, then will the whole cone and the immersed part be as a^3 to x^3 . But,

by the nature of specific gravities, the magnitude of the whole body, and that of the part immersed, are reciprocally as the specific gravities of the body and the fluid. Hence $a^3 : x^3 :: 8 : 5$, or $8x^3 = 5a^3$, and $x = a \sqrt[3]{\frac{5}{8}} = \frac{1}{2} a \sqrt[3]{5} = .8549a$.

The same by Mr. J. A. Dotchen, of Chelsea.

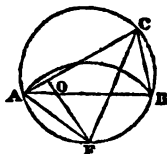
From the principles of hydrostatics, (see cor. 4, p. 220, vol. 2, Dr. Hutton's Course) the magnitude of the whole cone, is to the magnitude of the part immersed, as the specific gravity of the fluid to that of the body, or as 8 to 5; but the whole cone, and the part immersed being similar, their contents are as 8 to 5, therefore as $\sqrt[3]{8} : \sqrt[3]{5} :: 1 : \frac{1}{2} \sqrt[3]{5} = .854988$, which expresses the part of the axis immersed, when the whole is denoted by unity.

XI. QUESTION 1264, *by Mr. D. T. Sheridan.*

Given the area, the vertical angle, and sum of the including sides; to construct the plane triangle.

Answered by Mr. J. Collins, Hatton Garden.

Let ABC be the triangle required, inscribed in the circle ACBF, and CF a line bisecting the given vertical angle C, also FO perpendicular to AC. Then it is well known that CO is half the sum, and AO half the difference of the sides. Now, because the area and vertical angle are given, the rectangle of the sides is given also; and because the rectangle $AC \cdot BC = OC^2 - AO^2$, and that the half sum OC is given, therefore AO is given also, and consequently both AC and BC. Hence this



Constr. With CO half the given sum, and angle FCO = half the given one, construct the right-angled triangle FCO; take AO so that $AO^2 = OC^2 - AC \cdot BC$, and join AF; with centre F, and radius AF, describe the arc AB, to meet the line BC making the angle FCB = ACF; join AB, then ABC will be the triangle required, as is evident.

The same by Mr. John Dawes, Birmingham.

Let ACB be the triangle, circumscribed by a circle, and FH a dia-

sion, the common difference being 5° , and the least angle 120° .
 Quere how many sides has the figure?

Answered by Mr. J. Collins, Hatton Garden.

Since the inward angles increase, the outward angles must decrease by the same common difference. Now, by theor. 20, (Dr. Hutton's Geome. Course) all the outward angles of any plane figure are equal to four right angles $= 360^\circ$; hence 60 is the greatest term, 5 the common difference, and 360 the sum of the series, to find the number of terms, which denote by x ; then by nature of arithmetical proportion $125x - 5x^2 = 720$, or $x^2 - 25x = -144$; hence $x = 9$ the number of terms, which of course is = the number of sides required.

The same by Mr. M. Dawson, near Bromsgrove:

It is evident that the complements of the angles will form a decreasing series in arithmetical progression, the common difference being 5° , the first term 60, and the sum of the terms 360, or 4 right angles; whence we get the number of terms $= 9$, which is the number of sides.

The same by Mr. Joseph Hall, Hurdfield, near Macclesfield.

The sum of all the exterior angles of any rectilinear figure is $= 4$ right angles. Now, by the question, $180 - 120 = 60 =$ the first exterior angle; then $60 - 5 = 55$ the second, &c. Then we have given the greatest term $60 = a$, the common difference $5 = d$, and sum of the terms $= 360 = s$, to find the number of terms. Let x denote that number; then $a - d(x - 1) = a - dx + d$ the least term, and $2a - dx + d$ the sum of the extremes, hence $x(2a - dx + d) = 2s$ by the nature of the progression, the solution of which quadratic equation gives $x = 9$ the number of sides.

The same by Mr. W. D. Snooke, Woolbridge, near Wareham.

In any rectilinear figure, the sum of all the interior angles is equal to twice as many right angles, wanting 4, as the figure has sides: therefore let $x =$ the number of sides; then the sum of all the interior angles $= 180x - 360$; and by arithmetical progression that sum is also $= \frac{1}{2}(5x^2 + 235x)$, therefore $\frac{1}{2}(5x^2 + 235x) = 180x - 360$, hence $x = 9$ or 16: but it cannot be 16, for 120 being the least angle, and the common difference 5, the 12th angle $= 175$, the 13th would be 180° , which is absurd; therefore the figure has 9 sides.

XIII. QUESTION 1266, by Mr. John Smith, *Alton Park*.

Which is the greatest, and which the least, of these three means, viz. the arithmetical mean, the geometrical mean, and the harmonical mean, between the same two extremes, m and n ; and what ratio of ratios have those three means to each other?

Answered by Mr. John Davcy, St. Just.

Let m be the greater extreme, and n the less: then is the arithmetical mean $= \frac{1}{2}(m+n)$, the geometrical mean $= \sqrt{mn}$, and the harmonical mean $= \frac{2mn}{m+n}$; which three means are evidently in geometrical progression, viz. as $\frac{1}{2}(m+n) : \sqrt{mn} :: \sqrt{mn} : \frac{2mn}{m+n}$. Now,

$\sqrt{mn}$ being a geometrical mean between the other two, it can therefore be neither the greatest nor the least; so that we have only to find which is the greatest or least of the others. To effect which, I reduce them to a common denominator, and after taking their difference I find the arithmetical mean to exceed the harmonical mean by $\frac{(m-n)^2}{2(m+n)}$: consequently the former is the greatest and the latter is the least of the three proportionals:

The same by Mr. Alfred Fox, Falmouth.

Here $m+n$ is the arithmetical mean, $\sqrt{mn}$ the geometrical, and $\frac{2mn}{m+n}$ the harmonical mean. Now $\frac{1}{2}(m+n) - \sqrt{mn} = \frac{1}{2}(m - 2\sqrt{mn} + n)$, which being half the square of $(\sqrt{m} - \sqrt{n})$, is an affirmative quantity; therefore the arithmetical mean is greater than the geometrical. Again, $\sqrt{mn} \div \frac{2mn}{m+n} = \frac{m+n}{2\sqrt{mn}}$, which is more than 1, because $\frac{1}{2}(m+n)$ is more than $\sqrt{mn}$. Therefore the harmonical mean is the least, and the three are in geometrical progression, having the common ratio $\frac{2\sqrt{mn}}{m+n}$.

The same by Mr. Edward Riddle, Whitburn.

It is well known that $\frac{m+n}{2}$, $\sqrt{mn}$, and $\frac{2mn}{m+n}$ are the three means,

which are evidently in geometrical continued proportion, the common ratio being $\frac{2\sqrt{mn}}{m+n}$. Of those, the first or arithmetical mean is greater than the 2nd or geometrical mean; for if we take the diameter of a circle $= m+n$, the radius is the arithmetical and the sine the geometrical mean. Hence the arithmetical is the greatest of the three means, the harmonic the least, and the geometrical a mean proportional between them.

The same by Mr. Thomas Smith, of Hethersett.

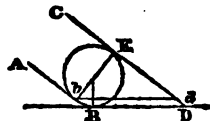
$\frac{m+n}{2}$, $\sqrt{mn}$, and $\frac{2mn}{m+n}$ are the three means, of which $\frac{m+n}{2}$ the arithmetical mean is the greatest, and $\frac{2mn}{m+n}$ the harmonic mean is the least (vide Hutton's Mathematical Dictionary, vol. 2, p. 88); and their common ratio $= \frac{m+n}{2\sqrt{mn}}$. For $\frac{2mn}{m+n} \times \frac{m+n}{2\sqrt{mn}} = \sqrt{mn}$, and $\sqrt{mn} \times \frac{m+n}{2\sqrt{mn}} = \frac{m+n}{2}$.

XIV. QUESTION 1267, by Mr. John Whitley, Rotherham.

What time, in the forenoon of the 16th of November, 1811, at Rotherham, in latitude $53^{\circ} 24'$ north, was the area of the shadow of a globe, placed on a horizontal plane, in the sunshine, equal to its superficial content?

Answered by Mr. J. A. Dotchen, Chelsea.

Considering the sun's rays as parallel, the shadow of the globe on the horizon will be an ellipse, being the section of the cylindrical shadow of the globe. And if AB , CD be the two extreme rays, in a vertical plane passing through the centres of the sun and globe, and meeting the horizontal plane in the points a , d ; then $ad = bd$ will be the transverse diameter of the ellipse, the conjugate of which will be $=$ the globe's diameter. Let this last be assumed $= 1$, then the superficies of the globe is $3.14159 =$ (per



question) the area of the ellipse, whence its transverse diameter must be $4 = ad = bd$ the cosine of the sun's elevation to radius 1, which by the tables is found $= 14^\circ 28' 39''$. The rest of the solution is a case of trigonometry, having given, in a spherical triangle zrs , $zr = 36^\circ 36'$ the colat. $rs = 108^\circ 36' = 90 +$ sun's south declination and $zs = 75^\circ 31' 21''$ his colat. to find the hour angle from noon $zrs = 26^\circ 25' 10''$, answering to 10hr. 14min. $19\frac{1}{2}$ sec. the time required before noon.

The same by Mr. Edw. England, Normanby School.

Let the diameter of the globe be denoted by 1; then its surface will be $3.1416 =$ the area of the ellipse; therefore $3.1416 \div .7854 = 4$ is its transverse diameter. Then as $4 : 1 :: \text{radius} : \frac{1}{4} = \text{sine of } 14^\circ 28' 39''$, the apparent altitude of the sun's upper limb; which being corrected for semidiam. parallax, and refraction, gives $14^\circ 8' 58''$ the true altitude of the sun's centre. Again, in a spheric triangle, there are given the sun's colat. $75^\circ 51' 2''$, the sun's declin. $18^\circ 36' S.$ and the lat. of the place $53^\circ 24'$, to find the hour-angle $= 27^\circ 39' \frac{1}{2}$, answering to 1h. 50min. 38sec. from noon, or 10h. 9min. 22sec. A.M. the time required.

The same by Mr. John Whitley, the Proposer.

Admitting that half the globe was illuminated by the sun's rays, the area of the shadow on the plane would be an ellipse, having its conjugate diameter equal to that of the globe $= c$, and let its transverse $= t$, also $n = .7854$; then its area will be $tcn = 4cn$ the superficies of the globe; hence $t = 4c$: therefore $t = 4c : c :: 4 : 1 :: \text{rad. } 1 : .25$ the nat. sine of $14^\circ 28'$ the sun's altitude at the time required. Hence, in a spheric triangle spz , we have given $sp = 108^\circ 36'$ the polar distance, $sz = 75^\circ 32'$ the sun's altitude, and $zp = 36^\circ 36'$ the colatitude to find the hour-angle $spz = 26^\circ 28'$, which gives the time required $= 10h. 14m. 8\text{seconds, A.M.}$

The same by Mr. James Wood, of Dudley.

Let $b\delta e$ be the globe, ad or bd the transverse axis of its elliptic shadow, the conjugate being $= be$ the globe's diameter, which put $= 1$; hence the transverse is found $= 4$; then in the triangle bed is found the angle $d = 14^\circ 28' 44''$ the sun's altitude. Hence, by an easy problem in spherical trigon. the time is found $= 10h. 14m. 39\text{secs. A.M.}$ But if we take refraction and parallax into the account, the true time will be found $= 10h. 13m. 39\text{sec.}$

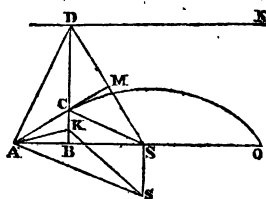
PRIZE QUESTION, *by Llerien.*

Given the base of an inclined plane, to find geometrically its angle of inclination to the horizon, such that a body projected up it, with a given velocity, may fall beyond it at the greatest horizontal distance?

N. B. This question, which was proposed in the *Diary* two years ago, is now re-proposed to be determined by geometrical construction.

Answered by Llerien, the Proposer.

Let AB be the given base, DB perpendicular to AB and $=$ the height due to the projectile velocity at A ; join AD , and in AB produced take $SA = AD$; join DS , and bisect it in M ; join AM cutting DB in C , which is the plane required.

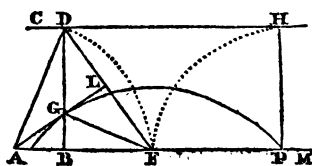


Join CS . Then the angles at M are equal, and therefore $CD = CS$, and the angle $DCM = SCM$. Now, because the velocity of the body at A is that acquired down DB , the velocity at C is that acquired down DC ; and DN , perpendicular to DB , is the directrix of the parabola described, and, from what has just been proved, S is its focus; taking, therefore, $SO = DB$, then O is the point where the body strikes the horizontal plane.—Now it is evident that, in any other plane, as AK , the focus of the parabola described cannot lie in AO , and therefore equally evident that, for the body to fall beyond O , the focus must lie beyond SS perpendicular to AS . But it cannot even lie so far as in it. For, if possible, let s be the focus, and join As : it will follow that $As = AD = AS$, which is impossible. Therefore AC is the plane required.

The same by Mr. Henry Milburn, Sunderland.

On the horizontal line AM cut off $AB =$ the given base of the inclined plane; at B erect the perpendicular $DB =$ the height due to the given velocity. Join AD , and bisect the angle DAB by the line AL , intersecting BD in G ; then is BAG the angle required.

For, on AM cut off $AP = AD + BP$; draw DH parallel to AB , and PH parallel to BD , intersecting each



other in H ; from centres A, P , with distances AD, PH , describe arcs to meet or intersect in F , and join PD and FG . Then is AB = the given base by construction, and because $AB = AF$, and the angle $DAL = FAL$, FD is bisected perpendicularly by AL ; consequently $GD = GF$; but $PF = PH$, therefore the points G and P are in the parabola, whose focus is F , and directrix BH ; and the velocity in G = to that of falling through GD , or in falling through CA and then ascending up AG : also since $AP = AD + BD = AD + PH$, the arcs DF, HF will touch each other in the point F ; therefore F is the maximum point: for if P were in any other point of the line AM , these arcs would cut each other both above and below the line AM , or else not meet at all.

The same by Mr. Edward Riddle, Whitburn.

At the extremity B , of the given base AB , erect the perpendicular BD = the altitude due to the given velocity, and draw DH parallel to the horizontal line AM ; then DH will be the directrix of the parabola. With centre A and radius AD describe the arc DF , join DF , and upon it let fall the perpendicular AGL ; then will BAG be the required angle. For $DL = LF$, and GL common to the triangles DGL, GLF , and the angles at L equal, therefore $DG = GF$, and the angle $LGF = LGB$, and this again is $= AGB$. Therefore the point G is in a parabola, of which the directrix is DH , focus F , and AG is a tangent: Take $PF = BD$, then P is the point through which the body will pass on the horizontal line, and it is evident that the distance BF is greater than if the focus had been in any other part of the arch DF .

The same by Mr. Wm. Wright, Bonby, near Barton.

Analysis. Imagine the problem solved; AB the given base, AG the inclined plane, P the point of fall on the horizontal plane; through G draw BD perpendicular to AP making BD = the height due to the given velocity; parallel to AP draw DH the directrix of the parabolic path of the projectile. Now if F be the focus of the parabola: then by conics $GF = GD$; join DF , and produce AG to cut DF in L . Now AL is a tangent to the parabola at G , and by conics the angle $DGL = FGL$, $DL = FL$, and AL is perpendicular to DF , hence $AF = AD$, but AD is evidently given; draw PH parallel to BD , then $PF = PH$ is known, consequently when P is given, the intersections of circles described on the centre A, P , with the radii AD, PH , will determine F , and it is obvious that when these arcs only touch (which they will do in the point where the arc DF cuts AP) that AP is then a maximum. Hence the following

Construction. With the centre A , and radius AD , describe an arc, cutting AP in F ; join DF , and perpendicular to DF draw AL , and the angle BAL is the inclination sought,

Questions proposed in 1814, and answered in 1815.

1. QUESTION 1269, by Mr. Joseph Williams, Canterbury.

If a globe, of $1\frac{1}{2}$ foot diameter, be put to float in common water; required the area of the section at the surface of the water, when the specific gravity of the globe, to that of water, is as 3 to 5?

Answered by Mr. T. Harleston.

Put d = diam. of globe, and x = height of segment without.

Then sol. of globe : sol. of segment without :: $d^3 : (3d - 2x)x^2$.

But sol. of globe : sol. of seg. without :: 5 : 2.

Theref. $d^3 : (3d - 2x)x^2 :: 5 : 2$; hence $10x^3 - 15dx^2 + 2d^3 = 0$;

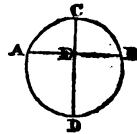
Or $40x^3 - 90x^2 + 27 = 0$; therefore $x = .65$; and 1.7357 feet the required area.

The same by Mr. W. G. Horner, Bath.

Let diameter = $d = 18$ inches; x = height of the segment dry. Then whole globe : dry segm. :: $d^3 : 3dx^2 - 2x^3 :: 5 : 2$; therefore $15dx^2 - 10x^3 = 2d^3$, or $x^3 - \frac{3}{2}dx^2 + \frac{1}{2}d^3 = 0$; hence $x = .43293d$; and $.43293 \times .56707 \times 3.1416 \times 18^2 = 249.89$ square inches, is the area required.

The same by Miss Susannah Jackson, Mile End.

By the question $18^3 \times .5236 \times \frac{2}{3} = 1832.18112$, the part immersed; and by Dr. Hutton's Dictionary, word segment, $.5236 x^2 (3d - 2x) = 1832.18112$, hence $x = 10.20724 = DE$ by approximation. Then $\sqrt{(CE \times ED)} = AE = 8.91866$; therefore the area of the section AB is = 249.88 inches.



The same answered by Mr. Wm. Wiseman, Hull.

Put x = the depth of the wet segment, $a = 3.1416$, and $d = 1\frac{1}{2}$; then $(3d - 2x) \times \frac{1}{2}ax^2$ = the content of the segment, and $\frac{1}{2}ad^3$ = the content of the globe; but, by the question, $(3d - 2x) \times \frac{1}{2}ax^2 \times 5 = \frac{1}{2}ad^3 \times 3$; hence $x = .8506$, and $(d - x) \times x \times a = 1.7353$ feet, the area required.

The same answered by Mr. Joseph Williams, of Neath.

Let $d = cb$, the whole diameter $= 1\frac{1}{2}$, $n = .5236$, and $x = dx$ the depth of the immersed segment. Then (by 323, Dr. Hutton's Hydrostatics) the segment's content $= \frac{1}{2}nd^3 =$ (by Dr. Hutton's first rule for spherical segments) to $nx^3 (3d - 2x)$; from which equation (by trial-and-error) $x = .85060$; hence $4(dx - x^3) \times .7854 = 1.735357$, the area required.

II. QUESTION 1270, by Mr. Tho. Smith, of Hethersett.

Given $5x^3 + y^3 = z^3$; to find x, y, z , in the least whole numbers?

Answered by Mr. John Craggs, of Hilton.

Put $rx^3 = y^3$, and $sx^3 = z^3$; then $5x^3 + rx^3 = sx^3$; this, divided by x^3 , gives $x = (5 + r) \div s$. If $r = 49$, and $s = 27$, then $x = (5 + 49) \div 27 = 2$, $z = 6$, $y = 14$, the numbers required.

The same by Mr. Wm. Makinson, Manchester.

To reduce the formula $ax^3 + cy^3 = z^3$, Euler furnishes us with the equations $z = ap^3 - 3cpq^2$, and $y = 3ap^2q - cq^3$; then $a = 5$, $c = 1$, and putting $p = 1$, and $q = 1$, then $x = 2$, $y = 14$, and $z = 6$.

After a similar Solution by Mr. Henry Atkinson, this gentleman says,

If it be admitted that x, y, z , may be equal, the value in that case

to equations of the form $ax^n = cx^m$, is $\left(\frac{a+b}{c}\right)^{\frac{1}{m-n}} = 6$, in the present case; so that x, y, z is each of them 6.

The same by Mr. Henry Milburn, Sunderland.

Put $rx = y$, and $sx = z$; then $5x^3 \div r^3x^3 = s^3x^3$, and $x = (5 + r^3) \div s^3$ a whole number. If r and s be taken each $= 1$; then x, y, z , will be each $= 6$; but if r be made 7, and $s = 3$; then will $x = 2, y = 14, z = 6$.

The same answered by Mr. S. Treeby, Schoolmaster, Plymouth.

The readiest way to solve questions of this nature is by the exten-

sive table of squares, cubes, and roots, in vol. 1, of Dr. Hutton's Mathematics; from which it soon appears that $x = 2$, $y = 14$, $z = 6$, the least possible whole numbers.

III. QUESTION 1271, by Mr. A. Nesbit, *Land Surveyor, Farnley.*

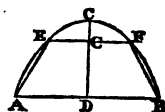
What is the area of the greatest trapezoid that can be described in an hyperbola, whose transverse axis is 120, the ordinate 48, and the abscissa 40 chains?

Answered by Mr. Henry Atkinson, Newcastle.

The conjugate is readily found to be 72. Now if t and c represent the semi-transverse and semi-conjugate diameters, and x the abscissa counted from the centre, the ordinate will be $(c \div t)\sqrt{(x^2 - t^2)}$, and the area of the trapezoid $[(c \div t)\sqrt{(x^2 - t^2)} + b] \times (t + a - x)$, putting $a =$ the given abscissa 40, and $b =$ the ordinate 48, this being a maximum, its fluxions being put $= 0$, gives $(t + a)cx - 2cx^2 + ct^2 - bt\sqrt{(x^2 - t^2)} = 0$; from which x is found $= 64.068317$ chains, and the area 2209.08 chains $= 220$ acres, 3 roods, 25.28 perches.

The same by Mr. Teasdale Bell, of Slaley.

Put $GD = x$; then $CG = 40 - x$, and by the property of the curve, $\sqrt{(120 + 40)40} : 48 :: 120 : 72 :: \sqrt{[(40 - x) \times (160 - x)]} : \frac{3}{2}\sqrt{(x^2 - 200x + 6400)} = EG$; consequently $(AD + EG) \times GD = 48x + \frac{3}{2}x\sqrt{(x^2 - 200x + 6400)}$ is the area, a maximum; this in fluxions, and reduced, gives $x^3 - 300x^2 + 27300x - 640000 = 0$, or $x = 35.9316$, and hence $(AD + EG) \times GD = 2209.078936$ square chains, the area, as required.



The same answered by Mr. T. Harlestone.

Put the semi-axis minor $36 = b$, semi-axis major $60 = a$, abscissa $cd = c$, and ordinate $= d$, also $x =$ abscissa to g from the centre. Then $GD = a + c - x$, and $EG = (b \div a)\sqrt{(x^2 - a^2)}$; whence $(AD + EG) \times GD =$ maximum, or $[(a + c)b \div a]\sqrt{(x^2 - a^2)} - dx - (bx \div a)\sqrt{(x^2 - a^2)} = \text{max.}$; the fluxion of which put $= 0$, gives $(a + c)x - (ad \div b)\sqrt{(x^2 - a^2)} - (2x^2 - a^2) = 0$; hence $x = 64.07$, and the area $= 2209.05$ chains.

IV. QUESTION 1272, by Mr. John Epps, *London.*

A certain sum of money consists of x pounds y shillings, but its half

consists of y pounds x shillings; quere the sum? Also if, instead of the half, any other part should be taken, whether a similar equality could be made to appear?

Answered by Mr. W. Burdon, Acaster Malbis.

By the question, $(20x + y) \div n = 2y + x$, therefore $(20 - n)x = (20n - 1)y$, or $x : y :: 20n - 1 : 20 - n$; hence, by expounding n by 2, 3, 4, &c. the ratio of x to y , expressed in numbers each less than 20, will be when $n = 2$, or 6. For 13l. 6s. divided by 2, gives 6l. 13s. and 17l. 2s. divided by 6, gives 2l. 17s. the only sums that will answer.

The same by Mr. J. A. Dotchen, Chelsea.

By the question, $20x + y = 2(20y + x)$ or $40y + 2x$, hence $18x = 39y$, or $6x = 13y$, and $x : y :: 13 : 6$; therefore, the sum is 13l. 6s. If the n th part of the sum be taken, then $x : y :: 20n - 1 : 20 - n$, and a similar relation to the first cannot subsist unless this ratio admit of being expressed by whole numbers that are both less than 20. Now, putting $n = 3, 4$, &c. successively, to 10, which is as far as necessary, (because if n be greater than 10, the ratios produced are all greater than that of $20 - 1$); we find that the only ratio that can be expressed by numbers below 20, is when $n = 6$, for then $20n - 1 : 20 - n :: 119 : 14$, or as $17 : 2$; and thus the sixth part of 17l. 2s. is 2l. 17s.

The same by Mr. Alfred Fox, of Falmouth.

By the question $\frac{1}{2}(20x + y) = 20y + x$, which equation reduced gives $6x = 13y$, which gives $x = 13$, and $y = 6$, and the sum required, is 13l. 6s. the half of which is 6l. 13s. Or, more generally, instead of half, if the n th part be taken we get $x = \frac{20n - 1}{20 - n}y$; where, by substituting 1, 2, and 6, for the values of n , we shall have 19l. 19s. and 13l. 6s. and 17l. 2s. which are the only sums that will properly answer the conditions of the question.

v. QUESTION 1273, *by Mr. John Davey, of St. Just, in Penwith.*

A gentleman desired his goldsmith to make him a hemispherical bowl, $\frac{1}{16}$ th of an inch thick throughout, and of such other dimensions, as to require seven times its weight put into it, to immerse it in common water, just to the brim; but when made, it was found on trial to require one drachm avoirdupois more than the weight proposed. Now,

if the specific gravity of the metal was 10 times that of the water, what was the internal diameter of the bowl?

Answered by Mr. John Charlton, North Shields.

Put $\frac{1}{16}$ inch = $\frac{1}{2}a$, $\cdot 5236 = n$, and x the internal diameter of the bowl; then $\frac{1}{2}(3x^2a + 3xa^2 + a^3)n \times \frac{100000}{1728}$ is the weight of the bowl, and $\frac{1}{2}(x^3 + 3x^2a + 3xa^2 + a^3)n \times \frac{1000}{1728} - \frac{1}{16}$ is the weight of the water displaced = $\frac{1}{2}(3x^2a + 3xa^2 + a^3)n \times \frac{100000}{1728} \times 8$, by the question, hence $x^3 - 237ax^2 - 237a^2x = 79a^3 + \frac{2}{123}n$, and $x = 11\cdot9027602$ the internal diameter.

The same answered by Mr. Anthony Cook, of Wooley, near Hexham.

Put $\frac{1}{16}$ inch = a , $\cdot 5236 = c$, $\frac{259}{2}$ drams = the weight of a cubic inch of water = w , that of the metal = $10w$, and x = the internal diameter of the bowl, then $(x+2a)^3 \times \frac{1}{2}c = \frac{1}{2}c(x^3 + 6ax^2 + 12a^2x + 8a^3)$ is the whole solidity of the bowl, and also of the water displaced, $\frac{1}{2}cx$ = the concavity; their difference is $(3ax^2 + 6a^2x + 4a^3)c$ = the solidity of metal; hence (by prop. 64, cor. 3, vol. ii. Dr. Hutton's Course) we have $\frac{1}{2}(x^3 + 6ax^2 + 12a^2x + 8a^3)cw = ((3ax^2 + 6a^2x + 4a^3)10cw \times 8) + 1$: this reduced, gives $x^3 - 474ax^2 - 948a^2x = 632a^3 + (2 \div cw) = \cdot 422395$. Hence $x = 11\cdot90276$ inches, the diameter sought.

The same answered by Miss Susannah Jackson, Mile End.

Put x = the extreme or outer diameter, and $a = \cdot 2618$; then ax^3 = the solidity of the hemisphere, and $\cdot 1515x^3$ = the weight, also $(x - \frac{1}{16})^3 \times a$ = the content of the vacuity, and $ax^3 - a(x - \frac{1}{16})^3$ = the bowl, which multiplied by $5\cdot 787 \times 8 + 1$ dram, gives $x^3 - 12x^2 + \cdot 5x = \cdot 422541$; hence $x = 11\cdot95275$, and $x - \cdot 05 = 11\cdot90275$, the internal diameter of the bowl.

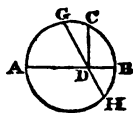
VI. QUESTION 1274, by Mr. John Dawes, Birmingham.

Required to determine geometrically, by two lines in a circle, the arithmetical, the geometrical, and the harmonical mean, between two given lines?

Answered by Mr. Teasdale Bell, of Slaley.

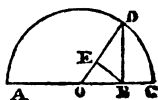
On AB the sum of the 2 lines AD, DB, as a diameter, describe a circle;

erect the perpendicular DC ; apply $DC = \frac{1}{2}AB$, or the radius, and produce GD to the circle at H . Then it is evident that the radius DC is the arithmetical mean, DC the geometrical mean, and DH is the harmonical mean, because $GD \cdot DH = AD \cdot DB = DC^2$; see theor. 61, 87, Dr. Hutton's Geom. with the solution to question 13, last year's Diary.



The same answered by Mr. Tho. Mundell, Three-mile Bridge.

The two given lines being AB, BC , then it is known that their half sum AO , or OC, OD is the arithmetical mean. On the diameter AC describe the semicircle, and erect the perpendicular BD , which it is known is the geometrical mean. Then erect BE perpendicular to OD , so will DE be the harmonical mean; because, by the equiangular triangles DBO, DBE , it is $DO : DB :: DB : DE$.



VII. QUESTION 1275, by Mr. D. T. Sheridan.

In any plane triangle, it is said that the three differences of the segments, made by the perpendicular, and by the contact of the inscribed circle, also by the line bisecting the vertical angle, are in continued geometrical proportion; required the demonstration?

Answered by Mr. Henry Atkinson, Newcastle.

If a, b, c , denote the sides of the triangle, of which b is the base, it is well known that the difference of the segments made by the perpendicular, is $\frac{a^2 - c^2}{b}$; the difference of the segments made by the contact of the inscribed circle is $a - c$, and the difference of the segments made by a line bisecting the vertical angle is $\frac{a - c}{a + c}b$. But $\frac{a^2 - c^2}{b} : a - c :: a - c : \frac{a - c}{a + c}b$.

The same answered by Mr. John Charlton, North Shields.

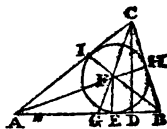
By theorem 3, Trigonometry, as $AB : AC + CB :: AC - CB : AD - DB = \frac{AC^2 - CB^2}{AB}$; and since CF bisects the $\angle ACB$, $CI = CH$, for the like reason $AI = AE$, and $BH = BE$; therefore $AE - EB = AC - CB$;

also by theor. 83, &c. Geom. as $AC + CB : AC - CB$

$:: AG + GB$, or $AB : AG - GB = \frac{AC - CB}{AC + CB} AB$. But

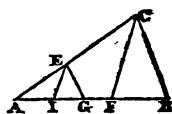
$$\frac{AC^2 - CB^2}{AB} : AC - CB :: AC - CB : \frac{AC - CB}{AC + CB}$$

$$\times AB = AG - GB.$$



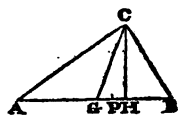
The same answered by Mr. John Davey, of St. Just.

In any triangle ACB , let AE be the difference of the sides, which is also equal to the difference of the segments made by the contact of the inscribed circle, as is well known; AG the difference of the segments made by the perpendicular; CF the line bisecting the vertical angle; EI drawn parallel to CF , and GE joined. Then by Dr. Hutton's Geom. theor. 83, as $AC : BC :: AF : BF$, and by division, as $AC : AC - BC (AE) :: AF : AF - BF$; but, by similar triangles, as $AC : AE :: AF : AI$; hence by equality $AI = AF - BF$; and since ACF , AGE , AEI are similar triangles, it will be as $AG : AE :: AE : AI$, and the proposition is manifest.



The same answered by Mr. John Whitley, Rotherham.

In the plane triangle ACB , let CG bisect the vertical angle C , draw the perpendicular CH , and let P be the contact of the inscribed circle with the base AB . Then, (Geom. 83,) $AG : GB :: AC : CB$, and by composition, division and alterna. $AB : AC + CB :: AG - GB : AC - BC$; but (theor. 3, Trigon. vol. ii.) $AB : AC + CB :: AC - CB : AH - BH$; therefore, by equality, $AG - GB : AC - BC :: AC - BC : AH - BH$; but, by a known property, $AC - BC = AP - PB$; therefore, $AG - GB : AP - PB :: AP - PB : AH - BH$. Q. E. D.

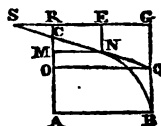


VIII. QUESTION 1276, by Mr. Geo. Ducket, of Northwich.

The middle of a tube of 10 feet long, fixed to the top of a pole, 8 feet in height, above a horizontal plane; quere its position, so that a ball being put in at its upper end, and descending by the force of gravity, may strike a point on the horizontal plane at the distance of 20 feet from the foot of the pole?

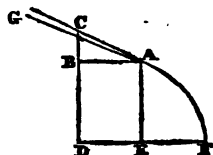
Answered by Mr. Anthony Cook, Wooley, near Hexham.

Let AC be the pole, perpendicular to the horizon AB , and scn the tube; then, after the ball quits the tube at N it will describe the parabolic curve NB , whose directrix is SRG parallel to AB : produce AC to R , and SN to Q forming NQ the tangent to the curve NB : draw BQG and NE parallel to AC , also MN and OQ parallel to AB . Put $AC = a$, $AB = b$, $SC = CN = c$, and $CQ = x$: then, $NQ = x - c$, $OC = \sqrt{(x^2 - b^2)}$, and by similar triangles, $cq : co :: cn : cm = (c \div x)\sqrt{(x^2 - b^2)} = \frac{1}{2}EN$; then, by prop. 20, cor. 3, vol. 2, of Hutton's Course, $EN \cdot 4BQ = NQ^2$, that is, $(2c \div x)\sqrt{(x^2 - b^2)} \times (4a - 4\sqrt{(x^2 - b^2)}) = (x - c)^2$, or reduced, $8ac\sqrt{(x^2 - b^2)} - 6cx^2 - x^3 - c^2x = -8b^2c$; an equation from which x may be readily found.



The same by Mr. Geo. Duckett, Northwich.

Let AC denote the tube, CD the pole, and P the given point in the horizontal plane. Draw AE perpendicular, and AB parallel to the horizon. Put $a = 5 = AC$, $b = 8 = CD$, $c = 10$ (instead of 20) = DP , $s = 16\frac{1}{2}$ feet, and $x = BC$. Then, by the laws of falling bodies, (schol. prop. 18, vol. 2, of the Course) the velocity of the ball at A , in direction CA , is $2\sqrt{2sx}$, and its relative velocities perpendicular and parallel to the ho-



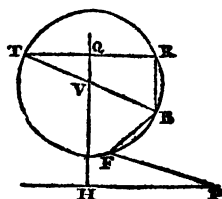
zison are $\frac{x}{a} \times 2\sqrt{2sx}$ and $\frac{\sqrt{(a^2 - x^2)}}{a} \times 2\sqrt{2sx}$, the last being constant after quitting the tube. Again, (by the same) $\frac{2x^3}{a^3}$ denotes the space a body must fall through from rest to acquire the velocity $\frac{x}{a} \times 2\sqrt{2s}$, and $\frac{x}{a} \sqrt{\frac{2x}{s}} =$ the time of falling. Hence the time of descending through AE ($b - x$) is $\frac{\sqrt{(a^2b - a^2x + 2x^2)} - x\sqrt{2x}}{a\sqrt{s}}$, which multiplied by $\frac{\sqrt{(a^2 - x^2)}}{a} \times 2\sqrt{2sx}$, gives $\frac{2\sqrt{(a^2 - x^2)}}{a} \times (\sqrt{(2bx - 2x^2 + \frac{4x^3}{a^2})} - \frac{2x^2}{a}) = EF$; therefore $\frac{2\sqrt{(a^2 - x^2)}}{a} \times (\sqrt{(2bx - 2x^2 + \frac{4x^3}{a^2})} - \frac{2x^2}{a} + \frac{a}{x}) = DP = c$; or putting $z = \frac{x}{a}$ = the sine of angle BAC , and $na = b$, it is $\sqrt{(1 - z^2)} [\sqrt{(2nz - 2z^2 + 4z^3)} - 2z^2 + 4z^3]$

$-2z' + \frac{1}{2}] = c \div 2a = 1$; this equation solved gives $z = .5048$ the sine $30^\circ 19'$, or $= .0897$ the sine $5^\circ 9'$, either of which angles the tube may form with the horizon for the position required.

Note. No position of the tube of the given length and height can cause the ball to strike 20 feet from the foot of the pole. For, putting the fluxion of $\sqrt{(1 - z^2)} \times (\sqrt{(2nz - 2z^2 + 4z')} - 2z' + \frac{1}{2}) = 0$, we find $z = .2605$, which gives $DP = 11.688$ feet, the maximum.

The same by Mr. W. G. Horner, Bath.

Let HV denote the pole perpendicular to the horizon HP , and TVB the tube inclined in an angle BTB to the horizon: draw BR perpendicular to TV ; then, because the velocity of the ball at B is the same as if it had fallen down RB , TB is the directrix of its parabolic trajectory. Draw $BF = BR$, and making the angle $TBF = TBR$, and F is the focus. Make $FF = HQ$, and P will be the point of impact.



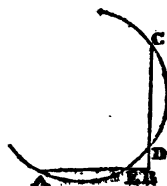
From the construction it is manifest that T, B, R, F are all in the periphery of a circle, whose diameter is TR , and centre V . And the impossibility of the ball's reaching the point assigned in the question appears from the simple consideration that the least distance of that point from the circle of foci is $\sqrt{(20^2 + 8^2)} - 5 = 16.54$ feet; whereas the greatest value of FF or HQ is $8 + 5 = 13$.

IX. QUESTION 1277, by Mr. Wm. Burdon, *Acaster Malbis.*

Two persons observing an object, 40 feet in height, placed on the top of York Minster, the one stood $66\frac{1}{2}$ yards, and the other $86\frac{1}{2}$ yards from it, when the object appeared to them both of the same magnitude. Required the height of the Minster, supposing the height of each person's eye to be $5\frac{1}{2}$ feet.

Answered by Mr. T. Bell, Slaley:

By the principles of Optics the object CD will appear of the same magnitude, and under the same angle, to the eyes at A and E , when the four points A, E, D, C , are in the circumference of a circle: hence by the property of the circle, $BD^2 + BDC = ABE$; the solution of which quadratic gives $BD = 207\frac{1}{2}$, or $BC + 5\frac{1}{2} = 213$ feet, as required,

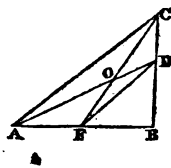


The same by Mr. John Collins, St. Andrew's School.

Let BD denote York Minster, DC the object, A and E the places of observation, and through the four points A, E, D, C , describe a circle, by the property of which $BA \times BE = BC \times BD$, all of which are given, except BD , for which put x , also $BE = 66\frac{1}{2}$ yards $= 198.6$ feet $= a$, $AB = 86\frac{1}{2}$ yards $= 258.6$ feet $= b$, and $DC = 40 = c$; then $BC = c + x$, hence the above equation is $x^2 + cx = ab$, or $x^2 + 40x = 51357.26$, from which $x = 207.5037$, to which adding $5\frac{1}{2}$ gives the height of the eye, 213.0037 feet $= 71.0123$ yards, the height required.

The same answered by Mr. John Davey, St. Just.

Suppose $BD =$ the height of the Minster, $CD =$ the object upon it, and A and E the places of the observers. Put $AB = 258.6$ feet $= a$, $EB = 198.6 = b$, $CD = 40 = c$, and $BD = x$. Then, since $\angle CAD = \angle CBD$, and $\angle COA = \angle DOF$, it follows $\angle ACF = \angle ADF$; hence the four points A, E, D, C , are in the circumference of a circle; therefore, by prop. 61, Geom. in the Course, $BC \times BD = AB \times EB$, or $x^2 + cx = ab$; whence is found $x = \sqrt{(\frac{1}{4}c^2 + ab)} - \frac{1}{2}c = 207\frac{1}{2}$ nearly, which added to $5\frac{1}{2}$ the height of the eye, gives 213 feet $= 71$ yards, for the height sought.

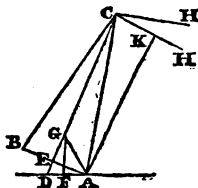


X. QUESTION 1278, by Mr. John Whitley, Rotherham.

A cone of marble, the axis of which is 20 feet, and base diameter 6 feet, stands on the edge of its base, the axis making an angle of 60° with the plane of the horizon: what must be the direction and quantity of the least power, applied to its vertex, that will just sustain the cone in that position?

Answered by Mr. Wm. Burdon, Acaster Malbis.

Let ABC be the given cone, CD the axis produced to the horizontal line AD , making the angle $= 60^\circ$, G the centre of gravity of the cone, and CH the direction of the power. Draw GF perpendicular to AD , and AK to CH . Then, by the property of the bended lever, $AF \times W = AK \times P$, where w denotes the weight of the cone, and P the required power. But AF and w are given, therefore the power P will be the least when AK is the greatest, that is, when $AK = AC$, or when the direction CH is perpendicular to the slant side AC .



Calculation. In the triangle ADE is found $AD = 2\sqrt{3}$, and $DE = \sqrt{3}$, also $EG = \frac{1}{2}CE = 5$, therefore $DG = 5 + \sqrt{3}$. Again, by similar triangles, $AD : DE :: DG : DF = \frac{1}{2}(5 + \sqrt{3})$, and therefore $AF = AD - DF = \frac{1}{2}(3\sqrt{3} - 5) = .0980762$. Also, $w = 284$ cwt. the weight of the cone, and $AC = \sqrt{409} = 20.2237484$: hence $AC : AF :: w : P = 1.37727$ cwt.

The same answered by Mr. Henry Milbourn, Sunderland.

Let ABC represent the section of the cone, CE its axis, G its centre of gravity, CH the direction of the power P, and AK perpendicular to it, and GF perpendicular to the horizontal line AD. Put $a = AE = 3$, $b = GE = 5$, and $x = DE$; then, since the angle ADE = 60° , and angles DAE and DGF are each = 30° ; therefore $AD = 2DE = 2x$, and $DF = \frac{1}{2}DG = \frac{1}{2}(b + x)$; but $AE^2 + DE^2 = AD^2$; that is, $a^2 + x^2 = 4x^2$, or $x = \frac{1}{3}a\sqrt{3} = \sqrt{3}$; hence $DF = \frac{1}{2}(5 + \sqrt{3})$, and $AF = \frac{1}{2}(3\sqrt{3} - 5) = .098087$. Now, since the power P, and the weight w, of the cone, acting in the line GF, are in equilibrio on the point A, we have, by the property of the bended lever, $AK : AF :: w : P$, or $AK \times P = AF \times w$; hence, since AF and w are given, the product $AK \times P$ is given; consequently the power P decreases as the perpendicular AK increases; therefore P will be the least when AK is the greatest, that is, when AK is equal to and coincides with AC; hence the direction CH of the power, when least, is perpendicular to AC the side of the cone. To determine P, we have $AF = \frac{1}{2}(3\sqrt{3} - 5)$, $AC = \sqrt{409}$, and $w = 284$ cwt. then as $AC : AF :: w : P = 1.3774$ cwt. the power required.

The same answered by Mr. John Whitley, of Rotherham.

Let ABC be the given cone, CE its axis, in which take $EG = \frac{1}{2}EC = 5$, then G will be its centre of gravity: draw GF perpendicular to the horizon AD, and AK perpendicular to CH, the direction of the power P, and produce CE to meet the horizon in D. Then, by trig. sin. $\angle D(60^\circ) = \frac{1}{2}\sqrt{3} : AE = 3 :: \text{rad. } 1 : AD = 2\sqrt{3}$: hence $DE = \frac{1}{2}AD = \sqrt{3}$, $DG = DE + EG = \sqrt{3} + 5$, $DF = \frac{1}{2}DG = \frac{1}{2}\sqrt{3} + \frac{5}{2}$, and $AF = DA - DF = \frac{1}{2}\sqrt{3} - \frac{5}{2} = .098076$. Now $AC = \sqrt{409}$, and the weight w of the cone = 284 cwt. very nearly; and since this weight, acting in the line GF, and the power P, acting in the direction CH, are in equilibrio on the point A, therefore by mechanics (see prop. 31, vol. 2, Dr. Hutton's Course) $AK : AF :: w : P$; now, as the two middle terms are constant, the last term varies invariably as the first; and since the last term P is to be a minimum, the first AK will be a maximum, which is evidently the case when it is equal to and coincides with AC; consequently the direction of the power, when least, is perpendicular.

lar to the side AC of the cone; hence $AC : AF :: W : P = 1.37727$
 cwt. = 1 cwt. 1 qr. 14½ lbs. the power required.

XI. QUESTION 1279, *by Mr. James North.*

It having been determined by Dr. Hutton, (in his new Tracts, vol. 2, page 62,) that the mean density of the whole mass of the earth, is to that of water nearly as 5 to 1, or to that of the rocks at the surface as 9 to 5; it is required then to determine the ratio in numbers, by which the density decreases from the centre to the circumference, supposing it to decrease as some power of the radius, viz. as r^n ; or to determine the numeral value of the exponent n , the radius being r ?

Answered by Mr. J. A. Dotchen, of Chelsea.

If we take the first proportion mentioned in the question, and call 1 the density at the surface of the earth, the value of n is already computed at cor. 2, page 325, vol. 3, Dr. Hutton's Course, it being $= \frac{1}{3}$, and the density decreases from the centre "inversely as the $\frac{1}{3}$ ths power of the radius;" but, if we consider $\frac{1}{3}$ ths of the mean density, as stated in the question, and as determined in page 62, vol. 2, of the

Tracts, then we have $\frac{2}{3} = \frac{3}{n+3}$, which gives the value of $n = -\frac{4}{3}$.

According to this supposition, therefore, the law of density is inversely as the $\frac{1}{3}$ ths power of the distance from the centre.*

* This opportunity may be taken to correct the numbers in the second corollary in the third vol. above referred to, where the number 5 has been inadvertantly used in the third line of the corr. instead of $\frac{2}{3}$, by which change the last three lines of the corollary will be thus :

" $\frac{2}{3}d = \frac{3d}{n+3}$, or $\frac{2}{3} = \frac{1}{n+3}$, and hence $n = -\frac{4}{3}$. So that in

this case the law of decrease is as $r^{\frac{4}{3}}$, or as $\frac{1}{r^{\frac{4}{3}}}$, that is inversely as

the $\frac{1}{3}$ ths power of the distance." By which those who possess the book may correct that passage in the said second corollary.

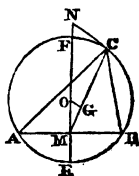
XII. QUESTION 1280, *by Mr. A. Glendenning, Yarmouth.*

Given the base, the vertical angle, and the distance of the centre of the circumscribing circle from the centre of gravity; to construct the plane triangle.

Answered by Mr. J. Cunliffe, jun. Bolton-le-Moors.

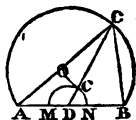
On the given base AB describe the segment of a circle to contain the

given vertical angle, and draw the diameter EF , bisecting AB in M , the point O being the centre of the circle. On MF take $MN = 3MO$, and apply $NC = 3$ times the given distance of the centres; then draw AC and BC , and ABC will be the required triangle. For, draw MC , and OG parallel to NC . Then because MN is $3MO$, and OG parallel to NC , by similar figures $MC = 3MG$, and $OG = \frac{1}{3}NC$ = the given distance. Therefore G is the centre of gravity of the triangle, and is at the given distance OG .



The same by Mr. Alfred Fox, of Falmouth.

On the given base AB , bisected in D , describe a segment of a circle to contain the given vertical angle; and on $MN = \frac{1}{3}AB$, and also bisected in D , describe a similar segment MCN , to which, from O , the centre of the former, apply OC = the given distance of the centre of gravity; join DC and produce it to the outer circle at c ; join AC , BC , so shall ABC be the required triangle, and c its centre of gravity. For the triangle is on the given base AB , and has the given vertical angle c by the construction. Also all the lines of the less circle are equal to $\frac{1}{3}$ of the similar lines in the greater, therefore $DC \approx \frac{1}{3}dc$, and consequently c is the centre of gravity (by Dr. Hutton's Course, prop. 43, Mechan.) and it is at the given distance OC by the construction.

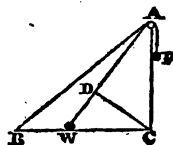


XIII. QUESTION 1281, by Mr. Richard Taylor, Carlisle.

$BC = 20$ feet is a horizontal plane; $AC = 15$ feet is perpendicular to BC ; w and P are two equal weights, connected by a string passing freely over the point A : determine the velocity with which w , drawn by P along the horizontal plane, arrives at C , as also the time in motion.

Answered by Mr. John Davey of St. Just.

In the annexed figure join AB , and draw CD perpendicular to AW . Put $s = AB = \sqrt{(AC^2 + BC^2)} = 25$, the length of the string, $a = AC = 15$, u = the velocity of P , v = that of w , $g = 16\frac{1}{2}$ feet, $x = AW$, and t = the time in motion. Then, by a known general formula, $pu^2 + wv^2 = 4g(py + wz)$, y and z being the perpendicular heights passed through by the weights P and w respectively; but here we have $P = w$, $y = s - x$, and z (because w has no perpendicular motion) = 0; hence, by correction, the formula



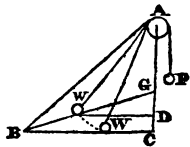
becomes $u^2 + v^2 = 4g(s-x)$. Now, $wc = \sqrt{(x^2 - a^2)}$, and by similar triangles as $\Delta W : wc :: wc : wd = (x^2 - a^2) \div x$, therefore by the resolution of forces, as $(x^2 - a^2) \div x : \sqrt{(x^2 - a^2)} :: u : v$, hence $u = (v \div x) \sqrt{(x^2 - a^2)}$, and $v = ux \div \sqrt{(x^2 - a^2)}$, and by substituting for u , its equal, in the corrected formula, we have $\frac{v^2(x^2 - a^2)}{x^2} + v^2 = 4g(s-x)$, hence $v = 2x \sqrt{\frac{g(s-x)}{2x^2 - a^2}}$, which, when w arrives at c , or $x = a$, becomes $2\sqrt{g} \times \sqrt{(s-a)} = 25.364$ feet per second, the last or greatest velocity.

Again, for the time, $t = \frac{-x}{u} = -x \div \frac{v \sqrt{(x^2 - a^2)}}{x} = \frac{-x}{2\sqrt{g} \sqrt{\frac{2x^2 - a^2}{(x^2 - a^2) \times (s-x)}}}$, which thrown into a series, and the fluent taken, gives $t = 1.358$ seconds, for the whole time in motion.

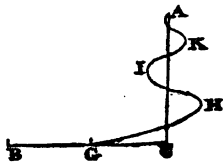
The same by Mr. Richard Taylor, Carlisle.

Let BC be any inclined plane, up which w is supposed drawn by P . Put $h = BA$, the direction of the string at first; $b = BC$, $c = AC$; $x = WA$ any other direction of the string; $y =$ the height ascended by w while P descends through AP ; $v =$ the velocity of w in direction WA or of P in ΔP ; $u =$ its velocity in the plane at w ; and $g = 32\frac{1}{2}$ feet, the force of gravity. Then by a well-known formula, $rvv + wu^2 = grx - gw y$, the fluents of which are $rv^2 + wu^2 = 2g(rx - wy) =$ (by correction) $2g \times [(h-x)r - wy]$.

Now let the plane BC coincide with bc , then the equation becomes $rv^2 + wu^2 = 2gr(h-x)$; but $u : v :: WA : wc :: x : \sqrt{(x^2 - a^2)}$, hence, $v = (u \div x) \sqrt{(x^2 - a^2)}$, therefore $r(u^2 \div x^2)(x^2 - a^2) + wu^2 = 2grx(h-x)$; hence $u^2 = \frac{2grx^2(h-x)}{r(x^2 - a^2) + wx^2}$, and $u = \sqrt{\frac{2grx^2(h-x)}{r(x^2 - a^2) + wx^2}} = \sqrt{\frac{2gx^2(h-x)}{2x^2 - a^2}}$ when $r = w$, and when $x = a$, this becomes $u = \sqrt{2g(h-a)} = \sqrt{64\frac{1}{2} \times 10} = 25.364$ the velocity sought.



Mr. Atkinson, at the end of his solution, bringing out the same conclusion, remarks "that the question is an impossible one in the form in which it is proposed; for the weight w will not touch the line BC along its whole length, but will leave it at some point G ; after which it will describe a curve resembling GHK in the annexed figure, and therefore will never arrive at the point c ."

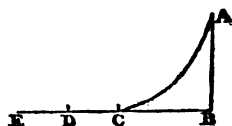


XIV. QUESTION 1282, by Mr. W. G. Horner, of Bath.

A flexible chain of uniform density, five feet seven inches in length, is suspended by one extremity, three feet four inches above a horizontal table: what length of the chain will lie on the table when there is an equilibrium; the friction being such, that a piece of similar chain did not begin to slide till the table was inclined in an angle of $16^{\circ}\frac{1}{2}$ to the horizon?

Answered by Mr. Henry Atkinson.

Let A be the point of suspension, BE the table, ACN the chain, and BC the part lying on the table. Put AB = a , the whole length of the chain = b , tang. of $16^{\circ} 40' = t$ (radius 1), and BC = x : then, by the question, tx = friction of the part x on the table, which is the tension of the curve AC at the point C. Now, by the nature of the catenary, the length of the curve CA = $\sqrt{2txa + a^2}$, which is = $b - x$, by the question, hence x is found = $b + ta - \sqrt{(t^2 + 1)a^2 + 2tab} = b + ta - \sqrt{s^2a^2 + 2tab} = 21.112574$ inches, putting $s = \sqrt{t^2 + 1}$ the secant of $16^{\circ} 40'$.



The same by Mr. W. G. Horner, the Proposer.

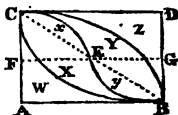
Let l = length of the chain; z = the length suspended; h = the height of the point suspended, above the table; x = the horizontal effect of the portion which remains on the table. By the catenary, $z^2 = h^2 + 2hx$; but, by the inclined plane, $x = (l - z)s$, where s is the sine of the angle of inclination, which makes the gravity of the chain a counterpoise to its friction, and the length being as the weight, by the question. Therefore $z^2 = h^2 + 2hls - 2hsz$; hence $z = \pm \sqrt{((s^2 + 1)h^2 + 2hls) - hs}$ = in the present case 45.7 inches = 3 feet 9.7 inches suspended; consequently, 1 foot 9.3 inches, lying on the table.

PRIZE QUESTION, by Llerien.

Four brothers having a rectangular piece of ground left equally between them, find that the sides of it are respectively the diameter and semi-circumference of a circle. This they wish to divide by curve lines running from any one corner to its opposite. How may this be done?

Answered by Mr. J. A. Dotchen, of Chelsea.

Admit AD to be the rectangular plot, and since $AC : AB ::$ diameter of a circle : its semi-circumference, a semicycloid may be inscribed in AD ; let two such be constructed towards the opposite sides as in the figure. Draw the diagonal CB , and FEG through its centre E parallel to AB , and on FE and EG describe two other semicycloids to generating circles, whose diameters are CF and EG , meeting in E , on the opposite sides of the diagonal. Then will the areas of the four spaces w, x, y, z , be all equal.



Demonstration. The area of the rectangle AD is by supposition = four semicircles described on the diameter AC ; but the area of the inscribed semicycloid is (by the art. Cycloid in Dr. Hutton's Dictionary) = three of these semicircles, therefore each of the trilineal spaces w, z , is = one such semicircle, or $\frac{1}{4}$ part of the whole rectangle; and, these excluded, the diagonal CB evidently divides the remaining middle space into two equal cycloidal segments; but, the smaller semicycloids x, y , are also equal, therefore the spaces x, y , are equal, and of course all the four areas w, x, y, z .

Remark. From what is done above, it follows that the lengths of the three dividing curves are all equal, each being = $2AC$; in which property the figures resemble Dr. Hutton's celebrated division of a circle into parts equal both in area and perimeter.

The same by the Rev. J. Furnass, of Ponteland.

In the same figure on AB constitute the semicycloid cwb , and on CD the semicycloid czb ; draw the diagonal BC , which bisect in E , and on CE and BE describe the equal circular arcs cxE and ByE ; by which means the given rectangle is evidently divided into four equal parts by curve lines, as required.—The demonstration is the same as above.

The same, by Mr. W. G. Horner, Bath.

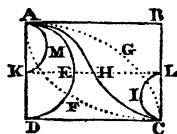
The given rectangle is that which circumscribes the common hemicycloid; and the areas of these two figures are as 4 to 3. Hence is deduced this obvious

Construction. Draw the diagonal; and also the two opposite semicycloids, having the diagonal for their common chord; and the figure will be divided as required.—Or, if the middle division also is required to be a curve, let the diagonal be divided into two, or any number of pairs, of equal parts; and on each division, as a chord, describe the

segment of a hemicycloid, disposing equal segments on opposite sides of the diagonal. The serpentine thus formed, together with the two principal hemicycloidal arcs, will complete the distribution, not only into equal shares, but also by fences of equal length, and consequently at equal rates of expence.

The same by Mr. Edward Riddle, Master of Trinity-house School, Newcastle, formerly of Whitburn.

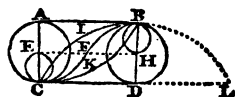
Let $ABCD$ be the rectangle, the semicircular arc AED being = the side DC .—If the semicircle AED be rolled from D to C , along the line DC , and then from B to A , along the line BA , the point A of the semicircle will trace out the cycloidal arcs AFC , AGC , cutting off the spaces AFC , AGC , each = $\frac{1}{4}$ of the rectangle $ABCD$, by the property of the cycloid. And, if the semicircles AMK , CIL (the diameters of which are = $\frac{1}{2}AD$) be rolled along the line KL , till the points A and C coincide in the middle point H , then will these points trace out the cycloidal arcs AH , HC , forming one continued curve line, and bisecting the curvilinear space $AGCFA$; and consequently the three curves, AGC , AFC , AHC , divide the rectangle as required.



Note. The curves AH , HC , may be two equal and similar curves of any kind, described on opposite sides of the diagonal AC , and having the same curvature at H ; provided they be included within the arcs AFC , AGC . It may be further observed, that, as above, the division may be effected when the rectangle is of the given species, and I presume that this solution is the one required.—But it may be observed, that whatever be the species of the rectangle, if the arcs AGC , AFC , be the arcs of a cubic parabola, the vertex of which is C or A respectively; these arcs, with the curve line AHC , drawn as above, will divide the rectangle in the manner required.—By adopting different parabolas, a rectangle may be divided into any number of equal parts, by lines drawn as in the question.

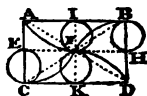
The same, by Mr. James Utting, of Brooke.

Draw the parallelogram $ABCD$, whose length and breadth are in the ratio of 1.5708 to 1; consequently the area is 1.5708, and each brother's share 0.3927. If the circles AC , BD , be made to rotate on the sides AB , DC , the points C and B will generate the semicycloidal arcs CKB , BIC , cutting off $\frac{1}{4}$ of the whole area towards each of the sides. It then remains only to bisect the middle space, which may be done by a similar method, as follows, viz. make the two smaller circles XC , BH , rotate on the middle line EFH , towards each other, till



the two generating points c , b , meet in F , forming the bisecting curve line CFB , which is also equal in length to each of the two former curves CIB , CKB ,

Second Solution. $ABCD$ being the rectangle as before : if the smaller generating circles EC , IF , FK , BH , be made to rotate on the lines EF , IA , KB , HF , they will describe the same cycloidal curves CF , FA , FD , FB ; and as the four small parallelograms are all equal, and the curves similar, the proposed rectangular piece of land will be thus equally divided, by the curve lines.

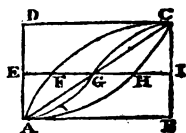


Note. Thus then the question admits of another answer. But I question whether it corresponds with the proposer's plan of solution, who says in his question, that the rectangular piece of land is to be divided by curve lines running from *any one corner* to its opposite.

But if it is divided as in the second figure, the solution is much easier than in the former case, as well as more just and convenient, since in the first case the owners of the two interior shares cannot pass and repass without infringing on their brother's property, as the entrance to each share is but a point, mathematically considered.

The same, by Mr. John Whitley, Rotherham,

This curious question admits of an indefinite number of solutions, if there was room in the Diary to shew them; but the following one may suffice. $ABCD$ being the rectangle, having the sides equal to the diameter and semi-circumference of a circle. Draw the diagonal AC , and describe the semicycloid AFC , having BC the diameter of its generating circle, and which divides the triangle ABC into two equal parts. In like manner the semicycloid AHC dividing the other triangle ABC equally in two; consequently the three lines AFC , AHC , AC , divide the whole rectangle into four equal parts; but, as AC is a right line, and therefore (by the question) not admissible, the middle space $AFCHA$, which is equal to half the rectangle, may be bisected by a curve line generated as follows: Draw any right line EF , parallel to AB or DC , cutting the curves in F , H ; then the middle point G , of the intercepted part FH , will describe the curve AGH , and which will obviously divide the space $AFCHA$ into two equal parts. Therefore the three curve lines AFC , AGC , AHC , divide the rectangle into four equal parts, as required.



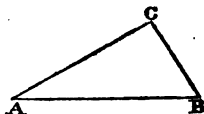
Questions proposed in 1815, and answered in 1816.

I. QUESTION 1284, by Mr. Wm. Wiseman, of Hull.

How must three trees A, B, C, be planted, so that the angle at A may be double the angle at B, the angle at B double the angle at C; and so that a line of 400 yards may just go round them?

Answered by Mr. John Craggs, of Hilton.

The sum of the proportional numbers $4 + 2 + 1 = 7$; then $180^\circ \div 7 = 25^\circ 42\frac{6}{7}' = \angle A$, its sine $\cdot 43388376$; the double $51^\circ 25\frac{2}{7}' = \angle B$, its sine $\cdot 7818343$; the double again $102^\circ 51\frac{4}{7}' = \angle C$, its sine $\cdot 97437944$; the sum of which sines is $2\cdot 1900975$. Then, as $2\cdot 1900975 : 400 :: \cdot 43388376 : 79\cdot 247 = BC :: \cdot 7818343 : 142\cdot 79 = AC :: \cdot 97437944 : 177\cdot 96 = AB$; the three sides of the triangle.



The same by Mr. Alfred Fox, Falmouth.

The angles are easily found to be as follows, viz. $\angle A = \frac{1}{7}$ of $180^\circ = 25^\circ \frac{4}{7}$, $\angle B = \frac{2}{7}$ of $180 = 51^\circ \frac{2}{7}$, $\angle C = 102^\circ \frac{4}{7}$. Then, from Hutton's Course, vol. 3, ch. 3, ex. 4, we deduce the following proportion, viz. as $\sin. A + \sin. B + \sin. C : 400 :: \sin. A : \sin. B : \sin. C \propto 78\cdot 23 : 142\cdot 76 : 178\cdot 01$, the distances between the three trees.

The same by Mr. W. G. Horner, of Bath.

The angles are $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{4}{7}$ of 180° , or $25^\circ 42\frac{6}{7}'$, $51^\circ 25\frac{2}{7}'$, and $102^\circ 51\frac{4}{7}'$; the n. s. of which are $\cdot 4338838x$, $\cdot 7818315x$, and $\cdot 9749279x$; the sum of which is $2\cdot 1906432x = 400$; therefore $x = 400 \div 2\cdot 1906432$, and the three sides $79\cdot 225$, $142\cdot 758$, $178\cdot 017$ yards, nearly.

N. B. This is ex. 14, Analyt. Plain Trig. Hutton's Course, vol. 3.

Scholium. Let $s = \sin$ of the smallest $\angle A$; then the sine of the largest $4A$, will be $(4s - 8s^3) \sqrt{(1 - s^2)}$, and that of its supplement $3A$ is $3s - 4s^3$. These must be equal, and, by making them so, we arrive at the equation $s^6 - \frac{1}{4}s^4 + \frac{1}{8}s^2 - \frac{7}{8} = 0$, the three positive roots of which are the sines of the three angles.

The same by Mr. John Smith, of Alton Park.

Since the three angles together contain 180° , it will be $1 + 2 + 4 = 7 : 180 :: 1 : 25^\circ 42' \frac{1}{4} = \angle A :: 2 : 51^\circ 25' \frac{1}{4} = \angle B :: 4 : 102^\circ 51' \frac{1}{4} = \angle C$. Then $\sin. A + \sin. B + \sin. C : BC + AC + AB = 400 :: \sin. A : BC = 77.225 :: \sin. B : AC = 142.758 :: \sin. A : AB = 178.017$.

After a scientific solution by Mr. A. Glendenning, he adds, For a neat synthetical solution, and an easy mode of calculation deduced from it, see the Mathematician, quest. 44.

II. QUESTION 1285, by Mr. Tho. Smith, of Hethersett.

Given $5x^2 + y^2 = z^2$, to find x, y, z , in the least whole numbers?

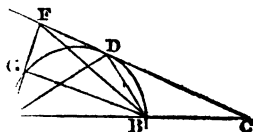
This question was, by mistake, inserted over again, after having been answered the former year.

III. QUESTION 1286, by Mr. John Abram, Teacher of Mathematics, King's Bridge, Canterbury.

Let ac be a given straight line any how divided in the point x , and cx another straight line making any given angle ACE with the former. It is required to find geometrically the point D in the line CE , where AB will appear under the greatest angle possible.

Answered by Mr. Newbegin, Norwich.

In CE take $CD = \sqrt{(AC \times CB)}$, a mean proportional between CA and CB ; then will CD be a tangent to the circle drawn through the three points, A, B, D (cor. to theo. 61 Hutton's Geom.). Draw AD, BD ; then the $\angle ADB$ is the greatest that can be subtended by AB , when D falls on CE .—For, take any other point F in CE ; and draw AF, BF . Now the exterior $\angle AGB$ (of the triangle BGF) $= \angle ADB$ in the same segment (geom. theor. 50) is greater than the interior $\angle AFB$ (geom. theor. 8). And the same may be proved of any other point in CE . Therefore ADB is the greatest angle that can be subtended by AB when D falls in CE .



The same by Miss Mary Weston, near Gainsborough.

Let AC be the given line, and CE another, making a given angle with it; also B the given point in AC . Then, by prop. 43, Simpson's Algebra, describe the circle ADB , touching CE in D ; and from the point of contact D , draw the lines DA, DB ; then will the $\angle ADB$ be the greatest possible.—For, from any other point F , in the line CD , draw the lines FA, FB ; as also from the point G where FA cuts the circle, draw GB . Then the $\angle AGB$, being external with regard to the triangle BGE , will be greater than the $\angle AFB$. Therefore ADB , on the same segment with AGB , will be also greater than AFB . Consequently $\angle ADB$ is the greatest possible.

The same by Mr. John Wills, Kentish Town.

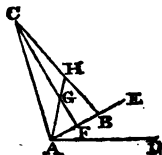
Let AC be the given line divided in the point B . By prob. 42, Simpson's Geometry, describe the circle ABD through the two given point A, B , and touching the line CE given by position, in the point D , which will be the point required.—For, if any other two lines be drawn, either above or below that point, the angle will fall without the circle, and will of course be less than the angle which falls in the circle.

IV. QUESTION 1287, by Mr. W. Trueman, Exeter.

Required the solidity of the greatest cone, the diameter of its base being 6 feet, to just stand on an inclined plane, which makes an angle of 30° with the horizon; and find the highest point in its slant side where a weight may be placed without overturning it?

Answered by Mr. John Baines, Mathematical Master, Reading School.

Let ABE represent the inclined plane, ABC the cone, its axis CF , AEM perpendicular to the horizon AD ; then H is the highest point on the side where a weight may be laid without oversetting the cone; G is the centre of gravity, $FC = 4FG$, $AF = FB = 3$, the $\angle BAD = 30^\circ = \angle G$, $\angle BAH = 60^\circ$. Then, by Trigonometry, $FG = 3\sqrt{3}$, $CF = 4FG = 12\sqrt{3}$. In the $\triangle CBF$ are given CF and BF , to find the sine $\angle B = \frac{3}{4}\sqrt{3}$; hence sine $\angle H = \frac{3}{4}\sqrt{3}$; therefore $AH =$



$8\frac{1}{2}$, the height of the highest point where the weight can be placed. Lastly, $6' \times 12\sqrt{3} \times .2618 = 195.89$ is the solidity of the cone.

The same by Mr. Anthony Cook, of Wooley.

Draw the horizontal line AD , and the inclined plane AE elevated 30° ; set off $AB = 6$, the cone's base, which bisect in F ; erect FC perpendicular to AB , and AG perpendicular to AD meeting FC in G , make $FG = 4FG$, join AC , BC ; then will G be the centre of gravity, and ABC a vertical section of the cone (see prop. 38, vol. 2, Hutton's Course). Now the $\angle AGF = \angle BAD = 30^\circ$, therefore $AG = 2AF = 6$, and $FG = AF/\sqrt{3} = 3\sqrt{3}$, hence $FC = 12\sqrt{3}$, and the solidity $AB^2 \times \frac{1}{3}FC \times .7854 = 195.89$. Also, if AG be produced to meet the slant side in H , it is obvious that H will be the highest point where a weight can be placed without overturning the cone.

The same by Mr. A. Glendenning, North Yarmouth.

Let AB be the inclined plane, making with the horizon the $\angle BAD = 30^\circ$; having made $AB = 6$ feet, let CF be perpendicular to and bisect AB in F , and intersect AH , drawn perpendicular to AD in G ; make $GC = 3GF$, and FC is the altitude of the required cone. For G is the centre of gravity of the cone, by construction, and Dr. Hutton's Dictionary under Centre: and since AH is perpendicular to the horizon, and passes through that centre, the body is supported (Mechan. prop. 38, course, vol. 2): and, because the base AB and inclination $\angle BAD$ are constant, and the cone is merely supported, it is consequently the greatest which can be constituted on that base under those limitations. It is further obvious that the point H , at which AG produced meets BC , is the highest point at which a weight may be placed without overturning the cone. For, were the weight placed higher than H , the common centre of gravity of that and the cone would not be supported.

Calculation. In the right-angled $\triangle AFG$, AF is given = 3 feet, and the $\angle GAF = 60^\circ$, to find $FG = 3\sqrt{3}$, therefore $FC = 12\sqrt{3}$, and the solidity $6^2 \times .7854 \times 4\sqrt{3} = .7854 \times 144\sqrt{3} = 195.89$, &c.

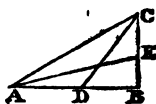
V. QUESTION 1288, by Mr. Joseph Williams, of Neath.

At Neath, in latitude $51^\circ 40'$ north, about 56 min. 8 sec. past 8 o'clock before noon; from the top of a perpendicular pole, in the sun-shine, to the middle of its shadow, measured just 20 feet; at

the same time, from the extremity of its shadow to the middle of the pole measured just 36 feet; required the time of the year 1814?

Answered by Mr. J. Collins, Teacher of Mathematics, Hatton Garden.

Let AB and BC be the shadow and the pole; also AE and CD the given lines drawn from the extremity of each to the middle of the other, viz. $CD = 20 = a$, and $AE = 36 = b$; put $x = AD = DB$, or $2x = AB$; then $BC^2 = a^2 - x^2 = 4(b^2 - 4x^2)$ by theor. 34, Hutton's Geometry; this equation reduced gives $x = \sqrt{\frac{4b^2 - a^2}{15}} = 4\sqrt{\frac{299}{15}} = 17.8587$,



or $AB = 35.7174$; hence $BC = \sqrt{CD^2 - DB^2} = 9.0037$ the height of the pole. Now, in the triangle ABC, are given AB and BC, to find the $\angle A = 14^\circ 3' 55''$ the sun's apparent altitude, which corrected by allowance for semi-diameter, refraction and parallax, gives $13^\circ 49' 36''$, its true altitude. Then we have two sides of a spheric triangle given, and an angle opposite to one of them, to find the third side, which, by prop. 25, pa. 167, Dr. Hutton's Tables, 5th edit.) is $103^\circ 17' 35''$; and hence the declination is $13^\circ 17' 35''$, answering to Feb. 14, and Oct. 29, 1814.

The same by Mr. A. Glendenning, North Yarmouth.

Let $AD = DB = x$, and $BE = EC = y$. Then $4x^2 + y^2 = 36^2$, and $x^2 + 4y^2 = 20^2$; hence $x = 4\sqrt{\frac{299}{15}}$, and $y = 4\sqrt{\frac{13}{15}}$; therefore $AB : BC :: \text{radius} : \sqrt{\frac{10}{3}}$, the tangent of $\angle A = 14^\circ 8' 54''$; the sun's altitude at the time of observation. Then in an oblique-angled spherical triangle, we have two sides and an angle opposite to one of them, viz. the complement of the latitude, the complement of the sun's altitude, and the time from noon, to find the sun's polar distance $= 102^\circ 59' 36''$; hence his declination was $12^\circ 59' 16''$ south; corresponding to Feb. 15, and Oct. 28, 1814.

The same by Mr. Jos. Williams, Schoolmaster, Cardiff.

Here is given $AE = 36$, and $CD = 20$. Then, by theor. 37, Dr. Hutton's Geom. $\frac{1}{2}(AE^2 - CD^2) + CB^2 = AB^2 = (\text{per fig.}) AC^2 - CB^2$; and by theor. 38, $2CD^2 + \frac{1}{2}AB^2 = AC^2 + CB^2$; hence, by subtraction, we have $2CD^2 - \frac{1}{2}AB^2 = 2CB^2$,

which, by substituting for AB^2 its equal, viz. $\frac{4}{3}(AE^2 - CD^2) + CE^2$, it gives

$$BC = 2\sqrt{\frac{4CD^2 - AE^2}{15}} = 9.0037, \text{ the perpendicular } BC, \text{ and}$$

$$AB = 2\sqrt{\frac{4AE^2 - CD^2}{15}} = 35.7034, \text{ the base } AB.$$

Then $35.7034 : 9.0037 :: \text{radius} : \text{tang. } 14^\circ 9'$ of the sun's altitude.

Hence, in the spherical triangle zsr , we have given

the coalt. $sz = 75^\circ 51'$

the colat. $zp = 38^\circ 20'$

the hour $\angle spz = 44^\circ 58'$

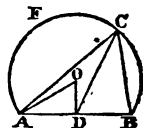
} to find $sr = 102^\circ 57'$ the polar distance,
or $12^\circ 57'$ south declin. answering to Feb. 15, or Oct. 28, 1814, the time.

VI. QUESTION 1289, by Mr. T. Harleston.

Given the sum of the squares of the sides containing the vertical angle, the line drawn from the vertex bisecting the base, and the perpendicular from the centre of the circumscribing circle to the base; to construct the triangle.

Answered by Messrs. John Charlton, Foster Charlton, John Davy, Rev. John Furnass, A. Glendenning, Henry Milburn, Edward Riddle, and D. T. Sheridan.

Suppose ACB is the triangle, o the centre of the circumscribing circle, od the given perpendicular, and cd the given bisecting line. Then, by theor. 38 in the Course, $AC^2 + BC^2 - 2CD^2 = 2AD^2$; but, $AC^2 + BC^2$, and $2CD^2$, being given, $2AD^2 = AB$ will also be given. Therefore, take $AB =$ to the given base, bisect it in d , erect the given perpendicular do , with the centre o and distance OA , describe the circle $AFCB$, apply $dc =$ the given bisecting line, join AC , BC and ABC is evidently the triangle required.



The same answered by Messrs. Wm. Burdon, Geo. Dalton, A. Fox, Wm. Makinson, and W. S.

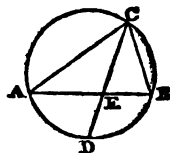
By Dr. Hutton's Geom. theo. 38, $2AD^2 + 2CD^2 = AC^2 + BC^2$; and since $AC^2 + BC^2$ is given, as also the line CD , therefore AD is also given, and the radius $AO = \sqrt{(AD^2 + DO^2)}$ is obtained; hence the construction.—Draw AB , which bisect by the given perpendicular do ; with radius AO and centre o describe a circle; then from d , with the given line bisecting the base, cut the circle in c , and ABC will be the required triangle.

VII. QUESTION 1290, *by Mr. T. Harleston.*

Given the diameter of the circumscribing circle, the vertical angle, and the ratio of the sides containing that angle; to construct the triangle:

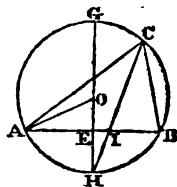
Answered by Messrs. John Collins, John Davey, James Newbegin, W. Putsey, D. T. Sheridan, and John Wills.

Since the vertical angle and diameter of the circumscribing circle are given, the base is given. Therefore having made AB = the base, and divided it at E in the given ratio of the sides, describe on it the segment of a circle to contain the vertical angle, by Geom. prob. 14; complete the circle and bisect the arc ADB in D ; draw DE , and continue it to meet the circle in C ; join AC , BC , and ABC will be the triangle required. For AB is the given base, and ACB the given vertical angle, by construction, and because the angles ACD , BCD , stand on equal arcs, DC bisects the vertical angle; also AE is to EB in the given ratio of the sides; therefore, by the Geom. theor. 83, in the Course, AC and BC are the sides.



The same answered by Messrs. William Burdon, Geo. Dalton, John Darley, Rev. J. Furnass, A. Glendenning, Alfred Fox, and W. S.

About the given diameter GH describe a circle, and let O be its centre; make the $\angle AOH$ = the given vertical angle; draw the chord AB perpendicular to GH ; take AI to IB in the given ratio of the sides; through the points H , I , draw a straight line to meet the circle in C ; join AC , CB and ABC is the triangle required. For that triangle is inscribed in the given circle; and because GH is perpendicular to AB , the arcs AH , BH , are equal (Geom. theor. 40), therefore the $\angle ACB$ = $\angle AOH$ (ibid. theor. 51) = the given vertical angle (by construction). Also, because the arcs AH , BH are equal, the $\angle ACB$ is bisected by CH , and consequently $AC : CB ::$ (ibid. theor. 83) $AI : IB$ the given ratio by construction.

VIII. QUESTION 1291, *by Mr. John Davey, of St. Just.*

To find two whole numbers such, that the difference of their squares shall be a cube, and the difference of their cubes a square.

Answered by Messrs. J. R. Ambler, Wm. Burdon, and W. S.

Let $x^3 + 2$ and $x^3 - 2$ denote the numbers. Then the difference of their squares, $8x^3$, is evidently a cube. It remains then to make the difference of their cubes $12x^6 + 16$, or $3x^6 + 4$ a square. For its root put $2x^3 - 2$; then $3x^6 + 4 = 4x^6 - 8x^3 + 4$; this gives $x^6 = 8x^3$, or $x^3 = 8$; hence $x = 2$; and the required numbers are 10 and 16.

The same answered by Mr. John Baines, Reading School.

Let $x + y$ and $x - y$ denote the two numbers; and put the difference of their squares $(x + y)^2 - (x - y)^2 = 8y^2$, which gives $x = 2y^2$; then the difference of their cubes $(x + y)^3 - (x - y)^3$ becomes $(2y^2 + y)^3 - (2y^2 - y)^3 = 24y^6 + 2y^3$ a square, or $24y^6 + 2y$ a square, which is the case when $y = 2$; hence $x = 2y^2 = 8$; and therefore $x + y = 10$, and $x - y = 6$, the least two numbers that answer the question.

The same by Mr. John Davey, the Proposer.

Let x be the greater number, and y the less: then is $x^2 - y^2 = a$ cube, and $x^3 - y^3 = a$ square. Put $x^2 - y^2 = z^2$, and assume $x = n^2z - y$; then will x^2 or $z^2 + y^2 = n^4z^2 - 2n^2zy + y^2$, which reduced gives $y = \frac{n^4z - z^2}{2n^2}$, consequently $x = n^2z - y = \frac{n^4z + z^2}{2n^2}$; therefore $x^3 - y^3 = \left(\frac{n^4z + z^2}{2n^2}\right)^3 - \left(\frac{n^4z - z^2}{2n^2}\right)^3 = (3n^4 + z^2) \times \frac{z^2}{4n^2} = a$ square, or, striking out square numbers, $3n^4 + z^2 = a$ square; let its root be $rn^4 - z$, then will $3n^4 + z^2 = r^2n^8 - 2n^4rz + z^2$, hence $z = \frac{r^2 - 3}{2r} n^4$. Now, to get an answer in whole numbers, let r and n be each taken $= 2$; then will $z = 4$, $y = 6$, and $x = 10$; which appear to be the least numbers that will answer the conditions.

The same answered by Messrs. Teasdale Bell, Thomas Chapman, Ant. Cook, Wm. Makinson, Tho. Smith, and Miss Mary Weston.

This, and indeed many other diophantine problems, may be expeditiously solved by the very useful and extensive tables of squares,

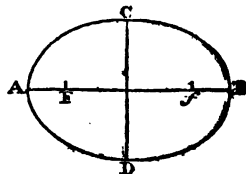
cubes and roots, in vol. 1, of Hutton's Course of Mathematics; from which it quickly appears that 10 and 6 will answer the conditions of the question; for $10^3 - 6^3 = 4^3$, and $10^3 - 6^3 = 28^2$.

IX. QUESTION 1292, by Mr. John Smith, *Alton Park*.

Determine the axes of a hollow prolate spheroid, whose capacity is 12 million feet, a sound emitted at one of its foci being heard at the other $\frac{1}{8}$ th of a second sooner than its echo.

Answered by Messrs. Foster Charlton, Thomas Collins, John Davy, Cha. Holt, S. Jones, R. Maffet, Hen. Milburn, Edw. Riddle, and Samuel Treby.

Let ABC be an elliptic section of the spheroid, AB the transverse axis, CD the conjugate, and F, f, the two foci. Then, by the question, a sound emitted at F, after arriving at f, will pass on to B, and by striking the curve there, reverberate from B to f again, in the 8th part of a second; hence, by the theory of sound, $2fb = 1142 \div 8 = 142\frac{1}{2}$ feet, which put $= a$; also put 12 millions $= b$, $\cdot 5236 = c$, and $rf = x$. Then, by the nature of the ellipse (see Dr. Hutton's Course and his Mensuration) we have $AB = x + a$, $AF = \frac{1}{2}a$, $FB = x + \frac{1}{2}a$, and $cd^2 = 4cf \times FB = 2ax + a^2$; hence, the solid content $= cd^2 \times AB \times c = ac(2x^2 + 3ax + a^2) = b$ by the question, this equation solved gives $x = 171\cdot 50309$: therefore $AB = 321\cdot 25309$, and $cd = 267\cdot 09585$.



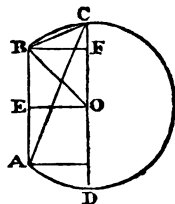
X. QUESTION 1293, by Mr. Wm. Burdon, *Acaster Malbis*.

From a point in the intersection of two planes inclined to each other in an angle of 15 degrees, two balls were let go, and at the instant the one of them was perpendicular above the other, they were 50 feet asunder. Required the inclination of each plane to the horizon, and the time of descent of each ball?

Answered by Messrs. H. Atkinson, W. Burdon, John Collins, John Davy, J. A. Dotchen, A. Fox, A. Glendenning, Edw. Page, Edw. Riddle, J. Smith, J. Whitley, and W. W. Wiseman.

Take $AB = 50$ feet, the given distance of the two balls; on which

describe a segment of a circle to contain an angle of 15° , the given angle between the two planes. Through o , the centre of the circle, draw the diameter CD parallel to AB ; join CA , CB , which will be the required planes. For by (prop. 25, cor. 4, Dr. Hutton's Course, vol. 2) the times of descent down CA , CB , will be each equal to the time down the diameter CD . Now draw BF , EO , perpendicular to CD ; then $\angle BOE = \angle BCA = 15^\circ$: therefore, by trigonometry, $OB = OC = 96.5926$: also $\angle FBC = \frac{1}{2} \text{ arc } BC = 37^\circ 30'$, the elevation of the plane BC above the horizon,



and $37^\circ 30' + 15^\circ = 52^\circ 30'$ that of the plane AC . Lastly, $\sqrt{\frac{CD}{16 \cdot \frac{1}{12}}}$
 $= \sqrt{\frac{2OB}{16 \cdot \frac{1}{12}}} = 3.4658$ seconds, is the time of descent of each ball.

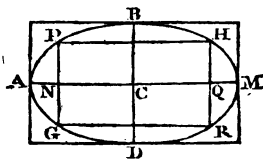
N. B. Mr. Atkinson, at the end of his solution, remarks that the balls would be continually in the same vertical line. And Mr. Davey adds, that, if the friction of the planes be such as will just cause the balls to roll, then will the time down each plane be $3.46576 \times \frac{1}{2} = 4.85206$ seconds.

XI. QUESTION 1294, by Mr. Wm. Burdon, *Acaster Malbis*.

In an ellipsis suppose a rectangle inscribed, the greatest possible, and within the rectangle the greatest ellipsis possible, again within the latter the greatest rectangle, and so on. Then will the sum of an infinite number of inscribed rectangles be equal to any parallelogram circumscribed about the given ellipsis. Required a demonstration?

Answered by Messrs. Wm. Bagshaw, Wm. Burdon, Tho. Collins, Geo. Duckett, Hen. Milburn, Miss Weston, and Ed. Riddle.

Put the transverse diameter $AM = t$, the conjugate $BD = c$, and the length of the greatest inscribed rectangle $NQ = x$; then, by conics, the breadth GP or $HR = \frac{c}{t} \sqrt{(t^2 - x^2)}$; therefore $t^2 x^2 - x^4 =$ a maximum, the fluxion of which gives $x = t\sqrt{\frac{1}{2}} = NQ$, and $GP = c\sqrt{\frac{1}{2}}$. Hence the area of the triangle CPH is $\frac{1}{2}tc$. In the same manner it is found that the next inscribed rectangle is $\frac{1}{4}tc$; and that of the next $= \frac{1}{8}tc$, and so on. The sum of which series is $\frac{1}{2}tc \times (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \&c.)$ to infinity $=$



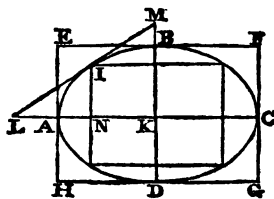
$\frac{1}{2}tc \times 2 = tc$. But, by Dr. Hutton's Conics, vol. 2, theor. 10, all parallelograms about an ellipse are equal to one another, and each equal to the rectangle of the two axes, and therefore equal to all the inscribed rectangles.

The same answered by Messrs. John Davy, W. G. Horner, S. Jones, and John Whitley.

All parallelograms circumscribing a given ellipse being equal, and double the greatest inscribed one, it follows that if a be put for the area of the circumscribing one, the series of rectangles will be $a + \frac{1}{2}a + \frac{1}{4}a + \frac{1}{8}a$, &c. which put $= z$; then by transposition, $\frac{1}{2}a + \frac{1}{4}a + \frac{1}{8}a$, &c. $= z - a$, and the double is $a + \frac{1}{2}a + \frac{1}{4}a$, &c. $= 2z - 2a = z$, therefore $z = 2a$, and $a = \frac{1}{2}a + \frac{1}{4}a + \frac{1}{8}a$ + &c. ad infinitum.

The same by Mr. W. Trueman, Schoolmaster, Exeter.

Let ABCD be a given ellipse, EFGH the least circumscribing parallelogram. By Dr. Hutton's 3d vol. page 35, the least triangle that can be described about, and greatest parallelogram that can be described in, any curve concave to its axis, is when the subtangent LM is equal to half the base of the triangle LMK, or to the whole base NK of the parallelogram IK, and that the triangle LMK and parallelogram IK are as 2 to 1. And, by the Conic Sections, vol. 2, page 107, all parallelograms circumscribing an ellipsis, are equal to one another, therefore the parallelogram AEBK $= \triangle LMK$, and the parallelogram IK $=$ half the parallelogram EK, so that the greatest inscribed $=$ half the least circumscribed parallelogram. Now, let a be the area of the least circumscribed parallelogram EG; then $\frac{1}{2}a$, $\frac{1}{4}a$, $\frac{1}{8}a$, $\frac{1}{16}a$, &c. ad infinitum, will express the areas of all the parallelograms inscribed: and the sum of the same series is well known to be equal to a , the area of the least circumscribed parallelogram, as required.



XII. QUESTION 1295, by Mr. John Whitley, Rotherham.

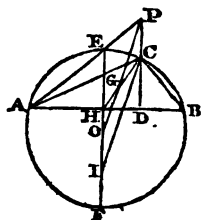
Given the base, the vertical angle, and the distance from the centre of gravity to the intersection of the perpendiculars from the angles upon the opposite sides: to construct the plane triangle

Answered by Messrs. Teasdale Bell, William Burdon, Alfred Fox, A. Glendenning, R. Maffett, and Sam. Treby.

In any triangle, if o denote the centre of the circumscribing circle, g the centre of gravity, and p the intersection of the perpendiculars from the angles upon the opposite sides; then it appears, from Diary quest. 826, and its solution, that the three points o , g , p , are all in the same straight line, and that $go = \frac{1}{2}gp$; but gp is given, therefore, og is a given line, and the problem is reduced to question 1280, the solution of which see in the Diary for 1815.

The same answered by Messrs. J. Darby, W. G. Horner, M. Lamplugh, D. T. Sheridan, Cha. Thompson, and John Whitley.

Let ABC represent the required triangle, o the centre of the circumscribing circle, g the centre of gravity, p the intersection of the perpendiculars. Now, since the base and vertical angle are given, the diameter EF is given, and o is a given point; also $og = \frac{1}{2}gp$ by parallels and the centre of gravity; $ho = \frac{1}{2}cp$ is given by similar triangles; therefore draw ic parallel to op , and i is a given point; hence this



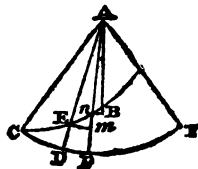
Construction. Describe the circle AEF given by analysis, take $oi = 2oh$, and from i as centre, with radius $= 3og = op$, intersect the circle in c ; join ac , bc , so is ABC the required triangle.—The demonstration is evident from the analysis.

XIII. QUESTION 1296, by Mr. Geo. Ducket, Northwich.

A right-angled triangle whose hypotenuse $AC = 6$ feet, and perpendicular $BC = 3$ feet, revolves once round its base AB , in the same time that a body x moves along the hypotenuse, viz from A to c . Required the length of the track described by the body, both motions being uniform?

Answered by Mr. Henry Atkinson, Newcastle.

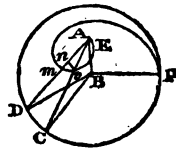
It is plane that the triangle, by revolving round its side AB , will form the cone $ACDP$, and that a curve of some kind will be described by the point x . Put $AC = 6 = a$, the circumference of the base $= 6 \times 3.1416 = c$, $AE = x$, and $CD = y$, also let mn represent $\dot{x}$, then will dp denote $\dot{y}$. Now, by similar triangles, $AD : AE :: dp : em$,



or $a : x :: \dot{y} : Em = \frac{x\dot{y}}{a}$, and the fluxion of the curve $En = \sqrt{(Em^2 + mn^2)} = \sqrt{\left(\frac{x^2\dot{y}^2}{a^2} + \dot{x}^2\right)}$. But, by the question, $a : c :: \dot{x} : \dot{y} = \frac{c\dot{x}}{a}$; this value of $\dot{y}$ substituted in the above fluxion, gives $En = \sqrt{\left(\frac{c^2x^2\dot{x}^2}{a^4} + \dot{x}^2\right)} = \frac{\dot{x}}{b}\sqrt{(x^2 + b^2)}$, putting $b = \frac{a^2}{c}$; the correct fluent of which is $\frac{1}{b} \times \left(\frac{x}{2}\sqrt{(x^2 + b^2)} + \frac{b^2}{2} \times \text{hyp.log.} \frac{x + \sqrt{(x^2 + b^2)}}{b}\right)$; which, when, $x = a$ and the value of b restored, becomes $\frac{1}{2} \times \left\{ \sqrt{(c^2 + a^2)} + \frac{a^3}{c} \text{hyp. log.} \frac{c + \sqrt{(c^2 + a^2)}}{a} \right\} = 11.6691$ feet, the length of the curve.

The same by Mr. John Davey, of St. Just, Penwith.

Let ABC be the triangle, AEDVP the spiral path described by the body, while the triangle in one revolution round the base AB describes, with the vertex C, the circle PCDF. Draw ADB indefinitely near ABC, and nv parallel to DC; put $PC = x$, $DC = \dot{x}$, $AV = An = y$, $mn = \dot{y}$, $AED = z$, $mv = \dot{z}$, $BC = r = 3$, $AC = 2r = 6$, and $3.1416 = c$. Now, while the vertex C, when at the point P, begins to move and describes the arc PC, let the body move from A to v, and by virtue of both motions describe the spiral AED; then, as $y : x :: 2r : 2rc :: \dot{y}$



$\dot{y} = c\dot{y} = DC$, and by similar sectors as $2r : c\dot{y} :: y : \frac{cy\dot{y}}{2r} = n\dot{v}$;

hence $\dot{z} = \frac{\dot{y}}{2r}\sqrt{(4r^2 + c^2y^2)}$; whose correct fluent z , when $y = 2r$, is

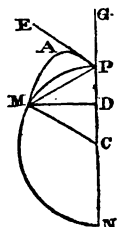
$r\sqrt{(c^2 + 1)} + \frac{r}{c} \times \text{hyp. log.} c + \sqrt{(c^2 + 1)} = 11.6691$ feet = the length of the whole spiral path.

XIV. QUESTION 1297, by Omicron, of Penrith.

In what *locus* must a person stand, in a vertical plane, so that throwing a perfectly elastic ball against a given point, in a line perpendicular to the horizon, it may always return to him again; the velocity with which it strikes the given point being supposed to be given?

Answered by Omicron, the Proposer.

Let P be the given point in a line PN perpendicular to the horizon, PMN the required locus in a vertical plane, M any point in it, and suppose the ball to be thrown in the direction MP , so as to strike the point P with a given velocity. Let $p = 4$ times the space due to the given velocity, and let PE be the line of direction after impact at P ; then the body will describe the parabola PAM , and p will be the parameter to the diameter PN . Draw MC parallel to PE , which will be an ordinate to the diameter PN . Then, since the $\angle MPC = EPG = MCP$, therefore $MC = MP$, and drawing MD perpendicular to PN , $PD = DC$, and $MC^2 = PC \times p = PD \times 2p$: therefore the locus is a circle of which the diameter is $2p = PN$.



Note. The principle of this solution, by the author of the question, is that the ball is thrown or passes in a direct line to the given point, or, at least, that it strikes the given point in that direction. And on this supposition the solution was given, similar to the above, by Messrs. *S. Jones, James Wood, and Wm. Wright.* But most of our other contributors have contemplated the question on the supposition that the ball thrown from the hand describes a parabolic curve in its approach to the given point, where it strikes that point in the vertex of the parabola, perpendicular to the given plane, and from whence, being elastic, the ball must necessarily return again in the same curve, and consequently arrive at the hand again. And in this way the solution is given scientifically by Messrs. *Abram, Atkinson, Butterworth, Charlton, Cook, Craggs, Dalton, Davey, Fox, Furnass, Glendenning, Horner, Maffett, Mason, Raby, Riddle, Sheridan, Treby, Whitley, &c.*

PRIZE QUESTION, by Llerien.

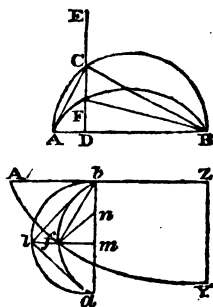
A cycloid being placed with the base horizontal, and vertex downwards, it is required to assign the straight line of shortest descent from any point in the base to the curve?

Answered by Llerien, the Proposer.

On any line AB , describe a semicircle ACB . Draw any ordinate DC , and produce it to E , making $CE = CA$. Take DF a third proportional to DE , CD ; AFB being the locus of F . Join BC , BF ; I say BF bisects the $\angle CBD$.

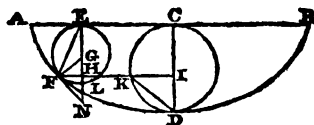
For, $DE : DC :: DC : DF$,
 therefore $AC + DC : DC :: DC : DF$,
 therefore $AC : DC :: CF : DF$,
 therefore $BC : BD :: CF : DF$,
 therefore (by Euclid) BF bisects the $\angle CBD$.

Let AY be the cycloid, b the point in the horizontal base AZ ; ba equal, and parallel to zy ; bfa a semicircle, bfa a curve similar to bfa in the first figure, cutting the cycloid in f ; then bf is the line required. For draw mfl an ordinate to the semicircle; join bl , la , bf , and draw fn parallel to bl . Then the $\angle nfb = \angle fbl = \angle nbf$, therefore $nf = nb$, therefore the circle described from the centre n , with the radius nf , passes through b ; it also touches the cycloid, for nf is parallel to bl , and bl is perpendicular to la ; therefore nf is perpendicular to the tangent at f . Hence bf may easily be shown to be the line required.



The same by Mr. Henry Atkinson, of Newcastle.

This ingenious gentleman first gives a solution by means of a mechanical curve similar to that of the ingenious proposer, above given. He then adds a solution analytically, thus: Let ACB be the base of the cycloid $AFDB$, and let CKB be the generating circle, and E the given point in the base. Now, if a circle be drawn, having its vertex in E , and just touching the cycloid in some point F , it is manifest that EF will be the line of shortest descent. Draw the other lines as in the figure, and



put $CD = a$, $CE = b$, $ID = x$, $IF = y$, and $GE = EF = z$; then will $IK = \sqrt{(ax - x^2)}$, $FH = y - b$; and $GH = a - x - z$: now $FH^2 + HG^2 = FG^2$, that is $(y - b)^2 + (a - x - z)^2 = x^2$, hence $z = \frac{(y - b)^2 + (a - x)^2}{2(a - x)}$. Again, from the nature of the cycloid, the

triangles KID and GMF are similar, hence $KI : ID :: GM : HF$, that is $\sqrt{(ax - x^2)} : x :: a - x - z : y - b$, hence we have $x(a - x - z) = (y - b) \cdot \sqrt{(ax - x^2)}$; then, by substituting the value of

z , and reducing, it is $x \left(\frac{a - x}{y - b} - \frac{y - b}{a - x} \right) = 2\sqrt{(ax - x^2)}$: but, from

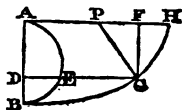
the nature of the cycloid, we know that $y = (\sqrt{(ax - x^2)} + \text{arc } DK)$; so that, if this value of y be substituted in the above equation, there will then be only one unknown quantity, and consequently the value of x may be found, and thence all the other parts be readily determined. For example, suppose a and b , to be each $= 2$, then x is

found to be $=.8645774$, the arc $KD = 82^\circ 13' 1''$, $y = 2.4257443$, and hence $EF = 1.21262$, and angle $FEH = \frac{1}{4}$ arc $KD = 21^\circ 33' 15\frac{1}{4}''$.

But any example in numbers, Mr. A. continues, may be worked much more expeditiously by trial-and-error, than by the above equation; and then Mr. A. works over the whole calculation by this method, bringing out the very same numbers as those above. And, though it be impossible to find room for this correspondent's calculations, we are greatly obliged by the correct and ample state in which his solutions are conveyed.

Another Solution by Mr. John Abrams, of the National School, Canterbury.

Let $ABGH$ be the given semicycloid, P a given point in the base, and PG the line of shortest descent. Suppose DG parallel to AH , and FG to AB , B being the vertex of the cycloid. About AB describe the semicircle AEB . Put $\frac{1}{2}AB = r$, $BD = v$, the arc $BE = z$, $DE = s$, $AD = FG = y$, and $AP = a$. Then, by the cycloid, $DG = AF = z + s$, therefore $PF = AF - AP = z + s - a$. Now, by



mechanics, the time of descent down PG varies as $\frac{PG}{\sqrt{FG}} = \frac{\sqrt{((z + s - a)^2 + y^2)}}{\sqrt{y}}$, a minimum, therefore $\frac{(z + s - a)^2 + y^2}{y}$

is also a minimum. But the fluxion of $z + s - a$ is $\dot{z} + \dot{s} = \frac{y}{s} \dot{v}$;

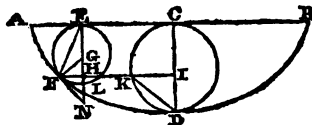
also, $y = 2r - v$, therefore $\dot{y} = -\dot{v}$. Hence $2y^2(z + s - a) \frac{1}{s} \dot{v} - y^2 \dot{v} + (z + s - a)^2 \dot{v} = 0$, therefore, by reduction $y^2 - (z + s - a)^2 = 2y^2(z + s - a) \frac{1}{s}$. Now, as the value of y and s depend

on that of z , y may be found by the method of trial-and-error, by assuming any value of z , till both sides of the equation become equal. Having determined y , make AD equal to it, and draw DG parallel to AH to meet the curve in G , then PG joined will be the line of shortest descent.

The same answered by Mr. Edward Riddle, Master of Trinity-House School, Newcastle-on-Tyne.

Let EFL be a circle inscribed in the cycloid touching the base in the

given point E, then is EF the straight line of shortest descent, as may easily be proved. Draw the diameter EL, and produce it to meet the tangent FN in N; join F and the centre G; draw FI parallel to AB, cutting the generating circle in K, and its diameter in I, and join DK. Then, by the property of the cycloid, FK = arch DK, and the chord DK is parallel to the tangent FN. Put CD = a, CE = b, IK = x, DI = v, and arch DK = z. Then FI = x + z, FH = x +



$$z = b, EH = a - v; EL = \frac{EH^2 + FH^2}{EH} = \frac{(a - v)^2 + (x + z - b)^2}{a - v},$$

$$FG = LG = \frac{(a - v)^2 + (x + z - b)^2}{2(a - v)}, \quad GH = EH - EG = a - v -$$

$$\frac{(a - v)^2 + (x + z - b)^2}{2(a - v)} = \frac{(a - v)^2 - (x + z - b)^2}{2(a - v)}. \quad \text{But } GH :$$

$$HF :: HF : HN :: IK : ID, \text{ or } \frac{(a - v)^2 - (x + z - b)^2}{2(a - v)} : x + z$$

— b :: x : v, hence $v((a - v)^2 - (x + z - b)^2) = 2x(a - v)(x + z - b)$. From this equation with the aid of Dr. Hutton's Tables, x, v, and z may very easily be determined; for if z be assumed in degrees, the tables immediately indicate the corresponding values of v and x, and the length of z; hence the true values may be readily approximated.—If a = 2, and b = 2, then z = 82° 13', and EF = 1.21262.

Many other solutions were given by our ingenious correspondents; some of them, however, will not fail to perceive, by observing the foregoing solutions, that their own are not always on the most accurate principles, particularly when they describe a circle through the given point and cutting the curve of the cycloid, whether that circle be equal to, or greater, or less than the generating circle of that curve. And, though those solutions are not incorrect which only direct to describe a circle through the given point and touching the curve of the cycloid, yet they are not to be considered as complete, since such a tangent circle cannot be described geometrically, or by the ruler and compasses only.

AN

APPENDIX,

CONTAINING

ADDITIONAL SOLUTIONS, REMARKS, &c.

To some of the Questions in the

LADIES' DIARY.

QUESTIONS 8, 295, 658, &c.

Remarks. By Mr. P. Barlow, R. M. Academy.

These questions all relate to finding the number of answers in indeterminate equations of the form

$$ax + by + cz + \&c. = d,$$

which are, in fact, all reducible to the more simple form $ax + by = c$; it is therefore proposed here to investigate a general theorem for determining the number of solutions that equations of the above form admit of in integral numbers.

THEOREM. *The number of integral answers in an equation of the form $ax + by = c$, is equal to the greatest integer contained in the expression*
$$\frac{a + b + c - 1}{ab}.$$

Investigation. Find the values of p and q in the equation $ap - bq = 1$; then will

$$apc - bq c = c, \text{ or } a(pc - mb) - b(qc - ma) = c,$$

$$\text{or } a(pc - mb) + b(ma - qc) = c,$$

m being here indeterminate, except that it must be such that $pc > mb$,

and $ma > qc$; or $m < \frac{pc}{b}$ and $> \frac{qc}{a}$; and consequently the num-

ber of different values of m , or different integral solutions, will be

equal to the difference of the two integral parts of $\frac{pc}{b}$ and $\frac{qc}{a}$. Now the difference of the integral parts of $\frac{pc}{b} - \frac{qc}{a}$, is the same as the integral part of the fraction $\frac{(ap - bq)c}{ab} = \frac{c}{ab}$, (because $ap - bq = 1$) except when the remainder of $cp \div b$, is less than the remainder of $cq \div a$, in which case the latter integer will be too little by an unit. Let us see, therefore, what must be added to c , in order to assure the same integral quotient.

For this purpose we may observe that the least remainder of $\frac{cp}{b}$ is 1, and the greatest of $\frac{cq}{a}$ is $a - 1$; and the least of $\frac{c}{ab}$ is 1. Therefore the difference between the remaining fractional parts of $\left(\frac{cp}{b} - \frac{cq}{a}\right)$ and $\frac{c}{ab}$ can never exceed $\left(\frac{1}{b} - \frac{a-1}{a}\right) - \frac{1}{ab} = \frac{a + b - ab - 1}{ab} = \frac{a + b - 1}{ab} - 1$.

If therefore we add $a + b - 1$ to the numerator of $\frac{c}{ab}$, the integral part will be the same as the integral part of $\frac{cp}{a} - \frac{cq}{b}$. It may also be shown that in the case in which $\frac{cp}{a}$ has its greatest remainder, viz. $\frac{a-1}{a}$ and $\frac{cq}{b}$ its least, viz. $\frac{1}{b}$: the difference $\frac{a-1}{a} - \frac{1}{b} = \frac{ab - a - b}{ab}$ will not when increased by $a + b - 1$ be equal to a unit, being in the extreme case $= \frac{ab - 1}{ab}$, so that here also the integral part of $\frac{a + b + c - 1}{ab}$ is the same as that of $\frac{cp}{a} - \frac{cq}{b}$, consequently in all cases the number of integral solutions will be $\frac{a + b + c - 1}{ab}$.

Cor. 1. Hence it follows that, when $c \leq ab - a - b$, the question is always possible, which is the theorem of Waring, page 349: *Meditationes Algebraicae*. This theorem is not demonstrated by Waring, but I have given the demonstration of it at page 78, of my

"*Theory of Numbers.*" I have also, at page 324 of the same work, shewn that the number of answers in the equation $ax + by = c$ is expressed by the difference of the integers of $\frac{cp}{b} - \frac{cq}{a}$; but the present theorem is more concise, and has the advantage of requiring no prior operation.

Examp. In how many different ways may £100 be paid in dollars and guineas? Here

$$11x + 42y = 4000:$$

$$\text{therefore } \frac{4000 + 42 + 11 - 1}{11 \times 42} = \frac{4052}{462} = 8,$$

the number required.

Cor. 2. When there are three quantities as $ax + by + cz = d$, then by transposing cz , we have $ax + by = d - cz$, and substituting successively 1, 2, 3, &c. for z ; and c' for $d + a + b - 1$, the number of integral answers will be equal to the sum of the integers of $\frac{c' - c}{ab} - \frac{c' - 2c}{ab} + \frac{c' - 3c}{ab} + \&c.$ continued till $c' - nc$ becomes less than ab .

As an example, required the number of solutions that the equation $5x + 7y + 41z = 179$ admits of in integers. Here $c' = 179 + 5 + 7 - 1 = 190$, whence the sum of the integers of $\frac{190 - 41}{35} + \frac{190 - 82}{35} + \frac{190 - 123}{35} + \frac{190 - 164}{35} = 4 + 3 + 1 + 0 = 8$, the number sought.

When this series consists of many terms, it will be more expeditious to sum it by the method explained at page 329 of my "*Theory of Numbers.*"



QUESTION 61.

In the magic square, fig. 9, page 76, vol. 1, the numbers fall in pairs, making 16 and 18 ($1 + 15$, $12 + 6$, $8 + 10$, &c.), but not alternately: if, however, the third and fourth vertical columns from the left change places, and the third horizontal column from the top be made the upper one:—or let the two left hand vertical columns take place of one another, and the second horizontal column from the top be shifted to the bottom, in either case the pairs will stand alternately, and consequently the sum of the numbers in any square of four cells will be $= 16 + 18 = 34$ the sum in any column or diagonal. Fig. 1, is according to the latter method of changing the ranks; and fig. 2, is the same square (fig. 1.) only the numbers making the pairs stand under one another as in the arrangement, page 80.

Now let the 16 numbers be set down alternately, in pairs of 16 and

16 each (similar to the arrangement, page 80), thus
 2 1 4 3 6 5 8 7
 16 15 14 13 12 11 10 9, and call the greater numbers the complements of the less : then arrange the less numbers as in the margin at (A).

The square, fig. 2, consists of 4 squares of 4 cells each, and the two upper figures (1 and 8) of the arrangement (A) stand in one of them ; the two next (2 and 7) in another ; the next two (3 and 6) in a third ; and 4 and 5 in the fourth, the sum in each square being 9.

By following the method of arrangement in this square, it is manifest we can make magic squares of 16 cells each with other series that shall have similar properties.

Now let the upper or less numbers (from 1 to 128) of the four divisions of pairs, page 80, be arranged as below :

1 16 17 32	33 48 49 64	65 80 81 96	97 112 113 128
2 15 18 31	34 47 50 63	66 79 82 95	98 111 114 127
3 14 19 30	35 46 51 62	67 78 83 94	99 110 115 126
4 13 20 29	36 45 52 61	68 77 84 93	100 109 116 125
5 12 21 28	37 44 53 60	69 76 85 92	101 108 117 124
6 11 22 27	38 43 54 59	70 75 86 91	102 107 118 123
7 10 23 26	39 42 55 58	71 74 87 90	103 106 119 122
8 9 24 25	40 41 56 57	72 73 88 89	104 105 120 121

where we begin at 1 and proceed downwards to 8 (half the side of the square of 256 cells), then upwards to 16, then down again, &c. &c. Next, let the vertical lines divide the 128 numbers into 4 parts of 32 numbers each ; and when these are again divided by the horizontal lines into 4 parts of 8 numbers each, the whole will be divided into 16 parts of 8 numbers each ; and if these numbers are placed in 16 squares according to the order in which the less numbers stand in fig. 2, and the vacant cells filled with their respective complements, we shall have the 16 magic squares which compose the square of 256 cells, fig. 4.

Fig. 3, is the first square of 16 cells in which the numbers 1, 2, 15, 16, 17, 18, 31, 32, (the first division) stand in the same order as 1, 2, 3, 4, 5, 6, 7, 8, are in fig. 2, these with the complementary numbers form the first square on the left at top in fig. 4. And the other divisions of 8 numbers each follow the same order in all the other squares.

In this square (fig. 3) the sum of the two numbers in each of the 4 squares is 33 (= 1 + 32) instead of 9, as in fig. 2, and the like necessarily follows in the 3 next squares downwards in fig. 4. In the next 4 squares on the right, the sum is 97 (= 33 + 64) ; and 65 + 96, and 97 + 128, in the other squares respectively.

Fig. 1.

15	1	14	4
10	8	11	5
3	13	2	16
6	12	7	9

Fig. 2.

6	9	10	15
12	13	8	1
7	2	11	14
9	16	5	4

(A) 1 8
 2 7
 3 6
 4 5

Fig. 3.

18	15		
		32	1
31	2		
		17	16

Fig. 4.

18	15	226	255	50	47	194	223	82	79	162	194	114	111	130	159
240	241	32	1	208	209	64	33	176	177	96	65	144	145	128	97
31	2	239	242	63	34	207	210	95	66	175	178	127	98	143	146
225	256	17	16	193	224	49	48	161	192	81	80	129	160	113	112
20	13	228	253	52	45	196	221	84	77	164	189	116	109	132	157
238	243	30	3	206	211	62	35	174	179	94	67	142	147	126	99
29	4	137	244	61	36	205	212	93	68	173	180	125	100	141	148
227	254	19	14	195	222	51	46	163	190	83	78	131	158	115	110
22	11	230	251	54	43	198	219	86	75	166	187	118	107	134	155
236	245	28	5	204	213	60	37	172	181	92	69	140	149	124	101
6	27	235	246	59	38	203	214	91	70	171	182	123	102	139	150
252	229	21	12	197	220	53	44	165	188	85	76	133	156	117	108
29	9	232	249	56	41	200	217	88	73	168	185	120	105	136	153
234	247	26	7	202	215	58	39	170	183	90	71	138	151	122	103
25	8	233	248	57	40	201	216	89	72	169	184	121	104	137	152
231	250	23	10	199	218	55	42	167	186	87	74	135	154	119	106

That the four numbers in any four cells forming a square in either of the 16 squares together make 514 ($= 258 + 256$) is evident from their similarity to the square, fig. 2, and to prove that the same property holds good at the junctions of the squares in fig. 4, take 4 numbers in the 4th and 5th horizontal ranks (for example) where the numbers belong to 4 pairs: thus $16 + 253 = 269$, $193 + 52 = 245$, now $269 - 256 = 13$, and $258 - 245 = 13$, therefore $269 + 245 = 258 + 256$; and so of any other 4 contiguous numbers in any two adjacent ranks.

This square of 256 cells is therefore composed of 16 magic squares of 16 cells each. The sum of the numbers in each column or diagonal of each of these squares is $= 258 + 256$ (two pairs) $= 514$: consequently each column or diagonal of the great square contains 514×4 .

The 4 numbers in any 4 cells forming a square together make 514:

Therefore,

$$\text{in any square of } \left\{ \begin{array}{c} 16 \\ 36 \\ 64 \\ 100 \\ 144 \\ 196 \end{array} \right\} \text{ cells, the sum is } \left\{ \begin{array}{l} 514 \times 4 \\ 514 \times 9 \\ 514 \times 16 \\ 514 \times 25 \\ 514 \times 36 \\ 514 \times 49 \end{array} \right.$$

Thus, if a square hole the size of 36 cells be cut in a paper, and the paper laid any where upon the square, the numbers appearing through the hole will together make 514×9 . If the square hole takes in 64 cells, the sum will be 514×16 , &c. &c. And the 4 corner numbers of either of those squares together make 514. These are the principal properties of this square, of which only two belong to that at page 82, vol. 1. The other properties (see pp. 82, 83) are common to both squares.

That the 4 corner numbers of either of the squares containing 4, 16, 36, 64, &c. cells, together make 514, will be evident if we consider that the corner numbers of any square whose root is an even number, are always two of the less numbers with two of the complementary ones, and the sum of the two less numbers belonging to those two complementary ones is always the same as the sum of the other two less numbers, and consequently the 4 numbers together must make two pairs or 514. Thus, suppose the 4 corner numbers are 18, 47, 238, 211, standing at the corners of a square of 36 cells; now the less numbers belonging to the complementary numbers 238 and 211 are 20 and 45, whose sum is 65, the same as $18 + 47$: and so of either of the other squares.

In the same manner other magic squares of this kind may be constructed when the sides are multiples of 4, as 8, 12, 20, &c. Suppose, for example, a square of 144 cells, with the series 1, 2, 3, &c. to 144.

2 1 4 3

The numbers in pairs will stand thus 144, 143, 142, 141, &c. the pairs making 146, 144, 146, &c. and the arrangement of the 72 less numbers as below.

1	12	13	24	25	36	37	48	49	60	61	72
2	11	14	23	26	35	38	47	50	59	62	71
3	10	15	22	27	34	39	46	51	58	63	70
4	9	16	21	28	33	40	45	52	57	64	69
5	8	17	20	29	32	41	44	53	56	65	68
6	7	18	19	30	31	42	43	54	55	66	67

Here the numbers begin at 1 and are continued downwards to 6 (half the side of the square). The vertical lines make 3 divisions because the square is to consist of 9 squares of 16 cells each; and the horizontal lines divide the 72 numbers into sets of 8 numbers each for each of the 9 squares; these numbers follow the same order in the

several squares respectively as the less numbers in fig. 2, or the numbers in fig. 3.

Fig. 5.

14	11	122	145	38	35	98	119	62	59	74	95
132	133	24	1	108	109	48	25	84	85	72	49
23	2	131	134	47	26	107	110	71	50	83	86
121	144	13	12	97	120	37	36	73	96	61	60
16	9	124	141	40	33	100	117	64	57	76	93
130	135	22	3	106	111	46	27	82	87	70	51
21	4	129	136	45	28	105	112	69	52	81	88
123	142	15	10	99	118	39	34	75	84	63	58
18	7	126	139	42	31	102	115	66	55	78	91
128	137	20	5	104	113	44	29	80	89	68	53
19	6	127	138	43	30	103	114	67	54	79	90
125	140	17	8	101	116	41	32	77	92	65	56

Fig. 5 is the square of 144 cells composed of 9 magic squares of 16 cells each. The columns and diagonals of these squares are each = 290 (146 + 144, two pair), consequently the columns and diagonals of the square itself, contain 290×3 each. And the numbers in any square of 4 cells together make 290; and therefore 290×4 in any square of 16 cells:— 290×9 in any of 36 cells:— 290×16 in any of 64 cells:—and 290×25 in any square of 100 cells. And the 4 corner numbers of either of those squares together make 290. This square also retains its properties when the halves change places: but it fails in the property of the bent rows, because the centre falls in the middle of one of the 9 squares, and therefore the sums in the half diagonals are unequal.

NUMERICUS.

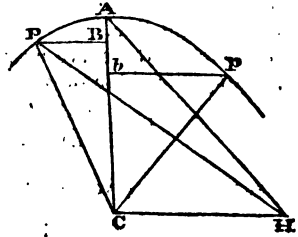
N.B. In the magic square, page 82, vol. 1. in the 5th horizontal rank, for 163 read 103.



QUESTION 87:

In order to resolve this question when the sun is not in the equinoctial, we must first investigate the nature of the curve described by the shadow of the top of the tree.

Let CH , perpendicular to the horizon, represent the tree, CA its shadow when the sun is on the meridian, CP its shadow at some other time, and PAP a portion of the curve required. Join PH , AH and draw PA perpendicular to CA , then will the angle PCA be the azimuth, and the angle CPH the altitude of the sun when the extremity of the shadow is at P . Put $z = CP$, $v = PH$, $h = CH$, $\phi =$ the angle PCA and θ the $\angle CPH$: Then to find the relation between z and ϕ , conceive a spherical triangle to have, for its angular points, the zenith, the pole, and the sun's place, then the side between the zenith and the sun will be the co-altitude; the side between the zenith and the pole the co-latitude; the side between the sun and the pole, the co-declination; and the angle at the zenith, the sun's azimuth. Put $a =$ the co-altitude, $b =$ the co-declination, and $c =$ the co-latitude; then by the common rule for finding an angle from the



three sides, we have $\frac{\cos b - \cos c \cos a}{\sin c \sin a} = \cos \phi$.

But $\sin a = \cos \theta = \frac{z}{v}$, and $\cos a = \sin \theta = \frac{h}{v}$, therefore by substitution and reduction,

$$v = \frac{z \cos \phi \sin c + h \cos c}{\cos b}.$$

But $v^2 = z^2 + h^2$, therefore

$$z^2 + h^2 = \frac{(z \cos \phi \sin c + h \cos c)^2}{\cos^2 b} \dots \dots \dots (1),$$

an equation which exhibits the relation between ϕ and z .

But to find the equation of the curve in terms of the rectangular co-ordinates CB and PA , or x and y , we have

$$\cos \phi = \frac{x}{z} \text{ and } z^2 = x^2 + y^2,$$

$$\text{therefore } x^2 + y^2 + h^2 = \frac{(x \sin c + h \cos c)^2}{\cos^2 b},$$

and by reduction, &c.

$(\cos^2 b - \sin^2 c)x^2 + \cos^2 b y^2 - 2hx \cos c \sin c + h^2(\cos^2 b - \cos^2 c) = 0$, which is an equation to a conic section, viz. to an ellipse when the co-efficient of x^2 is positive, or $\cos b > \sin c$; to an hyperbola when it is negative, or $\cos b < \sin c$; and a parabola when it is $= 0$, or $\cos b = \sin c$. When $b = 90^\circ$, or the sun is in the equinoctial, the co-efficient of y^2 is $= 0$, and x is a constant quantity $= h \cot c$, which shews that the path is then a straight line.

Hence it appears that the path of the extremity of the shadow is an ellipse when the sun's declination is greater than the co-altitude, a parabola when they are equal, and a hyperbola when the declination is less than the co-altitude, except on the equinoctial days when it is a straight line.

The axes of the curve may be determined from the above equation by finding the value of x when $y = 0$, in which case

$$x \text{ is } = \frac{h \sin b \cos b}{\cos^2 b - \sin^2 c} \mp \frac{h \cos c \sin c}{\cos^2 b - \sin^2 c},$$

the sum of which values of x or $\frac{2h \sin b \cos b}{\cos^2 b - \sin^2 c}$ is = the transverse axis, when the curve is an ellipse, and half the difference, or $\frac{h \cos c \sin c}{\cos^2 b - \sin^2 c}$ = the distance of the point c from the vertex of the

curve. Whence the conjugate axis is found = $\frac{2h \sin b}{\sqrt{(\cos^2 b - \sin^2 c)}}$.

But when the curve is a hyperbola or $\sin c$ greater than $\cos b$, the transverse axis is $\frac{h \cos c \sin c}{\sin^2 c - \cos^2 b}$, and conjugate $\frac{2h \sin b}{\sqrt{(\sin^2 c - \cos^2 b)}}$.

To apply what has been done to the solution of the question, let cp , cp be the extreme shadows of the tree, then we have to find the area of the space $cpapc$ included between those lines and the curve. Now, because the angles pca and pca are given, pc and pc may be found from formula (1), and then pb and bc , $pb = pc \sin pcb$, $bc = pc \cos pcb$, $pb = cp \sin bcp$, and $cb = cp \cos bcp$. The axes of the curve may also be found as above, and then the areas of the parts pba , pba by the rules of mensuration. If to these the areas of the two triangles be added, the sum will be the area sought.



PRIZE QUESTION 1721.

As the first article in Dr. Hutton's *Miscellanea Mathematica* is very frequently referred to in the course of this work, we shall insert it here. It contains a very ingenious and complete tract on the exhaustion of vessels of a fluid extended to the different figures as far as lines of the second order, according to the hypothesis that the velocity is that due to *half* the height of the fluid above the aperture, which is supposed to be very small compared with the surface of the fluid. If the velocity is supposed to be that due to the *whole* height of the fluid above the orifice, then that author desires it to be observed, that the time of exhaustion, or the quantity of fluid issuing in any time, will still be the same, as the area of the stream is less than that

of the orifice, in the same ratio as the two velocities from those pressures, viz. the ratio of $\sqrt{2}$ to 1, so that it comes to the same thing whether we use the whole height and the reduced area of the orifice, or half the height and the whole area of the orifice.

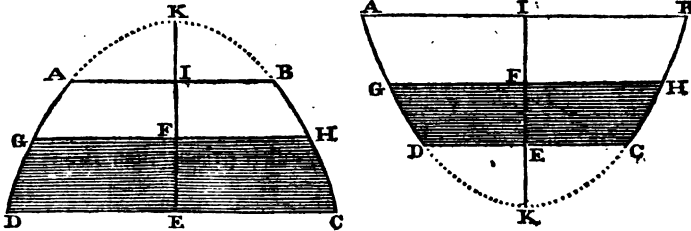
A dissertation on the times of exhausting vessels of a fluid, through a hole or aperture in their bottoms, parallel to the horizon.

THEOREM I.

Supposing ABCD to be any vessel containing a fluid ; and putting $m = 32\frac{1}{2}$ feet = 386 inches, n = the area of the aperture in the bottom DC, x = EF the altitude of the surface of the fluid above the bottom, and z = the area of the descending surface GH : then, the time of exhausting the fluid CDGH will be equal to the fluent of

$$\frac{zx^{\frac{3}{2}}}{n\sqrt{mx}}$$

Demonstration. By Sir I. Newton's Principia. lib. II. prop. 36, the velocity of the issuing fluid at E, is equal to that which is acquired



by a body in falling through $\frac{1}{2}EF$ or $\frac{1}{2}x$; but the velocities of falling bodies are as the roots of the spaces fallen, and m is the velocity (per second) acquired by falling through the space $\frac{1}{2}m$; hence $\sqrt{\frac{1}{2}m} : \sqrt{\frac{1}{2}x} :: m : \sqrt{mx}$ = the velocity of the issuing fluid at E ; but the velocities at the orifice and of the descending surface GH will be inversely as their areas (for since the quantity, or solidity is the same, the velocities, or altitudes of the equal column will be inversely as

the sections) ; therefore $z : n :: \sqrt{mx} : \frac{n\sqrt{mx}}{z}$ = the velocity (per second) of the descending surface GH, or the space through which GH would uniformly descend in one second with the velocity it hath at the altitude x . Now in descending the space $\frac{1}{2}x$, the velocity

may be considered as uniform; and uniform descents are as their times, wherefore $\frac{n\sqrt{mx}}{z} : \dot{x} :: 1 \text{ second} : \frac{zx}{n\sqrt{mx}}$ = the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. Q. E. D.

Scholium. In the above investigation, the resistance made by the air to the issuing fluid, is neglected, the exhaustion being supposed to be made in a vacuum; which will occasion a small difference (though scarce perceptible) between the calculations and experiments made in the air. But this consideration must by no means be neglected, when the density of the medium, into which the fluid issues, bears any considerable proportion to that of the fluid.

It will make no difference in the account, in whatever part of the base the aperture is placed, the altitude of the surface above it being the only consideration; nor is it material what the figure of it is, whether circular, triangular, square, &c. regular or irregular, the area of it alone being the only necessary consideration.

As we have above found the fluxion of the time, so by taking the fluent of it we obtain the time itself, substituting first the value of x instead of it, as found in terms of x from the equation of the figure of the vessel, as in the following problems, except when it is a prism, for then x is constant, and does not affect the fluent.

PROBLEM I.

To find the time of emptying a prism.

When the vessel is a cylinder or any other prism, then the section qm or x is constant, and considering it as such, the fluent of $\frac{zx}{n\sqrt{mx}}$ is $\frac{2x}{n} \sqrt{\frac{x}{m}}$, which is the time of exhausting a prism whose base is x , and altitude x .

Cor. 1. If the apertures of two prisms therefore be equal, the times of emptying them, will be as their bases drawn into the square roots of their altitudes. If their bases be also equal, the times will be as the square roots of the heights. But when their altitudes are equal, the times will be as their bases.

Cor. 2. Putting a for the altitude xi of the prisms $abcd$, and b for the area of the base cd ; then the time of emptying the part $abmg$ at x or D is $\frac{2b}{n} \times \frac{\sqrt{a} - \sqrt{x}}{\sqrt{m}}$. For, by the problem, the

time of emptying ABCD is $\frac{2b}{n} \sqrt{\frac{a}{m}}$; and the time of emptying GHCD is $\frac{2b}{n} \sqrt{\frac{x}{m}}$; their difference is the time of emptying ABHG as above.

Cor. 3. By the problem, the time of exhausting ABHG at G or F is $\frac{2b}{n} \sqrt{\frac{a-x}{m}}$; comparing this therefore with the last corollary, it appears that the times of emptying the same part ABHG, at the depth D and G, will be respectively as $\sqrt{a} - \sqrt{x}$ to $\sqrt{a-x}$.

Cor. 4. By writing, in the general theorem $\left(\frac{zx}{n\sqrt{mx}}\right)$, b for x , a for x , and $a-x$ for x , we obtain $\frac{a-x}{n\sqrt{ma}} \times b$ for the time in which AB would descend to GH with the first velocity, or the time in which a quantity equal to ABHG would run out supposing the vessel to be kept always full by a supply from without; comparing this therefore with corollary 2, it will appear that the time of emptying the part ABHG at D, when there is no supply from without, is to the time in which an equal quantity would run out when the vessel is kept always full, as $\sqrt{a} - \sqrt{x}$ to $\frac{a-x}{2\sqrt{a}}$, or as $2 \times (a - \sqrt{ax})$ to $a - x$. And when x is nothing, or when ABHG becomes the whole prism, the proportion is that of $2a$ to a or 2 to 1.

PROBLEM II.

To determine the time of exhausting any pyramidal vessel.

Putting, as before, a = the altitude EI, b = the base or area of the bottom DC, t = the area of the top AB; and n = area of the aperture, $m = 32\frac{1}{2}$ feet; also x = any other altitude EF, and z the area of the section GH.

Then, by the nature of the figure, whether the greater or less end is uppermost, we shall have as IE : EF :: the difference between the lines AB, CD, to the difference between the lines GH, CD; that is

$a : x :: \sqrt{b} - \sqrt{t} : \sqrt{b} - \sqrt{z} = \frac{\sqrt{b} - \sqrt{t}}{a} \times x$; hence $\sqrt{z} = \sqrt{b} - \frac{\sqrt{b} - \sqrt{t}}{a} \times x$, and $z = b - \frac{b - \sqrt{bt}}{a} \times 2x + \frac{b - 2\sqrt{bt} + t}{aa} \times x^2$: this being substituted for z in $\left(\frac{zx}{n\sqrt{mx}}\right)$ the fluxion of the time, in the theorem, and the fluent taken, we obtain $\frac{2\sqrt{x}}{n\sqrt{m}}$

$\times : b - \frac{b - \sqrt{bt}}{3a} \times 2x + \frac{b - 2\sqrt{bt} + t}{5aa} \times s'$ for the time of emptying the part $cdgh$, for either end up.

Cor. 1. When $xf = xi$, or $x = a$, the above expression becomcs $\frac{2\sqrt{a}}{n\sqrt{m}} \times : b - \frac{2b - 2\sqrt{bt}}{3} + \frac{b - 2\sqrt{bt} + t}{5}$, or $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{8b + 4\sqrt{bt} + 3t}{15}$ for the time of emptying the whole frustum $abcd$.

Cor. 2. When t and b are equal to each other, the above expression in corollary 1 becomes $\frac{2b\sqrt{a}}{n\sqrt{m}}$ for the time in the prism, the same as in the last problem.

Cor. 3. When t is = nothing, the expression in the first corollary becomes $\frac{16b\sqrt{A}}{15n\sqrt{m}}$ for the time of emptying the whole pyramid dxk by a hole at the base; A being here the whole altitude ek .

Cor. 4. When b is = nothing, the expression in the first corollary becomes $\frac{2t\sqrt{A}}{5n\sqrt{m}}$ for the time of emptying the whole pyramid $k\lambda k$ at the vertex, k ; A being the whole altitude xi , and t the base or the same with b in the last corollary. And hence the first corollary to the last problem will hold also in this.

Cor. 5. By comparing the last three corollaries together, it appears that the time of emptying the prism, the pyramid at its base, and at its vertex, are to one another as 15, 8, and 3; the apertures, altitudes, and bases being equal in each figure.

Cor. 6. Putting A = the axe ek of the pyramid, with either end up, or the distance of the vertex from dc , and the other letters as in corollary 1: Then, by the nature of the pyramid, $A : A \mp a :: \sqrt{b} : \frac{A \mp a}{A} \times \sqrt{b} = \sqrt{t}$; which being substituted for it in the first corollary, we have $\frac{2b\sqrt{a}}{n\sqrt{m}} \times \frac{15AA \mp Aa + 3aa}{15AA}$ for the time of exhausting $abcd$ at cd ; viz. either — or + according as the greater or less end is downward.

But if A represent xi when the vertex is downward, and xe when upward, or the distance from the vertex to the greater end; then, as above, $\frac{2b\sqrt{a}}{n\sqrt{m}} \times \frac{15AA - 10Aa + 3aa}{15AA}$ will be the time of empty-

ing ABCD with the greater end down. But when the less end is down,

then $A : A - a :: \sqrt{t} : \frac{A - a}{A} \times \sqrt{t} = \sqrt{b}$; which being writ-

ten for it in corollary 1, we have $\frac{2t\sqrt{a}}{n\sqrt{m}} \times \frac{15AA - 20Aa + 8aa}{15AA}$ for

the time of emptying ABCD at the less end; where A and a have the same value as at the other end, and t here = b there.

Cor. 7. By taking the difference between two times for two different altitudes, there will be obtained the time of emptying the part next the top of the vessel whose altitude is equal to the difference of the other two next the bottom.

By problem 1 and its 4th corollary, $\frac{b\sqrt{a}}{n\sqrt{m}}$ is the time in which a prism of water would run out with the first velocity, and the prism being to the frustum of a pyramid (whose base is b, top t, and altitude a) as b to $\frac{b + \sqrt{bt} + t}{3}$, therefore $\frac{\sqrt{a}}{n\sqrt{m}} \times \frac{b + \sqrt{bt} + t}{3}$ is the time in which a quantity would run out equal to the frustum of a pyramid when it is kept full by a supply at the top. And by comparing this with corollary 1, it will appear that the time of emptying the frustum of a pyramid when it has no supply from without, is to the time in which an equal quantity would run out when kept always full, as $16b + 8\sqrt{bt} + 6t$ to $5b + 5\sqrt{bt} + 6t$.

Cor. 8. By supposing either b or t to vanish in these last expressions, we shall obtain the proportions of the times when there is no supply from without to that when there is, for the whole pyramid. Thus, when $t = 0$, the expressions becomes as 16 to 5 by running out at the base; and when $b = 0$, they are as 6 to 5 at the vertex: so that they are to one another as 16, 6, and 5.

THEOREM II.

If x be the altitude of any part of a vessel formed from a line of the second order, viz. a conic section, $m = 32\frac{1}{2}$ feet, n = the area of the aperture in the bottom, and A, B, c constant quantities: Then the time of emptying the part CDEH whose altitude is x will be

$$\frac{2\sqrt{x}}{n\sqrt{m}} \times (A + \frac{1}{2}Bx + \frac{1}{3}Cx').$$

Demonstration. By theorem 1, the fluxion of the time is $\frac{zx}{n\sqrt{mx}}$; and by page 222 of my Mensuration z is as $A + Bx + Cx^2$, or x may

be supposed $= A + bx + cx^2$: Substituting therefore this value of z instead of it, and taking the fluent, we obtain the time as is expressed in the theorem.

Scholium. The particular values of A , B , and C will be determined from the nature of the curve in question, and from the situation of x with regard to the vertex or to the centre of the curve. In general, however,

Corollary. When x begins at the vertex of the curve, then A is $= 0$, and the above expression becomes $\frac{\sqrt{x}}{n\sqrt{m}} \times (\frac{1}{3}bx + \frac{1}{5}cx^2)$ for the time in the whole solid.

PROBLEM III.

To find the time of emptying a parabolic vessel, or any part of a paraboloid.

In the paraboloid, putting, as before, t for the area of the top AB of the vessel, b for that of the bottom DC , a for its altitude EF , $x = EF$, and $z =$ the section GH . Then, by the property of the figure, $a : x :: t - b : z - b$; hence $z = b + \frac{t-b}{a} \times x$: But the general value of z is $A + bx + cx^2$: Comparing these two therefore together, we have $A = b$, $b = \frac{t-b}{a}$, and $c = 0$. By substituting therefore these values in theorem 2, we obtain $\frac{2\sqrt{x}}{n\sqrt{m}} \times (b + \frac{t-b}{3a}) \times x = \frac{2\sqrt{x}}{n\sqrt{m}} \times \frac{3ab + tx - bx}{3a}$ for the time of emptying $GHCD$, whether the greater or less end be upward.

Other expressions for the time might be found, by bringing into the form the whole axe of the parabola and its parameter.

Cor. 1. When x is $= a$, the above form becomes $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{2b + t}{3}$ for the time of emptying the whole frustum $ABCD$.

And the difference between this and the former would give the time of emptying $ABHG$ at DC .

Cor. 2. When t is $= 0$, the last becomes $\frac{4b\sqrt{a}}{3n\sqrt{m}}$ for the time of emptying the whole paraboloid at the base; A being the whole axe.

Cor. 3. When $b = 0$, the first corollary becomes $\frac{2t\sqrt{a}}{3n\sqrt{m}}$ for the time of emptying the same at the vertex; t in this being equal b in the last, and $a =$ the whole axe as in the last corollary. So that the times are as 2 to 1. Any time is universally as the base into the root of the altitude, &c. in every particular as in corollary 1 to problem 1.

Cor. 4. When $b = t$, corollary 1 gives $\frac{2b\sqrt{a}}{n\sqrt{m}}$ for the prism as in the first problem.

Cor. 5. A parabolic frustum whose base is b , top t , and altitude a , is to a prism of $=$ base and altitude, as $\frac{b+t}{2}$, to b ; hence, by proceeding as in corollary 8 to prob. 2, we shall obtain $\frac{\sqrt{a}}{n\sqrt{m}} \times \frac{b+t}{2}$ for the time in which a quantity equal to the whole frustum ABCD would run out, when it is kept full by a supply at the top; and is therefore to the time, in corollary 1, of emptying the same when there is no supply, as $b+t$ to $\frac{2b+t}{3}$; or as $3b+t$ to $8b+4t$.

Cor. 6. When $t = 0$, the last proportion becomes as 3 to 8 for the times at the base: And when $b = 0$, it becomes that of 3 to 4 at the vertex. So that all the three times for the whole paraboloid, viz. time at the base, and time at the vertex when there is no supply from without, and the time of running an equal quantity at either end when it is kept full by a supply from without, are respectively as the numbers 8, 4, and 3.

PROBLEM IV.

To find the time of emptying any part of the sphere.

Putting r for the radius of the sphere, $p = 3.14159$, &c. q for the distance of the centre of the sphere above the bottom of the vessel, x for the altitude of the vessel from the bottom; and z , m , n , as before. Then, by the nature of the circle, $z = (r + q - x) \times (r - q + x) \times p = [r^2 - (q - x)^2] \times p = (rr - qq + 2qx - xx) \times p$: Comparing this with the general expression $z = A + Bx + Cx^2$, we have $A = (rr - qq) \times p$, $B = 2pq$, and $C = -p$;

which values being substituted in theorem 2, we get $\frac{2p\sqrt{x}}{n\sqrt{m}} \times (rr -$

$qq + \frac{2}{3}qx - \frac{1}{3}xx$) for the time of emptying the spherical vessel whose altitude is x .

The above supposes the bottom of the vessel to be below the centre, but the expression will hold in every case; for when the bottom is above the centre, then q is negative, and the term $+\frac{2}{3}qx$ will become $-\frac{2}{3}qx$; and when the bottom passes through the centre, then $q = 0$, and the terms qq and $+\frac{2}{3}qx$ vanish.

Cor. 1. When $q = 0$, the expression becomes $\frac{2p\sqrt{x}}{n\sqrt{m}}(rr - \frac{1}{3}xx)$ for the time of emptying at the base any altitude x of a hemisphere.

Cor. 2. And when $x = r$ the whole altitude, the last expression becomes $\frac{8pr\sqrt{r}}{5n\sqrt{m}}$ for the time of emptying the hemisphere at the base
 $= \frac{8b\sqrt{r}}{5n\sqrt{m}}$, putting $b = prr$ the area of the base or greatest circle.

Cor. 3. When the bottom of the vessel or the aperture is at the vertex of the figure, then $q = -r$, and the expression in the problem becomes $\frac{2p\sqrt{x}}{n\sqrt{m}} \times (\frac{2}{3}rx - \frac{1}{3}xx)$ for the time of emptying at the vertex any altitude x of the hemisphere.

Cor. 4. When $x = r$, the last expression becomes $\frac{14pr\sqrt{r}}{15n\sqrt{m}}$
 $= \frac{14b\sqrt{r}}{15n\sqrt{m}}$ for the time of emptying the hemisphere at the vertex.

Cor. 5. When $x = 2r$, the expression in corollary 3 becomes $\frac{16pr\sqrt{2r}}{15n\sqrt{m}} = \frac{16b\sqrt{2r}}{15n\sqrt{m}}$ for the time of emptying the whole sphere.

Cor. 6. Subtracting the time in cor. 4 from that in cor. 5, we obtain $\frac{2pr\sqrt{r}}{n\sqrt{m}} \times \frac{8\sqrt{2}-7}{15} = \frac{2b\sqrt{r}}{n\sqrt{m}} \times \frac{8\sqrt{2}-7}{15}$ for the time of emptying the upper hemisphere at the vertex of the lower one.

Cor. 7. By comparing cor. 1, with cor. 3, it appears that the times of emptying equal altitudes of a hemisphere at the base and vertex, are to each other respectively as $3rr - \frac{2}{3}xx$ to $2rx - \frac{1}{3}xx$, or as $15rr - 3xx$ to $10rx - 3xx$.

And when x becomes equal to r , the times of emptying the whole hemisphere at the base and vertex, are as 12 to 7.

Cor. 8. By cor. 4, to prob. 1, $\frac{2pr\sqrt{r}}{3n\sqrt{m}}$ or $\frac{2b\sqrt{r}}{3n\sqrt{m}}$ is the time of running out a quantity equal to the hemisphere with the first velocity, or when it is kept full by a continual supply from without; by comparing this with cor. 2 and 4, we find that the above time and the times of emptying at the vertex and base when there is no supply, are as the numbers 5, 7, 12.

PROBLEM V.

To determine the time of emptying any spheroidal vessel.

Putting r for the revolving and f for the fixed semi-axe, and the other quantities as in the last problem. Then in this problem will be two cases, viz. one with the revolving and the other with the fixed axe horizontal.

CASE I.

When the revolving axe is horizontal,

In this case the horizontal sections are circles, and the circle z is $= \frac{pr}{ff} \times (ff - qq + 2qx - xx)$; and then proceeding as in the last problem, we shall obtain $\frac{2pr\sqrt{x}}{ffn\sqrt{m}} \times (ff - qq + \frac{2}{3}qx - \frac{1}{3}xx)$ for the time of emptying any part whose altitude is x .

Cor. 1. When the bottom of the vessel passes through the centre of the spheroid, then $q = 0$, and the above expression becomes $\frac{2pr\sqrt{x}}{ffn\sqrt{m}} \times (ff - \frac{1}{3}xx)$ for the time of emptying at the circular base any altitude x of a hemi-spheroid.

Cor. 2. And when $x = f$ the whole altitude, the last expression becomes $\frac{8pr\sqrt{f}}{5n\sqrt{m}} = \frac{8b\sqrt{a}}{5n\sqrt{m}}$ for the time of emptying the hemi-spheroid at the base; putting $b = pr$ the base, and a for the whole altitude or vertical semi-axe.

Cor. 3. When the water issues at the vertex, then $q = f$ and the expression in the problem becomes $\frac{2pr\sqrt{x}}{ffn\sqrt{m}} \times (\frac{2}{3}fx - \frac{1}{3}xx)$ for the time of emptying at the vertex any altitude x .

Cor. 4. When x is $= f$, the last expression becomes $\frac{14prr\sqrt{f}}{15n\sqrt{m}}$
 $= \frac{14b\sqrt{a}}{15n\sqrt{m}}$ for the time of emptying the semi-spheroid at the vertex.

Cor. 5. And when $x = 2f$ the whole vertical axe, then it becomes $\frac{16prr\sqrt{2f}}{15n\sqrt{m}} = \frac{16b\sqrt{A}}{15n\sqrt{m}}$ for the time of emptying the whole spheroid, A being the whole vertical axe.

CASE II.

When the fixed axe is horizontal.

In this case the horizontal sections are ellipses, and the section x is $= \frac{pf}{r} \times (rr - qq + 2qx - xx)$; and therefore $\frac{2pf\sqrt{x}}{rn\sqrt{m}} \times (rr - qq + \frac{2}{3}qx - \frac{1}{3}xx) =$ the time of emptying the frustum whose altitude is x .

Cor. 1. When $q = 0$, the above expression becomes $\frac{2pf\sqrt{x}}{rn\sqrt{m}} \times (rr - \frac{1}{3}xx)$ for the time of the frustum whose altitude is x , and its bottom the greatest ellipse or section through the centre.

Cor. 2. And when $x = r$, the last becomes $\frac{8pfr\sqrt{r}}{5n\sqrt{m}} = \frac{8b\sqrt{a}}{5n\sqrt{m}}$ for the time of emptying the semi-spheroid at the base; b and a being as in the last case.

Cor. 3. When $q = r$, the general expression becomes $\frac{2pf\sqrt{x}}{rn\sqrt{m}} \times (\frac{2}{3}rx - \frac{1}{3}xx) =$ the time of the segment at the vertex.

Cor. 4. And when $x = r$, this last becomes $\frac{14pfr\sqrt{r}}{15n\sqrt{m}} = \frac{14b\sqrt{a}}{15n\sqrt{m}}$ for the time of the semi-spheroid at the vertex.

Cor. 5. Also when $x = 2r$, the same becomes $\frac{16pfr\sqrt{2r}}{15n\sqrt{m}} = \frac{16b\sqrt{A}}{15n\sqrt{m}}$ for the time of the whole spheroid, A being the whole vertical axe.

And the times for the hemi-spheroid when there is no supply is to that when there is, the very same as the times for the hemisphere in the 8th cor. to the last problem, viz. as the numbers 12, 7, 5.

PROBLEM VI.

To determine the time of emptying any part of a hyperboloid.

Putting t for the semi-transverse, c the semi-conjugate, x for any altitude from the bottom of the vessel, z the section at the top of x , q for the altitude of the centre of the generating hyperbola above the bottom of the vessel, and which therefore must be negative when the less end of the vessel is downward, and m, n, p , as before,

Then by the nature of the hyperbola, $tt : cc :: p \times (q - x + t) \times (q - x - t) : \frac{pcc}{tt} \times [(q - x)^2 - t^2] = \frac{pcc}{tt} \times (qq - tt - 2qx + xx)$. Hence, as in the last problem, $\frac{2pcc\sqrt{x}}{ttn\sqrt{m}} \times (qq - tt - \frac{2}{3}qx + \frac{1}{3}xx)$ will be the time of emptying any frustum whose altitude is x .

Cor. 1. In a whole hyperboloid with the vertex upward, q is $= t + x$; which being substituted for it in the above expression, we obtain $\frac{8pcc\sqrt{x}}{15ttn\sqrt{m}} \times (5tx + 2xx)$ for the time of emptying the whole hyperboloid at the base whose altitude is x .

But since the base b is $= pcc \times \frac{2tx + xx}{tt}$, the time of emptying the same will also be $\frac{8b\sqrt{x}}{15n\sqrt{m}} \times \frac{5t + 2x}{2t + x}$.

Cor. 2. And when $x = t$, the above expression becomes $\frac{56cc\sqrt{t}}{15n\sqrt{m}} = \frac{56b\sqrt{a}}{45n\sqrt{m}}$ for the time of emptying a whole hyperboloid at the base when the altitude is equal to the semi-transverse; b being the base, and a the altitude.

Cor. 3. When the smaller end is downward, then q is negative, and the general expression becomes $\frac{2pcc\sqrt{x}}{ttn\sqrt{m}} \times (qq - tt + \frac{2}{3}qx + \frac{1}{3}xx)$ for the time in a frustum with the less end downward.

Cor. 4. And when q is $= t$, this last expression becomes $\frac{2pcc\sqrt{x}}{ttn\sqrt{m}} \times (\frac{2}{3}tx + \frac{1}{3}xx) = \frac{2b\sqrt{x}}{15n\sqrt{m}} \times \frac{10t + 3x}{2t + x}$ for the time of emptying

a whole hyperboloid at the vertex, the altitude being x , and the area of the end b .

Cor. 5. And when x is $= t$, the last form becomes $\frac{26pcc\sqrt{t}}{15n\sqrt{m}}$
 $= \frac{26b\sqrt{t}}{45n\sqrt{m}} = \frac{26b\sqrt{a}}{45n\sqrt{m}}$ for the time of emptying at the vertex the hyperboloid whose altitude is equal to the semi-transverse.

Cor. 6. Since $\frac{b\sqrt{x}}{n\sqrt{m}} =$ the time of running out a quantity equal to the prism bx with the first velocity; and, by page 384 of my Mensuration, $bx \times \frac{t + \frac{1}{2}x}{2t + x}$ being the content of the hyperboloid; wherefore $\frac{b\sqrt{x}}{3n\sqrt{m}} \times \frac{3t + x}{2t + x}$ will be the time of running out a quantity equal to the hyperboloid with the first velocity, or when it is kept full by a continual supply from without. Comparing therefore this with cor. 1 and 4, it appears that the time above and the times of emptying at the vertex and base when there is no supply from without, are to one another as the three quantities, $5 \times (3t + x)$, $2 \times (10t + 3x)$, and $8 \times (5t + 2x)$; and when the altitude x is equal to t , the same times are as the numbers 10, 13, and 28.

General Scholium. In all the foregoing compleat solids, viz. the prism, the whole pyramid, the whole paraboloid and hyperboloid whose altitude is equal to the semi-transverse, and hemi-sphere and hemispheroid, it appears that the time of emptying each, either at the base or vertex, is always as the base drawn into the square root of the altitude, and that all that is said in cor. 1 to prob. 1 belongs in common to them all. And, collecting all the rules for the times of exhausting at the base and vertex, and the time of running out an equal quantity with the first velocity, or when the vessel is kept always full by a continual supply from without, we shall have the time for each equal to $\frac{\text{base} \times \sqrt{(\text{altitude})}}{\text{apert.} \times \sqrt{(32\frac{1}{2} \text{ feet})}}$ drawn into these corresponding numbers:

	At the base.	At the vertex.	With the 1st. veloc.
For the prism	2	2	1
Hemisphere and Hemispheroid..	$\frac{8}{5}$	$\frac{14}{15}$	$\frac{2}{3}$
Paraboloid	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
Hyperboloid	$\frac{56}{45}$	$\frac{26}{45}$	$\frac{4}{9}$
Pyramid	$\frac{16}{15}$	$\frac{2}{5}$	$\frac{1}{3}$

QUESTIONS 100, 290, 1013.

Additional Solution from Cagnoli's Trigonometry.

Let A, B, C be the three given points, m and n the observed angles at D . It is required to find the distances DA, DB , and DC . Since the three sides of the triangle ABC are given, the angles may be found.

From the triangle ABD , $\sin m : \sin x :: AB : BD$.

From the triangle BCD , $\sin(m+n) : \sin z :: BC : BD$.

Therefore, equating the values of BD , we have

$$\frac{AB \sin x}{\sin m} = \frac{BC \sin z}{\sin(m+n)}$$

and therefore $\sin z = \frac{AB \sin(m+n)}{BC \sin m} \times \sin x$.

But $z = 180^\circ - n - y = 180^\circ - n - (180^\circ - B - x) = x + B - n$;

therefore $\sin z = \sin x \cos(B-n) + \cos x \sin(B-n)$;
wherefore, equating the values of z ,

$$\sin x \cos(B-n) + \cos x \sin(B-n) = \frac{AB \sin(m+n)}{BC \sin m} \times \sin x;$$

dividing both sides by $\sin x$, gives

$$\cot x = \frac{AB \sin(m+n)}{BC \sin m \sin(B-n)} - \cot(B-n),$$

$$\text{or, } \cot x = \cot(B-n) \left\{ \frac{\sin C \sin(m+n)}{\sin BAC \sin m \sin(B-n)} - 1 \right\}.$$

The angle x (BAD) being given by this equation, we shall have also the angle CAD ; and the resolution of the triangles ABD, ACD , will give the distances sought.

If the angle $B < n$, the $\cot(B-n)$ will be negative, and this is the only change that can take place in the equation.

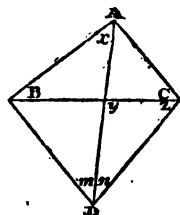
If the point D is within the triangle $(m+n) > 180^\circ$, and $\sin(m+n)$ also negative.

If one of the angles A, B, C is found to be obtuse, it will be better to find a segment of one of the acute angles, in order to avoid all embarrassment and all uncertainty with respect to the species of the angle x .

If the three points B, A, C are in the same straight line, then $B = 0$, A coincides with y , and the angle x with the angle xyD : in this case,

$$\cot x = \cot n \left(1 - \frac{AB \sin(m+n)}{BC \sin m \cos n} \right) = \frac{AC \cot n - AB \cot m}{BC}.$$

Finally, if $B = n$, the problem will be indeterminate, $\cot(B-n)$ being infinite; which shews that the point D will have an infinite number of positions, all satisfying the question.



QUESTION 110.

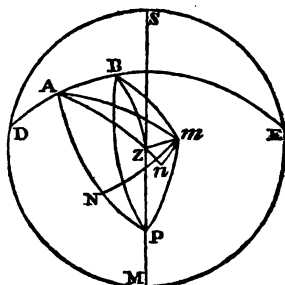
If the ship were supposed to move over a given distance during the interval of the observations, the solution may be as follows:

ARTICLE VII. BURROW'S DIARY 1776.

A method of finding the latitude at sea from having the altitudes of the sun or a star and the intermediate time.

By Reuben Burrow.

It must first be observed that if it be a star whose altitudes are taken, the watch ought to be regulated according to sidereal time, or the interval measured by it must have such corrections applied, as to give the angle the same, as the watch would do if it was exactly regulated to sidereal time: but if the sun be observed the watch ought to be regulated to apparent time: or to have an equivalent reduction; the manner of doing which I shall not here insist on. This being premised, let MDSE represent the horizon, DABE the sun or star's parallel of declination, and P the elevated pole; then PA and PB will be the polar distance, which must be corrected for parallax, &c. A and B being the places of the object when its altitudes were taken. Now if Z be the ship's place at the time of taking the first altitude, and if the ship move the given distance Zm in the interval of the observations, making a given angle AZm with the sun's bearing at the time of taking the first observation, the zenith distance at the time of taking the second altitude at m will be Bm; and to find what the first zenith distance would have been, if taken at the same place, let fall the perpendicular mn on AZ produced if necessary; and say, as radius: is to the cosine of AZm the sun's bearing at the first observation:: so is Zm the distance run between the observations: to zn, the correction of the first zenith distance, which must be added to it when the ship sails on a course which is more than 90 degrees from the sun, otherwise subtracted; and this reduces the first zenith distance nearly to what it would have been if taken at the same place as the last.



Then in the triangle PAB, $PA = PB =$ the polar distance is given, and also the included angle PAB measured by the interval of time;

therefore $\text{rad} : \cos PA :: \tan \frac{1}{2}P : \cot PAB,$

and $\text{rad} : \cos PAB :: \tan PA : \tan \frac{1}{2}AB;$

therefore AB, Am and Bm are given, and consequently the angle.

mAB may be found. But PAB is given, and therefore PAM is also given; wherefore in the triangle PAM , we have given PA , AM and the angle PAM , consequently the side PM , which is the complement of the latitude, may be found by drawing MN perpendicular to AP , and saying,

As $\text{rad} : \cos PAM :: \tan AM : \tan AN$,
and $\cos AN : \cos PN :: \cos AM : \cos PM = \text{sine of the latitude required.}$

QUESTION 113.

Colonel Silas Titus' problem having excited successively the attention of some of the first algebraists of this country, the public may not be displeased with a solution of it without substitutions, and free from the unnecessary complication of those of Dr. Wallis and Kirkby, of which this may be considered as a simplification. At the time this problem was solved by the Savilian professor, De Beaune and Hudden's method of elimination (*Cartesii Geom.* vol. 2, p. 114, 492, an. 1659) had probably not reached England, and there was no method of exterminating quantities from equations universally known as yet, besides that of substituting their values instead of them, and it is evident that this method required the algebraic resolution of equations, which is not yet extended beyond biquadratics, and that at last, even where this could be practised, the labour of clearing the final equation from surds remained a difficulty, in many cases, not much less than that of the original problem. It is true, that long before the time we speak of Monsieur Fermat had discovered the general method of extermination, which is found in his works, published in 1679, under the title "*Novus secundarum et ulterioris ordinis radicum in analyticis usus*," and "*Appendix ad superiorem methodum*." But this (which enunciated as a theorem, I hold to be one of the most general and elegant in all the analysis of finite quantities, namely, that "if there be ever so many algebraical equations among as many unknown quantities, any one of these may be found by an algebraical equation, and the rest are rational functions of it") was kept by Fermat a profound secret, or at least communicated only to a few select friends. Notwithstanding the publication of these methods and of that of Newton (the same in fact as De Beaune's) in the *Arithmetica Universalis* in 1707, this problem has been considered difficult by one of the most learned and laborious algebraists of the present day, who has condescended to explain Dr. Wallis's solution, and to verify his numbers by more expeditious methods of approximation. Before I proceed to the solution, I shall give Dr. Wallis's account of the problem in his own words, from which I have been induced to think that it may have been intended by

the first proposer, Dr. Pell, as a trial of the skill of the author of the arithmetic of infinites, of whose algebra it occupies four whole chapters, viz 60th, 61st, 62nd, and 63rd, extending over three and thirty folio pages.

"For a further explication of Dr. Pell's method, I here subjoin another question, not undetermined as the former; but (in itself determined) proposed to myself, long since, by Colonel Silas Titus (then of his majesty's bed-chamber) a very ingenious person, and well skilled in affairs, civil and military, and very well accomplished in mathematical and other learning.

THE PROBLEM.

Supposing $\left\{ \begin{array}{l} aa + bc = 16 \\ bb + ac = 17 \\ cc + ab = 18 \end{array} \right\}$ What are the numbers a, b, c ?

And it was desired, that I would at least in decimal parts, inquire the near value of the numbers sought.

To this, (in a letter of June 12, 1662) I returned him this solution,

The numbers are $\left\{ \begin{array}{l} a = 2, 5255 + \\ b = 3, 9692 - \\ c = 3, 2406 - \end{array} \right\}$ proximè.

The nearest squares and rectangles of which numbers are these,

$aa = 6,3782 +$	$bb = 8,8161 -$	$cc = 10,5015 -$
$bc = 9,6220 -$	$ac = 8,1841 -$	$ab = 7,4987 -$
$aa + bc = 16,0002 -$	$bb + ac = 17,00002 -$	$cc + ab = 18,0002 -$

The process of which (because I understood from the colonel, it was a question proposed by Dr. Pell) I drew up in general terms, (after Dr. Pell's method, with which the colonel was well acquainted,) in this form; (as I find it yet amongst my loose papers)".

The 62nd chapter concludes with the following paragraph:

"And thus I have pursued the problem, through all its cases and varieties, to an absolute and universal solution and determination of them. Which I have done more at large and distinctly, to shew a method how the like may be done in other problems in like manner proposed."

I think it must now appear highly probable that this was a sort of challenge from Dr. Pell to Dr. Wallis, and that the latter considered the problem worthy of his best attention, and the solution as fit to be a pattern to succeeding analysts: that he retained this opinion in 1693 is a strong presumption, that his co-temporaries had given

no reason for changing it, yet this solution has been pronounced 'remarkably operose and inelegant,' and so it certainly is, but for patient labour, exactness and completeness almost beyond example or imitation.

New solution to Colonel Silas Titus's problem.

PROBLEM.

The letters l, m, n representing any given numbers, as for instance, 16, 17 and 18, it is required to find a, b, c such, that

$$(1.) \quad a^2 + bc = l,$$

$$(2.) \quad b^2 + ac = m,$$

$$(3.) \quad c^2 + ab = n.$$

Solution.

By transposing a^2 in the first, and b^2 in the second equation, we obtain,

$$(4.) \quad bc = l - a^2,$$

$$\text{and } (5.) \quad ac = m - b^2,$$

then by the multiplication of equals, there results

$$(6.) \quad abc^2 = m(l - a^2) - (l - a^2)b^2,$$

and by transposing ab in the third equation,

$$(7.) \quad n - ab = c^2,$$

whence by multiplying the two left and the two right-hand members of (6) and (7), and omitting the common factor c^2 , we shall obtain the following quadratic in b , viz. $nab - a^2b^2 = m(l - a^2) - (l - a^2)b^2$, which arranged according to the powers of b becomes

$$(8.) \quad (l - 2a^2)b^2 + nab = m(l - a^2).$$

Another equation in b and a without c will enable us to solve the problem, by finding an equation in a , and the given quantities; this we proceed to find as follows. We have seen that $bc = l - a^2$, and that $m - b^2 = ac$, now multiply their left, and also their right-hand members together, and omit the common factor c , there results, for a second equation in a and b , the following cubic, viz.

$$(9.) \quad mb - b^3 = (l - a^2)a.$$

But by (8.) $m(l - a^2) = (l - 2a^2)b^2 + nab$, therefore, by multiplying their left, and also their right-hand members, and omitting the common factor $(l - a^2)b$ there will result $m^2 - mb^2 = a(l - 2a^2)b + na^2$, and by transposition, we obtain

$$(10.) \quad m^2 - na^2 = mb^2 + a(l - 2a^2)b.$$

But by (8.) $(l - 2a')b' + nab = m(l - a')$ therefore multiply-
ing as before, and omitting the common factor b ,

$(m^2 - na')(l - 2a')b + na(m^2 - na') = m^2(l - a')b + m(l - a')(l - 2a')a$
by transposition,

$$\begin{aligned} \{ (m^2 - na')(l - 2a') - m^2(l - a') \} b &= \\ \{ m(l - a')(l - 2a') - n(m^2 - na') \} a. \end{aligned}$$

Whence, by division, we obtain the following value of b , viz.

$$\begin{aligned} (11.) \quad b &= \frac{a \{ m(l - a')(l - 2a') - n(m^2 - na') \}}{(l - 2a')(m^2 - na') - m^2(l - a')} \\ &= \frac{2ma' + (n^2 - 3lm)a^2 + m(l - mn)}{a(2na' - ln - m^2)}. \end{aligned}$$

Again by multiplying (10.) by $l - 2a'$, and (8.) by m , we obtain,

$$m(l - 2a')b^2 + a(l - 2a')^2b = (l - 2a')(m^2 - na')$$

and $m(l - 2a')b^2 + mnab = m^2(l - a')$, whence, by subtraction,

$$\{ (l - 2a')^2 - mn \} ab = (l - 2a')(m^2 - na') - m^2(l - a').$$

Whence, by division, b is found, viz.

$$\begin{aligned} (12.) \quad b &= \frac{(l - 2a')(m^2 - na') - m^2(l - a')}{a \{ (l - 2a')^2 - mn \}} \\ &= \frac{a(2na' - ln - m^2)}{4a^4 - 4la^2 + ll - mn}. \end{aligned}$$

Now by equating the values of b in (11.) and (12.) we obtain the following final equation in a , viz.

$$(13.) \quad 8a^8 - 20la^6 + 2(9ll - mn)a^4 + (5lmn - 7lll - m^3 - n^3)a^2 + (ll - mn)^2 = 0,$$

if this equation be multiplied by 2, and then ee substituted for $2aa$, there will result

$$(14.) \quad e^8 5le^6 + (9ll - mn)e^4 + (5lmn - 7lll - m^3 - n^3)e^2 + 2(ll - mn)^2 = 0,$$

which coincides with that found by Dr. Wallis, Algebra, pa. 230, edit. 1685, also pa. 250, lat. edit. 1693.

Now then e^2 being found from (14.) $a = \sqrt{\frac{ee}{2}}$; then b is found by (11.) or (12.), and lastly $c = \frac{l - a^2}{b}$.

When 16, 17, 18 are substituted for l, m, n in the final equation, it becomes $e^8 - 80e^6 + 1998e^4 - 14937e^2 + 5000 = 0$, and hence, Wallis finds the approximate value of e^2 to be 12.7564, 4179, 4460, 744; or 0.350, 987, 046; or 34.832, 280, 28; or 32.060, 290, 88; and the corresponding values of a, b, c are

$$\begin{aligned}a &= 2525,513,986,744,158 \\b &= 2969,152,768,619,848 \\c &= 3240,580,681,617,174\end{aligned}$$

$$\begin{aligned}a &= 0.418,919,470 \\b &= 3912,226,866 \\c &= 4044,884,670\end{aligned}$$

$$\begin{aligned}a &= + 4.173,264,926 \\b &= + 4.287,022,553 \\c &= - 0.330,331,815\end{aligned}$$

$$\begin{aligned}a &= + 4.003,766,407 \\b &= - 0.007,099,744 \\c &= + 4.245,989,3\end{aligned}$$

And he observes that if the signs of every one of any set be changed they will solve the problem, because the signs of the planes $aa\ bb\ cc\ ab\ ac\ cb$ are not changed.

As it was not generally known, when the Doctor solved this problem, that b and c might be expressed by rational functions of a , he determines b from a quadratic equation in a , and l, m, n , as also from a cubic, and is at pains to show that only one root of the quadratic will serve, and that the cubic furnishes none serviceable but this

same; c he finds from the equation $c = \frac{l - a^2}{b}$. The quadratic

the Doctor found is less simple than mine; a ascending in it to the 4th power: the quadratic and cubic resulting for the extermination of b after c has been expell'd are the same in Kirkby; as in this of mine; but his solution is complicated with the unnecessary factors a^2 and $l - a^2$ as that of Wallis is by the square of this last: and I think they both seem to have imagined these factors somehow connected with the general problem, though it seems evident, from the solution here given, that they are not in the least. I have only to add that Dr. Wallis's solution is, in my opinion, as curious a morsel as can easily be found, as from its copiousness and minuteness, it enables us to ascertain very exactly the analytical skill of one who made so great a figure in the world of science, during the latter half of the seventeenth century, being along with Lord Brouncker, the unfoiled champion of England in the mathematical contests with M. M. Frenicle, Pascal, and Fermat, and in the doctrine of series and quadrature of curves, the precursor of Newton.

N.



QUESTION 167.

The problem of describing a circle to touch three circles, given in position and magnitude in the same plane, is reduced by Vieta, of whose Apollonius Gallus, Lawson's Tangencies is a translation, with additions, to the finding of a circle which shall pass through a given point, and touch two circles given in position and magnitude in the same plane as the given points. The construction of Vieta is singularly elegant. Newton's, it is true, is the same, but, his analysis is

very different, it having, apparently, been his intention, to shew that he could arrive at that construction, by the help only of a few of the most ordinary theorems in geometry. There has since been published, a new analysis of this problem, in the appendix to the *Opera Reliqua* of the late Dr. Simson of Glasgow, not less elegant even than that of Vieta, and as it leads directly to one of Pappus's lemmas for the Tactions, it is, probably, that of the great geometer Apollonius himself. Dr. Simson's book being very scarce, the reader may perhaps be pleased to see an extract from it here, which we proceed to give after the following lemmas from the seventh book of Pappus.

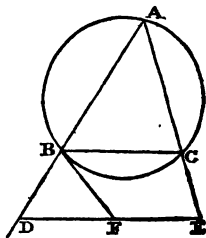
We shall afterwards insert Vieta's solution, as well as some others, which may not, perhaps, be in the possession of the generality of our readers.

I. Dr. Simson's Solution.

Lemma. Problem VIII. Prop. 107, 108, 109, Book VII. Pappus.

The circle A, B, C , being given in position, as also the two points D, E , to inflect DAE and make BC parallel to DE .

Suppose it done: And from either point of BC as B , draw the tangent BF : therefore, since this touches, and BC cuts the circle, the angle FBC , that is DFB , is equal to the angle A : therefore the points A, B, F, E are in a circle, and therefore the rectangle ADB is equal to the rectangle EDF . But the rectangle ADB is given because it is equal to the square of the tangent FB . Therefore the rectangle EDF is given, and DE is given, and therefore DF : and in position too: and the point D is given, therefore the point F is also given. Therefore, from the given point F , the tangent FB is drawn to a circle given in position: therefore FB is given in position, and the point B is given, and D is also given: therefore BD , is given in position, and since the circle ABC is given in position, the point A will also be given. But C is also given. Therefore, AD, AE are each given in position.

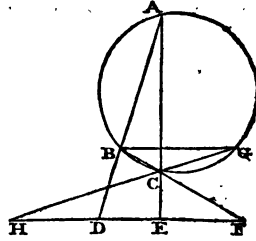


Lemma. Problem XI. Prop. 117, Book VII. Pappus:

The circle ABC being given in position, and the three points D, E, F in the same straight line being given, to inflect DAE so that BC may be in a straight line with CF .

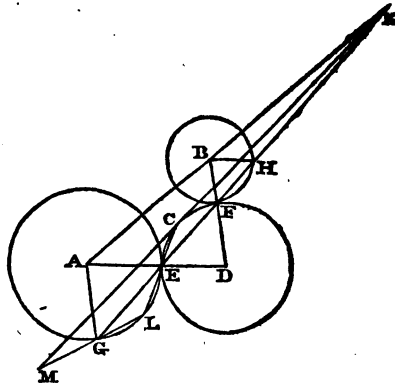
Suppose it done, and through B draw BC parallel to DF , and let CG joined, be produced to meet DF in H : therefore the angle BGC , that is, the angle A is equal to the angle CHF . Therefore the points

A, C, D, H are in a circle, and therefore the rectangle AEC is equal to the rectangle DEH : but the rectangle AEC, being equal to the square of the straight line drawn from E to touch the circle, is given ; therefore also the rectangle DEH, is given : and DE is given, therefore EH is given in magnitude as well as position, therefore H is given because E is given. Therefore HCF is inflected from the two given points H, F, so that HG is parallel to HF. But this has been constructed above : therefore the point C is given as well as E. Therefore CE is given in position as well as the circle, therefore the point A is given, and AD.



PROBLEM. The two circles whose centres are A, B being given in position and magnitude, and the point C being given : to describe a circle to touch the given circles, and pass through the given point.

Suppose it done, viz. Let the circle CEF touch the given circles in E, F, and let FE joined meet the circles again in G, H, and AB joined in K. Join AD, BD which will pass through E, F, join also AG, BH ; and since in the isosceles triangles AEG, DFE the angles at the base AEG, DEF are equal, the angles AGE, DFE are equal, therefore AG is parallel to DFB, therefore $AG : BF :: AK : KB :: GK : KF$ and AG, BF are given in magnitude, therefore their ratio, and that of AK to KB, and of GK to KF is given, but AB is given in position and magnitude, therefore the point K is given. Join CF and CE to meet the circle EG again in L : and let LG joined meet KC joined in M, and since the circles EGL, EFC touch each other in E, through which are drawn FEG, CEL, LG will be parallel to CF, by prop. 102 or 106, lib. 7, Pap. Alex. Therefore $GK : KF :: MK : KC$, but the ratio of GK to KF has been proved to be given, therefore that of MK to KC is given, but KC is given in position and magnitude, therefore KM and the point M are given. Therefore since there are given three points K, C, M in a straight line, and from two of them C and M is inflected CLM to a circle given in position, making GE in a straight line with EK, CL, LM are given in position, by 117, lib. 7, Pap. Alex. but the circle LGE is given in position ; wherefore the points G, E are given and the position of GE. But the circle FH is given in position,



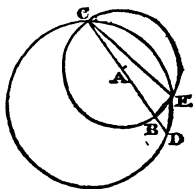
therefore the point r is given : and therefore the circle erc which passes through three given points is given in position.

Pappus and Simson add the composition, which for the sake of brevity are here omitted.

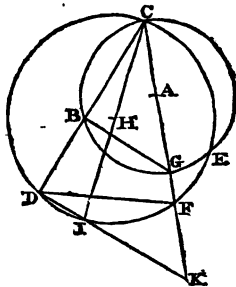
II. Vieta's Solution.

Lemma 1. If the two circles cxb and ced cut one another in c , then, I say, a line drawn from the point of section cbd cutting both circles, will cut off dissimilar segments from those circles.

1. Suppose cb to be the diameter of one of them : then draw to the other point of section e the line ce , and joining eb, ed , the angle ceb will be a right one, and the angle ced either greater or less than a right one, and consequently cd cannot be a diameter of the other.



2. Suppose cbd not to pass through the centre of either ; then through c draw a diameter cag , and continue it to meet the other circle in f , and join bg, df : then the angle cbg is a right one, and the angle cnf is either greater or less than a right one : and therefore the lines bg and df are not parallel : let h be the centre of the other circle, and let a diameter chn be drawn : draw di and continue it to meet cg in k : then dix will be parallel to bg : hence $cb : cd :: cg : ck$. But ci and ck are unequal (being both applied from the same point in a right angle) and therefore it cannot be as $cb : cd :: cg : ci$: and hence it appears that the segments cb and cd are dissimilar.



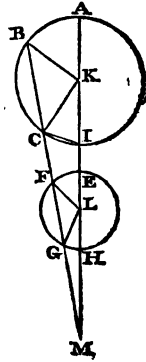
Lemma 2. If through the legs of any triangle edf , a line xi be drawn parallel to the base df , so that there be constituted two similar triangles about the same vertex ; and a circle be circumscribed about each of these triangles ; these circles will touch one another in the common vertex e .

It is plain that they will either touch or cut each other in the point e : if they cut each other, then by the preceding lemma, the segments be and de would be dissimilar ; but, they are similar, and they must therefore touch each other.

Lemma 3. Having two circles abc and $efgh$ given, it is required to find a point m in the line joining their centres, or in that line continued, such, that any line drawn through the said point m , cutting both the circles, shall always cut off similar segments.

Let the line kl joining the centres be so cut or produced, that km may be to lm in the given ratio of the radii ak to el (in the case

of producing KL we must make it as $AK : L : EL :: KL : LM$, for then, by composition it will be $AK : EL :: KM : LM$ and then I say that the point M , will be the point required. For, from it drawing any line $MGFCB$ cutting the circle $ABCI$ in B and C , and the circle $EFGH$ in F and G ; and let B and F be the correspondent points most distant from M , and C and G the correspondent points that are nearer; and let be joined BK , CK , FL , GL , and thereby two triangles will be formed BKC , FLG . Now, because, by construction, $KM : LM :: KB : LF$, KB and LF will be parallel, by Euc. vi. 7, and for the same reason KC and LG will be parallel; and therefore the triangles BKC and FLG will be equiangular, and hence the segment BC will be similar to the segment FG .

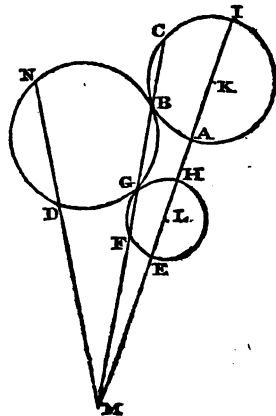


Lemma 4. The point M being found as in the preceding lemma, I say that it is a property of the said point, that $MG \times MB = MH \times MA$; and also that $MF \times MC = ME \times MI$.

For joining CI and GH , it is evident that these lines will also be parallel. Hence, $MI : MC :: MH : MG$, but $MI : MC :: MB : MA$, therefore $MH : MG :: MB : MA$. and $MG \times MB = MH \times MA$. Again $MH : MG :: MF : ME$, but $MH : MG :: MI : MC$, therefore $MF : ME :: MI : MC$, and $MF \times MC = ME \times MI$.

PROBLEM. Having two circles given in magnitude and position, whose centres are K and L , and also a point D ; to draw a circle which shall touch the two given ones, and also pass thro' the point D .

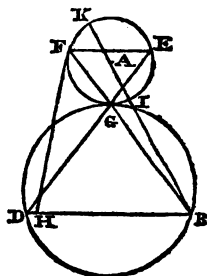
Join the given centres by drawing KL and in KL , or KL produced, find the point M (by lem. 3) such, that all the lines drawn from it, cutting the given circles shall cut off similar segments; and let KL cut the circumferences, one of them in the points A and I , and the other in the points E and H ; and joining MD , make it as $MD : MA :: MH : MN$. Then through the points D and N draw a circle which shall also touch the given circle whose centre is K , (by the following problem) and I say that this circle will also touch the other given circle whose centre is L . For let B be the point of contact, and BM be drawn cutting the circle K in C , and the circle L in F and G ; then, by lemma 4, $MB \times MG = MA \times MH$: but $MA \times MH = MD \times MN$, by construction, therefore $MB \times MG = MD \times MN$, and the points B, G, N, D , are in a circle. But the



point G is also in the circle L ; therefore these circles either touch or cut each other in the point G . Now the circles BND and BCI touch one another in B , by construction, therefore the segment BC is similar to the segment BG ; and also, by construction, the segment BC is similar to the segment FG ; and therefore the segment FG is similar to the segment BG ; and hence the circles FGE and BGD touch one another in the point G .

PROBLEM. Having two points given B and D , and likewise a circle whose centre is A ; to describe another circle which shall pass through the given points and touch the given circle.

Let DB be joined, as also AB , and let AB be produced to cut the given circle in the points I and K , then let BH be taken a fourth proportional to DB , BK , BI ; so that $BD \times BH = BI \times BK$: from H let a tangent HF be drawn to the given circle, and BF be joined and cut the circle again in G : and let DG be drawn cutting the given circle again in E ; and lastly, through the points D , B , G , let a circle be drawn, I say it will touch the given circle in G .



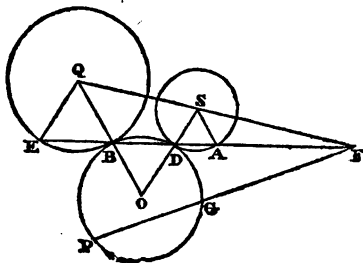
For joining EF ; because the rectangle $DBH =$ the rectangle KBI , i. e. the rectangle GBF , therefore the four points D , H , F , G , are in a circle; and hence the angle $HFB =$ the angle GDB . But, the angle $HFB =$ the angle GEF , by Euc. iii. 32; hence $GEF = GDB$: therefore the triangles GEF and GDB are similar, and under the same vertex, and therefore, by lemma 2, the circles described about them will touch each other in the common vertex G .

III. Two Solutions from the Mathematical Companion. (1811.)

First Solution.

PROBLEM. To describe the circumference of a circle through a given point F , which shall touch two other circles q and s given in position and magnitude.

Analysis. Suppose the circle FBD described through F , touching the circles q and s in the points B and D as required. Join the centres Q , S , O ; and through B , D , draw EF meeting q and s in E , A , and QS produced in F ; draw QE , SA and FF , meeting the circle FBD again in G . The isosceles triangles QBE , BDO , DA , are similar; hence $\angle QEF =$

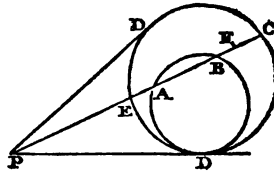


$\triangle SDF$; therefore the triangles FQE , FSD are also similar: wherefore $EQ : SD :: QF : SF$; dividendo, $EQ - SD : SD :: QS : SF$. Now, the three first terms of this analogy are given, therefore the fourth SF , or the point F is given, therefore the line PF is given in position and magnitude. Again, $EQ : SD :: EF : DF :: EF \times FB : DF \times FB$; but, since F is a given point, and Q a given circle, the rectangle $EF \times FB$ is given; and because the radii EQ , SD are also given, it follows that the rectangle $DF \times FB$, and its equal the rectangle $GF \times FP$, will be given; therefore, since EP is given, GF is given, or F and G are given points, and the problem is reduced to that of describing the circumference of a circle through two given points, and which shall touch another circle given in position and magnitude.

This admits of two cases :

1. When the given points A and B are within the given circle DEC .

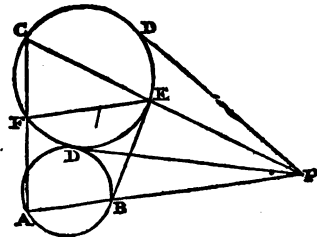
Construction. Let AB joined meet the circle in E , c ; take BF in BC , equal to AE ; and produce CE to F , so that $CF : FB :: BE : EP$; draw PD touching the circle in D ; then through the points A , B , D , describe the required circle ADB .



Demonstration. By construction, $CF : FB :: BE : EP$, and by composition and alternation, $BC : BF :: BF (AE) : EP$; and again, by composition, $PC : PB :: PA : EP$; wherefore $PB \times PA = PE \times PC = PD^2$; therefore (Euc. iii. 37) the circles touch each other in the point D .

2. When the given points A and B are without the given circle DEC .

Construction. From B one of the given points, draw BE a tangent to the circle; and from the point of contact E , draw EF parallel to AB , meeting the circle in F ; draw ARC meeting the circle again in C , and CEF meeting AB produced in F ; draw PD touching the circle in D ; then through the points A , B , D , describe the required circle ADB .



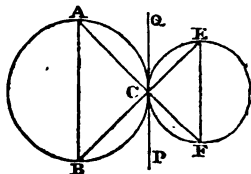
Demonstration. Since FE is parallel to AP (construction) the angle $PAC = EFC = FEB$ (Euc. iii. 32); therefore the points A , B , E , C are in a circle; wherefore $BP \times PA = PE \times PC = PD^2$; therefore the circles touch in the point D .

Remark. Since two tangents may be drawn from P to the given circle, it is evident that the problem admits of two solutions.

Second Solution.

Lemma. Let two circles touch each other at C , and through the point of contact draw the lines AFC , BCE meeting the circles again in A and F , and B and E ; then AB and EF when drawn are parallel.

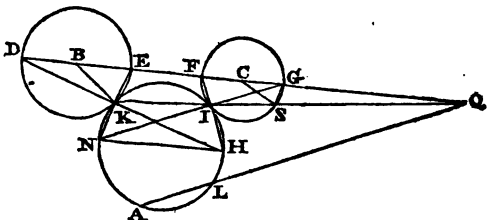
For let qC be drawn to touch the circles at C ; then the angle $ABC = ACQ = QCF = CEF$ (Euc. iii. 32); therefore EF is parallel to AB .



Corollary. When AB is a diameter, EF will also be a diameter.

PROBLEM. To describe the circumference of a circle, through a given point, which shall touch two other circles given in position and magnitude.

Let the given point be A , and the two given circles B , C . Suppose the required circle is described through A to touch the given circles at K and I . Through the centres B , C , draw the line nq meeting the circles in D , E , and F , G , and let KI be drawn to meet nq in q ; also draw QA meeting the circle in L . Then, by the lemma, if DK , EK be drawn to meet the circle in N and H , the line NH is a diameter of the circle KIA and parallel to DE ; also, if FI , GI be drawn they will meet the circles in the extremities of the diameter which is parallel to FG , that is in the points N , H .

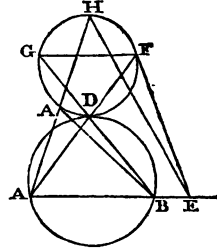


Draw the radii BK , CS and join sq ; then the angle $KB'E = 2KDE = 2KHN = 2KIN = 2GIS = SCG$; therefore the triangles BKQ , CSQ are similar, and $BK : CS :: BQ : CQ$; therefore Q is a given point, and QF , QE given lines. Moreover the angle $EKI = NHI = IFQ$, therefore the points E , K , I , F are in a circle; consequently, $QF \times QE = QI \times QK = QL \times QA$. But QF , QE , and QA are given lines; therefore QL is a given line, and L a given point. Hence the problem is reduced to describing the circumference of a circle through two given points, and which shall also touch another circle given in position and magnitude.

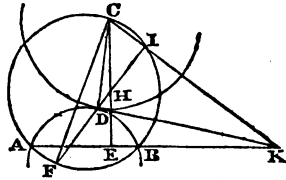
A neat construction of this may be seen in the Mathematical

Repository,* page 16, part 2, vol. 1, New Series. It may be also constructed as follows:

First method of Construction. Let D be the point of contact, and through the given points A, B , draw AD, BD to meet the given circle in F and G ; then by the lemma, GF is parallel to AB . Let the tangent FE be drawn to meet AB produced in E ; then, the angle $ABD = DGF = EFD$; therefore the points B, D, F, E are in a circle. From A draw any line AH to meet the given circle again in I and H , and join IB, HE ; then, $AI \times AH = AD \times AF = AB \times AE$; therefore the points I, H, B, E are in a circle, and therefore the angle $AHE = ABI$. Hence this construction: Draw any line AH to meet the given circle in I and H , join IB , and draw HE to make the angle $AHE = ABI$; from E draw the tangent EF , and draw AF to meet the circle in D , the point of contact.

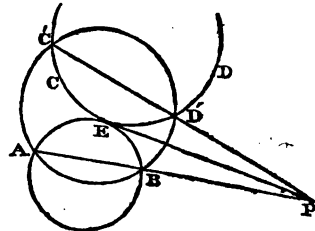


Second method of Construction. From c , the centre of the given circle, draw CE perpendicular to AB , the line joining the given points, and let the tangent to the given circle at D meet AB produced in K , and draw DI perpendicular to CK ; then because CDK is a right angle, $BK \times AK = DK^2 = KI \times KC$, therefore the points A, B, I, C are in a circle, also $CD^2 = CI \times CK = CE \times CH$. But CE and CD are given lines; therefore CH is a given line, and we have this construction: Through the points A, B, C , describe a circle, and draw the diameter CF ; take CH a third proportional to CE and CD , and draw FH , which will intersect the circle in D , the point of contact, as is evident from the analysis.



* The following is the construction alluded to above, and a similar one may be seen in the Monthly Review for October 1764.

Let A and B be the given points, and CEP the given circle. Take any point C' in the circumference of the given circle, and describe a circle through the points ABC' , which will touch the given circle in a point D , (in which case the thing required is done) or it will cut it again in a point D' ; join $C'D'$ meeting AB in E , and draw FE touching the given circle in E ; a circle described through A, B, E , is evidently that required.

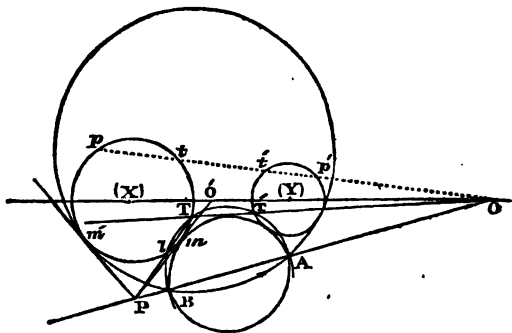


Demonstration. Because FE touches the circle CEP , $FE^2 = FC' \times PD'$, but $FC' \times PD' = FA \times PB$, therefore $FE^2 = FA \times PB$; and therefore the circle AEB touches FE in the point E , and consequently it also touches the given circle in that point.

11. *Two Solutions from Correspondance sur L'Ecole Impériale Polytechnique.*

First Solution, by M. Poncelet. (No. 3, vol. 2.)

Let o be the point of intersection of the exterior tangents common to the given circles x and y ; from o draw any line ot to meet the given circle in t and t' ; then if through the points t , t' , and A , a circle be described, the circumference of this circle will always cut the lines joining the points o , A , produced in the same point B , whatever be the direction of the line ot .



For ob and ot being secants to the same circle ABT , we have

$$AO \times OB = OT \times OT'.$$

And if any other line ot' be drawn to meet the given circles in t and t' , we have also

$$OT \times OT' = ot \times ot';$$

$$\text{therefore, } AO \times OB = ot \times ot';^*$$

wherefore the points t , t' , A and B , are in the circumference of a circle, that is to say, the circle which passes through the points t , t' , and A , passes also through the point B . But if any circle whatever touch the two given circles x and y , the line joining the points of contact will pass through the point o ;* therefore, it follows from what has been demonstrated above, that the circle which touches the given circles x and y , and passes through the point A , passes also through the point B . Thus the problem is reduced to that of describing a circle through the points A and B to touch either of the circles x or y .

As this last problem admits of two solutions, it is necessary to shew that the property corresponding to the case when the circle passing through A , touches the given circles exteriorly, appertains also to the case when they touch it interiorly:

For proving this, it will be sufficient to shew that the circle passing through the point A , and the two points p and p' , where any secant ot cuts exteriorly the circles x and y , passes also through the point B :

* This may be proved as in the preceding solution.

for then the circle which passes through A, and touches exteriorly the circles x and y , having the points of contact in the direction of the point o , evidently passes through the points A and B.

$$\text{Because } ot \times ot' = op \times op';$$

$$\text{therefore, as before, } op \times op' = Ao \times ob;$$

wherefore, the points A, B, p , and p' are in the circumference of the same circle.

Whence the following solution of the problem :

Having described the circle ATT' cutting the given circle x again in L , draw the chord TL cutting AO in the point P ; through this point draw the tangents pm and pm' to the circle x : m and m' are the points of contact of two circles, one of which touches interiorly, and the other exteriorly the circle x .

$$\text{For } pm^2 = pl \times pt,$$

$$\text{but } pl \times pt = pb \times pa;$$

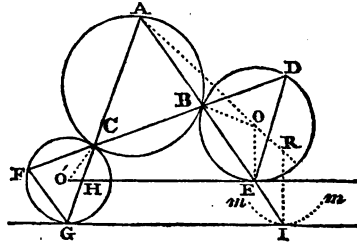
$$\text{therefore } pm^2 = pb \times pa.$$

This last equation proves that the line pm touches the circle which passes through A, B and m in the point m . And because pm' is equal to pm , we conclude also from the same equation, that the circle ABm' touches the circle x in the point m' .

If the point o' be the intersection of the interior tangents common to the circles x and y , we obtain, by a similar construction, two other solutions of the problem.

Second Solution, by M. Cauchy. (No. 6, vol. 1.)

Let A be the given point ; ob , $o'c$ the radii of the two given circles. Suppose the problem resolved, and that the circle ABC touches the two given circles in B and C. Let the lines AB, BC meet the given circles in E and G, and let BC meet them in F and D, and join F, G ; D, E : the three triangles ABC, BDE, CEG are equiangular.* Let tangents EH and GI be drawn to the given circles, at the points E and G ;



$$\text{then } \angle AEH = \angle BDE = \angle BCA,$$

$$\angle AGI = \angle CFG = \angle ABC;$$

therefore the triangles AGI, AEH are equiangular with the triangle ABC ; and therefore

$$\angle AIG = \angle ACB = \angle AEH ;$$

wherefore the tangents GI, EH are parallel.

* See Vieta's second lemma.

If t and t' denote tangents drawn from the given point A to the two given circles, we shall have

$$\left. \begin{aligned} t'^2 &= AC \times AG = AB \times AI \\ t^2 &= AB \times AE. \end{aligned} \right\} \text{whence } \frac{t'^2}{t^2} = \frac{AI}{AE}.$$

Therefore, if upon AO , the line joining the given point and the centre of one of the given circles, AR be taken a fourth proportional to t'^2 , t^2 and OE (or to AO , AR , and OE), and from the centre R with the radius RI , the circle mim be described, this circle will touch the line GI , because RI is parallel to OE , which is perpendicular to ME , which has been shewn to be parallel to GI .

Hence, if a common tangent be drawn to the two circles, whose radii are o/c and RI , the lines drawn from the points of contact g and i to A will meet the two given circles in the points B and C ; through which, and the given point A , a circle being described, will touch the given circles in the points B and C , as required.

Having shewn in a variety of ways how a circle may be described through a given point, to touch two other circles given in position and magnitude, we shall now give, from Vieta, the solution of the problem in the Diary, viz.

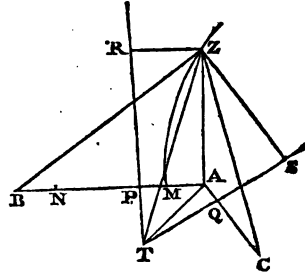
Having three circles given whose centres are A , B , and D ; to draw a fourth which shall touch all three.

Let that whose centre is A be called the first, that whose centre is B the second, and that whose centre is D the third. Then with centre B , and radius equal to the *difference*, or *sum*, as the case requires, of the semi-diameters of the first and second circles, let an auxiliary circle be described; and likewise with centre D , and radius equal to the *difference*, or *sum*, as the case requires, of the semi-diameters of the first and third circles, let another auxiliary circle be described; and lastly, by the preceding problem, draw a circle which shall touch the two auxiliary ones, and likewise pass through the point A , which is the centre of the first circle. Let the centre of this last described circle be E , and the same point E will likewise be the centre of the circle required; as will appear by adding equals to equals, or taking equals from equals, as the case requires.

It has been observed in Dr. Hutton's note to the original solution, that Newton had considered this problem, both in the *Principia*, and in the *Arithmetica Universalis*; the construction in the latter work being the same as Vieta's, it would be unnecessary to give it here; we shall, however, insert that in the *Principia* where the problem is proposed as follows:

V. PROBLEM. From three given points to draw to a fourth point that is not given, three right lines whose differences shall be either given or none at all. (Lem. 16, lib. 1, Princip.)

Case 1. Let the given points be A, B, C , and z the fourth point, which we are to find; because of the given difference of the lines Az, Bz , the locus of the point z will be an hyperbola, whose foci are A and B , and whose principal axis is the given difference. Let that axis be MN . Taking PM to MA , as MN is to AB , erect PR perpendicular to AB , and let fall ZR perpendicular to PR ; then from the nature of the hyperbola, ZR will be to Az as MN is to AB . And, by the like argument, the locus of the point z will be another hyperbola, whose foci are A, C , and whose principal axis is the difference between Az and Cz ; and qs a perpendicular on AC may be drawn, to which (qs), if from any point z of this hyperbola a perpendicular zs is let fall, this (zs) shall be to Az as the difference between Az and Cz is to AC . Wherefore the ratios of ZR and Zs to Az are given, and consequently the ratio ZR to Zs one to the other; and therefore if the right lines RP, sq meet in T , and TZ and TA are drawn, the figure $TRZs$ will be given in specie, and the right line TZ , in which the point z is somewhere placed, will be given in position. There will be given also the right line TA , and the angle ATZ ; and because the ratios of Az and TZ to Zs are given, their ratio to each other is given also; and thence will be given likewise the triangle ATZ whose vertex is the point z . Q. E. I.



Case 2. If two of the three lines, for example, Az and Bz , are equal, draw the right line TZ so as to bisect the right line AB ; then find the triangle ATZ as above. Q. E. I.

Case 3. If all the three are equal, the point z will be placed in the centre of a circle that passes through the points A, B, C . Q. E. I.

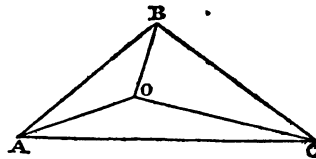
VI. A new solution by Trigonometry.

Let A, B and C be the centres of the given circles, r, r' and r'' their respective radii; O the centre of the required circle and x its radius: then AO is $= r + x$, $BO = r' + x$, and $CO = r'' + x$. Put $AB = a$, $AC = b$, $OC = c$, $\sin \angle BAC = m$, its $\cos = n$, angle $BAO = \phi$, and $CAO = \psi$.

Then, by trigonometry,

$$(r' + x)^2 = BO^2 = a^2 + (r + x)^2 - 2a \times (r + x) \cos \phi,$$

$$(r'' + x)^2 = CO^2 = b^2 + (r + x)^2 - 2b \times (r + x) \cos \psi;$$



$$\text{therefore } \cos \phi = \frac{a^2 + r^2 - r'^2 + 2(r - r')x}{2a(r + x)},$$

$$\text{and } \cos \psi = \frac{b^2 + r^2 - r''^2 + 2(r - r'')x}{2b(r + x)}.$$

But, by the formula for finding the cosine of the difference of the two arcs,

$$\cos \bar{\phi} = n \cos \psi + m \sin \psi,$$

$$\text{or } m \sin \psi = \cos \bar{\phi} - n \cos \psi;$$

$$\text{therefore } m^2 \sin^2 \psi = \cos^2 \bar{\phi} - 2n \cos \bar{\phi} \cos \psi + n^2 \cos^2 \psi;$$

add $m^2 \cos^2 \psi$ to each side of this equation;

then, because $\sin^2 \psi + \cos^2 \psi = 1$, and $n^2 + m^2 = 1$,

$$m^2 = \cos^2 \bar{\phi} - 2n \cos \bar{\phi} \cos \psi + \cos^2 \psi.$$

Substitute, in this equation, for $\cos \bar{\phi}$ and $\cos \psi$, their values found above, and, after proper reduction, we get

$$\left. \begin{aligned} & 4b^2(r - r')^2 \\ & 4a^2(r - r'')^2 \\ & - 4nab(r - r')(r - r'') \\ & - 4a^2b^2m^2 \end{aligned} \right\} x^2$$

$$\left. \begin{aligned} & - 4b^2(a^2 + r^2 - r'^2)(r - r') \\ & - 4a^2(b^2 + r^2 - r''^2)(r - r'') \\ & - 4nab(a^2 + r^2 - r'^2)(r - r'') \\ & - 4nab(b^2 + r^2 - r''^2)(r - r') \\ & - 8a^2b^2m^2r \end{aligned} \right\} x$$

$$\left. \begin{aligned} & + b^2(a^2 + r^2 - r'^2) \\ & + a^2(b^2 + r^2 - r''^2) \\ & - 2nab(a^2 + r^2 - r'^2)(b^2 + r^2 - r''^2) \\ & - 4a^2b^2m^2r^2 \end{aligned} \right\} = 0.$$

And, putting for n and m^2 their values,

$$\frac{a^2 + b^2 - c^2}{2ab} \text{ and } \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2b^2},$$

and, resolving the quadratic, we shall have x in terms of the given radii, and the distances of the given centres.

VII. *An Analytical Solution of the Problem by M. I. Binet.*

From the Journal de l'Ecole Polytechnique, 17 Cahier, Tome x.

Let a, β, γ denote the radii of the circles; and also the position of their centres on the plane. Let a, b, c be the distances of these points, a being that of α and β , b that of β and γ , and c that of β and α ; and let x be the radius of the circle which touches the circumferences α, β, γ . By supposing that these are touched outwardly

by the circle x , and putting d, e, f , for the distances $xa, x\beta, x\gamma$, we have

$$d = a + x, \quad e = \beta + x, \quad f = \gamma + x.$$

At the centre γ , suppose two perpendiculars a', b' , drawn to the sides a, b of the triangle $a\beta\gamma$ formed by the centres of the given circles, and draw through the centre x , two other straight lines parallel to a', b' . Thus there will be formed a parallelogram of which f is the diagonal. Let a' and b' denote the sides of this parallelogram which are perpendicular to a and b ; then we have

$$f^2 = a'^2 + b'^2 + 2a'b' \cos \widehat{a'b'}.$$

Here the expression $\cos \widehat{a'b'}$ denotes the cosine of the angle contained by a' and b' which is the supplement of that contained by a and b , or of $\widehat{ab}$.

It is evident that

$$a' : \sin \widehat{b'f} = b' : \sin \widehat{a'f} = f : \sin \widehat{a'b'},$$

and that

$$\sin \widehat{a'f} = \cos \widehat{af} = \frac{a^2 + f^2 - d^2}{2af}, \quad \sin \widehat{b'f} = \cos \widehat{bf} = \frac{b^2 + f^2 - e^2}{2bf},$$

$$\sin \widehat{a'b'} = \sin \widehat{ab}, \quad \cos \widehat{a'b'} = -\cos \widehat{ab}.$$

It is also known that four times the area of the triangle $a\beta\gamma$, is

$$\sqrt{(2b^2c^2 + 2a^2c^2 + 2a^2b^2 - a^4 - b^4 - c^4)}.$$

This quantity which we shall call p , is equal to $2ab \sin \widehat{ab}$.

In the expression for f^2 , substitute for a' and b' their values deduced from the above, viz.

$$a' = \frac{a}{p} (b^2 + f^2 - e^2) \quad b' = \frac{b}{p} (a^2 + f^2 - d^2),$$

we shall then have between d, e, f , this relation,

$$(a^2 + f^2 - d^2)^2 b^2 + (b^2 + f^2 - e^2)^2 a^2$$

$$- 2(a^2 + f^2 - d^2) \times (b^2 + f^2 - e^2) ab \cos \widehat{ab} = f^2 p^2 \dots (1)$$

If in this expression, we substitute instead of $2ab \cos \widehat{ab}$, its value $a^2 + b^2 - c^2$; and replace p^2 by its value given above, the result will contain only the quantities a, b, c, d, e, f . This relation expresses that the point x is in the plane of the three others a, β, γ . And to have the equation of the problem we have only to substitute instead of d, e, f , their equals

$$a + x, \quad \beta + x, \quad \gamma + x.$$

The expression then becomes

$$[a^2 + (\gamma + x)^2 - (a + x)^2]^2 b^2 + [b^2 + (\gamma + x)^2 - (\beta + x)^2]^2 a^2 \\ - 2[a^2 + (\gamma + x)^2 - (a + x)^2][b^2 + (\gamma + x)^2 - (\beta + x)^2] ab \cos \widehat{ab} = (\gamma + x)^2 p^2.$$

By developing this expression, and arranging the terms according to the powers of x we get this equation of the second degree,

$$0 = L + xL' + x^2L'' \dots \dots \dots (2)$$

where L , L' , and L'' denote the complex co-efficients of x , viz.

$$L = \begin{cases} (a^2 + \gamma^2 - a^2) b^2 + (b^2 + \gamma^2 - \beta^2) a^2 \\ - 2(a^2 + \gamma^2 - a^2)(b^2 + \gamma^2 - \beta^2) ab \cos \widehat{ab} - \gamma^2 p^2, \end{cases}$$

$$L' = 4 \begin{cases} (a^2 + \gamma^2 - a^2) [(\gamma - a)b^2 - (\gamma - \beta) ab \cos \widehat{ab}] \\ + (b^2 + \gamma^2 - \beta^2) [(\gamma - \beta)a^2 - (\gamma - a) ab \cos \widehat{ab}] \end{cases} - 2\gamma p^2,$$

$$L'' = 4 \begin{cases} (\gamma - a)^2 b^2 + (\gamma - \beta)^2 a^2 - 2(\gamma - a)(\gamma - \beta) ab \cos \widehat{ab} \end{cases} - 2p^2.$$

In this equation in which, α, β, γ were considered positive, L alone changes its sign when they become

$$-a, -\beta, -\gamma,$$

thus the equation will have the same roots, except the signs in both cases; but this change answers to the substitution of

$$-a + x, -\beta + x, -\gamma + x,$$

instead of d, e, f in equation (1), that is, when the circle which touches the given circumferences encompasses them all three; and this cannot happen if the two roots of the equation (2) have contrary signs; in the case when they have the same sign, the circumferences whose radii are found by this equation are both exterior, or both enveloping, according as the roots are positive or negative.

If in the expressions L', L'' the sign of only one of the radii α, β, γ be changed, for example γ , then equation (2) will have for its roots the radii of two circumferences, one of which will leave the circles α, β exterior, and will encompass the circle γ , and the other will encompass the circles α, β , and leave the circle γ exterior.

These cases happen if the roots of the equation (2) have different signs; when they are positive the two tangent circumferences envelope γ , and touch externally α and β ; lastly, if they are negative the circle γ will be exterior, and the two others α, β interior to the two circumferences described with the radii given by the values of x .

The change of the sign of only one of the radii may happen in three ways, and hence there may be six solutions which, with the two first, furnish all that the problem will in general admit of.

We shall have an example of the case in which the solutions are two and two of the same kind, if we suppose the centres α, β, γ in a straight line. Let us suppose that γ is between α and β ; we have then,

$$\cos \widehat{ab} = -1,$$

and the triangle α, β, γ being reduced to one of its sides its surface is nothing: thus we have $p^2 = 0$, a condition which is also a consequence of the preceding. It is by this last equation, that we express, by the mutual distances of three points, that they are in a straight line, as it is by equation (1) that four points are determined to be in the same plane: or that the volume of a tetraedron, of which these points are the summits, is $= 0$. In the present case we have

$$L = (a^2 + \gamma^2 - a^2)b^2 + (b^2 + \gamma^2 - \beta^2)a^2 + 2(a^2 + \gamma^2 - a^2)(b^2 + \gamma^2 - \beta^2)ab \\ = [(a^2 + \gamma^2 - a^2)b + (b^2 + \gamma^2 - \beta^2)a]^2,$$

$$L' = 4[(a^2 + \gamma^2 - a^2)b + (b^2 + \gamma^2 - \beta^2)a][(\gamma - a)b + (\gamma - \beta)a],$$

$$L'' = 4[(\gamma - a)b + (\gamma - \beta)a]^2.$$

These values put in the equation

$$0 = L + xL' + x^2L''$$

γ render the second member the square of

$$(a^2 + \gamma^2 - a^2)b + (b^2 + \gamma^2 - \beta^2)a + 2x[(\gamma - a)b + (\gamma - \beta)a],$$

hence we have,

$$x = \frac{a^2b^2 + \beta^2a - \gamma^2(a+b) - (a^2b + ab^2)}{ab + \beta a - \gamma(a+b)},$$

a formula easy to construct, and which gives also eight solutions, by the variations of the signs α, β, γ . It always gives a real value for x whatever be the radii of the circles α, β, γ , and their respective position: However, there are cases of manifest impossibility. For example, one of the circles comprehending a single one of the other two: Then, although x be real, the construction of the triangles $x\beta\gamma, x\gamma\alpha, x\alpha\beta$ is impossible, because one of the sides surpasses the sum of the other two. And if we wish to calculate the cosine of one of the angles of the triangles, we shall find it greater than unity, thereby indicating analytically this impossibility.

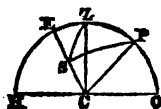
PRIZE QUESTION 1736:

The original solutions to this question appear to be much too concise, many particulars having been suppressed, or not properly noticed, which are nevertheless necessary to render them intelligible.

In what follows, we have endeavoured to supply all the material defects and omissions: We shall begin with finding a general expression for the sun's azimuth, at any time in the forenoon of the day of the journey, this being necessary in determining the nature or equation of the curve which the traveller describes.

By the question, the sun rose in the east on the day of the journey; he must, therefore, be in the equator at that time, or 90° from the pole.

Let $HEZPO$ represent the meridian or hour line of 12 o'clock; HO the horizon; CP the hour line of 6 o'clock; C the equator; s the place of the sun; and imagine the great circles rs , and zs to be described. By the question rz , the complement of the latitude, is an arc of 60° , therefore the sine of rz is $= \frac{1}{2}\sqrt{3}$; sine $ez = \frac{1}{2}$. Put sine of the angle ers , the hour from 6 $= s$; and its cosine or sine of the angle $zrs = c$; radius 1. Also put sine of the azimuth $ezs = v$. Then by the principles of



spherical trigonometry, $\cos zs = \frac{s}{2}\sqrt{3}$; and therefore the sine of

$sz = \sqrt{(1 - \frac{3s^2}{4})}$. In the right-angled spherical triangle zes ,

$\sin sz = \sqrt{(1 - \frac{3s^2}{4})} : \sin zs = c :: 1 : \sin \angle ezs = v$;

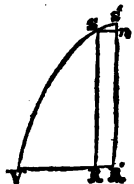
whence $v = \frac{c}{\sqrt{(1 - \frac{3s^2}{4})}}$; and hence $\sqrt{(1 - v^2)} = \frac{\frac{1}{2}s}{\sqrt{(1 - \frac{3s^2}{4})}}$,

which expresses the cosine of the sun's azimuth:

Having obtained an expression for the sun's azimuth, we shall now proceed to determine the equation of the curve, and here we shall remark that the journey is supposed to be performed upon an horizontal plane; and the meridians parallel to each other, which can cause no very material error, when the whole journey, comparatively speaking, is performed in a small compass.

Let vs represent a portion of the curve described by the traveller in his journey; vr the axis of the curve being a north and south line, or meridian; let $rs, r's'$ be two ordinates indefinitely near to each other, and at right angles to vr . Put $vr = x$, then $rr' = sn = s'$; and the angle $ss'n$ the complement of the azimuth: we have before put s the sine of the hour angle from six; and by the principles

of fluxions $\frac{s}{c}$ will express the fluxion of the hour



arc from six. Now as the traveller's motion is uniform, the fluxion of the space or curve described will have a constant ratio to the fluxion of the time: We may therefore denote the fluxion of the

curve or of ss' by $\frac{as}{c}$, where a denotes a constant determinable quantity.

By the principles of trigonometry we shall have $ss' = \frac{as}{c} : sn$

$= s :: 1$; cosine of azimuth $= \frac{\frac{1}{2}s}{\sqrt{(1 - \frac{3s^2}{4})}}$; whence $\dot{s} =$

$\frac{\frac{1}{2}ass}{c\sqrt{(1 - \frac{3s^2}{4})}}$; writing $1 - c^2 = s^2$, and $-cc = \dot{s}$, we shall have

$\dot{s} = \frac{-a\sqrt{(\frac{1}{4})} \times c}{\sqrt{(c^2 + \frac{1}{4})}}$; and taking the fluents $x = a\sqrt{\frac{1}{4}} \times \text{hyp. log.}$
 $\frac{1}{c + \sqrt{(c^2 + \frac{1}{4})}}$.

But the expression for x ought to become $= 0$, when $c = 1$, therefore the correct equation will be $x = \frac{1}{4}a\sqrt{3} \times \text{hyp. log.}$

$\frac{2 + \sqrt{3}}{\sqrt{3} \times (c + \sqrt{(c^2 + \frac{1}{4})})}$, and at 10 o'clock, that is after travelling

4 hours, $c = \frac{1}{2}$, and then $x = \frac{1}{4}a\sqrt{3} \times \text{hyp. log.} \frac{4 + 2\sqrt{3}}{\sqrt{7} + \sqrt{3}} =$

$a \times .5773503 \times \text{hyp. log.} 1.7049883 = a \times .5773503 \times .533565$

$= a \times .3080539 = 8$: whence $a = \frac{8}{.3080539} = 25.96947$, and

the length of the curve vs , described in 4 hours is an arc of $60^\circ \times a$, that is, $vs = a \times 1.0472 = 27.19522898$ miles, the distance travelled in 4 hours, which is 6.7988 miles an hour, nearly. c .

We shall now add another problem of the same kind, taken from the Mathematical Companion.

PROBLEM, by W. H. (No. 7.)

A man set out from latitude $53^\circ 10'$ north, and longitude $25'$ east, when the sun rose due east, and travelled uniformly with his face directly towards the sun five hours, and then found himself arrived at a well famous for its excellent water, in latitude $51^\circ 10'$ north. Required how far he travelled, and the longitude of the well?

Solution by β . Cygni. (No. 8.)

Let p be the north pole of the equator, z the place from which the man set out, $zz'z''$ his path, $zB, z'C$, parallels of latitude, and pz', pz'' , the meridians of z', z'' indefinitely near to each other. Put the angle zpz' , or the difference of longitude between z and z' , $= x$; $z'A$, their difference of latitude, $= y$, $t =$ the time he takes to describe zz' , expressed in degrees of the equator; and $\lambda =$ the latitude of z . Then will the latitude of $z' = \lambda - y$, $z'C = dx \cos. (\lambda - y)$, $z''C = dy$; and, because the man travels uniformly, $z'z''$ will be $= mdt$, where m is a constant quantity. Also, by the question, the angle $z''z'C$ is equal to the sun's easting at the time $t + x$ in the latitude $\lambda - y$; and consequently, by spherics, the tangent $z''z'C = \sin. (\lambda - y) \text{ tang.}$



$(t + x)$, which by the differential triangle is $= \frac{dy}{dx \cos. (\lambda - y)}$

Whence we have the two following differential equations :

$$dy^2 + dx^2 \cos. (\lambda - y)^2 = m^2 dt^2,$$

and $\frac{dy}{dx} = \sin. (\lambda - y) \cos. (\lambda - y) \text{ tang. } (t + x)$, by the in-

tegration of which the question is resolved. This may be done by assuming y equal to a proper function of t , and making this substitution in the above equations; but this method is laborious, and the resulting series would, perhaps, converge but slowly. The answer, however, may be found sufficiently accurate by the following approximation: From the nature of the question, y and x must necessarily be small; let us suppose them, therefore, to vanish, and the follow-

ing equations will result: $dy^2 + dx^2 \cos. \lambda^2 = m^2 dt^2$, and $\frac{dy}{dx} =$

$\sin \lambda \cos \lambda \text{ tang } t$; whence $dy = \frac{m \sin \lambda \text{ tang } t \times dt}{\sqrt{(1 + \sin \lambda^2 \text{ tang } t^2)}} = m \text{ tang}$

$\lambda \times \frac{\sin t \times dt}{\sqrt{(\text{tang } \lambda^2 + \cos t^2)}}$, of which the correct integral is $y = m$

$\text{tang } \lambda \times \text{hyp. log. } \frac{1 + \secant \lambda}{\cos t + \sqrt{(\text{tang } \lambda^2 + \cos t^2)}}$. Also from the

equations we have $dx = \frac{mdt}{\cos \lambda \sqrt{(1 + \sin \lambda^2 \text{ tang } t^2)}} =$

$\frac{m \cos t \times dt}{\cos \lambda \sqrt{(1 - \cos \lambda^2 \sin t^2)}} = \frac{m}{\cos \lambda^2} \times \frac{\cos t \times dt}{\sqrt{(\secant \lambda^2 - \sin t^2)}}$,

and therefore $x = \frac{m}{\cos \lambda^2} \times$ the arch whose sine is $\cos \lambda \sin t$, and

radius unity. Consequently, if t , λ , and y , be given, the difference of longitude will be found by means of the following formula: $x =$

$$\frac{y}{\sin \lambda \cos \lambda} \times \frac{\text{arch whose sine is } \cos \lambda \sin t}{\text{hyp. log. } \frac{1 + \sec \lambda}{\cos t + \sqrt{(\tan \lambda^2 + \cos t^2)}}}. \quad \text{In the pre-}$$

sent case y is $= 120'$, $t = 75^\circ$, and $\lambda = 53^\circ 10'$; which substituted in the formula, gives $20' 37''$ of time for the difference of longitude, if no mistake has been made in the calculation.

Remark. This question, in one or other of its cases, has been considered by Simpson,* Emerson, Coughron,† and Witchell;‡ but none of them have taken notice that their solutions are only approximations to the truth, being all grounded on the hypothesis that the man remains in the same latitude: The true answer may be found from the differential equations given above, by means of an infinite series.



QUESTION 206.

Many writers of eminence on the subject of Dynamics, both English and Foreigners, have considered this question (with the immediately connected one of the deviation of a falling body from a line perpendicular to the earth's surface) as worthy their particular attention. It is presumed, however, that, though, from the minuteness, and possibly still incorrectness, in some parts of the requisite *data*, precise accuracy may be scarcely attainable in this enquiry; yet, a much nearer approximation to truth may certainly be obtained, by introducing a circumstance in the investigation, which is unavoidably connected therewith, and which appears to have been hitherto totally neglected as too trivial to be noticed, (as was, within these few years, the resistance of the air in the doctrine of projectiles) by every author, who is, at present, generally known; that is, the *Spheroidal Figure* of the Earth. And what renders this omission the more remarkable is, that among the various writers alluded to on this subject, are found those accurate scientific men of our own country, *Emerson* and *Simpson*; and, among the *Illuminati* on the Continent, those eminent mathematicians, *Guglielmini*, *Benzenbergh*, and *La Place*; the latter of whom, in common with all the others, has, in his ultimately corrected theorem for this purpose (*vid.* "*Bulletin de Sciences.*" No. 75.) formed his investigation on the supposition of the earth being a perfect sphere,

* As above.

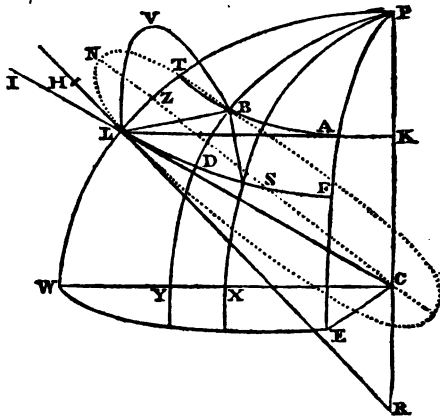
† See Gentleman's Diary, 1772.

‡ See question 458 of this work.

though this gentleman has himself lately given "a new method to ascertain the ellipse generating the terrestrial spheroid."

In the annexed scheme, CPWE represents the western half of the northern hemisphere of our spheroidal earth; PC the

semi-axis; cw the equatorial semidiameter; LF the parallel of the place L from whence the body is projected; and RLH the position of the radius of curvature at L, produced both ways towards H and R. Now, it is manifest, that if rw be the meridian of L, LX is there perpendicular to the horizon, and is the initial direction of the ball's flight in the plane of that meridian; but as gravity



acts uniformly towards the earth's centre, or in lines nearly parallel to ILC, during that time, the effect will evidently be the same on the projected ball, as if the angle of projection with the horizon at L was equal the complement of the angle (CLR) HLI: Hence, are known the angle of elevation, and the given impetus or height LU (ZN), to determine the elliptic (or *parabolic*, being here sufficiently correct) track of the ball, its amplitude LT, and the time of its flight. If, therefore, the earth had no diurnal motion, the ball would evidently fall on the same meridian with L, and on a parallel TA due north (if in that hemisphere) of LF, by the distance LT. But, during the ball's flight, the place L, by the earth's rotation from w, towards the east, will have arrived at s on its parallel LF, while the ball by the compound motion communicated to it by the projectile force in the direction LH (being the produced radius of curvature, and therefore perpendicular to the horizon at L), the action of gravity nearly uniform in the direction IL, and the centrifugal force arising from the earth's rotation in direction LF (which is evidently always as the cosine of the given latitude, or more accurately, as the ordinate LK, on the conjugate PC), the ball, instead of describing the portion of the ellipsis LNT, will now proceed in that of Lvs; and, therefore, when the projected body has again arrived at the earth at s, in the meridian PY, the place L will have gained the position s, in the meridian circle PX; which must necessarily from the established laws of projected bodies from the earth's surface, be to the east of PY, that is Ls will exceed LD, and this is demonstratively the immediate result of the equality of the contemporaneous sectoral

areas in all curves formed by projected bodies from the earth, which, in this case, are the elliptical one of the ball's trajectory, and the circular one of L from the earth's diurnal motion; the centre of the earth being their common focus (vid. the *Principia of Newton*, and every writer of eminence on Physico-mathematical science). Hence the ball will fall north or south *westerly* (according to the hemisphere) of the place from which it had been perpendicularly projected.

The mode of computation may evidently be deduced from what has been now premised, as by the attention of the first scientific characters of the present and preceding century to the form and dimensions of our earth, we are furnished with variety of calculated *data*, by which, we can with facility, resolve every rational enquiry appertaining to the subject. Thus, as the figure of the earth is now universally agreed to be an oblate spheroid, and similar to that generated by the revolving of an ellipsis on its conjugate diameter $2pc$; and that, not only the proportion of pc to cw (that is of the axis to the equatorial diameter) is known, but also their actual dimensions, every other part becomes known also from the demonstrative properties of the curve. Hence, we have sufficient *data* from these known properties, and the proposed question, to find the time of the ball's flight, and its amplitude LT ; the angle of elevation being *virtually* as the complement of ILH , and the *impetus*, or given perpendicular ascent LH . And hence immediately follows from the known diurnal angular motion of the earth in the same time, the angle wpx , or the position of the meridian of s , and thence the cir-

cular arc $LS = \frac{wx}{wc} \times \cos. WL$; or, $\frac{wx \times LK}{wc}$. And the arc LD may

be obviously deduced from comparing the elliptic sector of the ball's track with that of the circle arising from the earth's rotation in the same time; the difference of which ds , will therefore show how much the meridian of B falls to the west of that of s . And to determine the true distance and bearing of B from s , the right-angled triangle BDS may be conceived from its smallness to be rectilineal, and

thence $(ud^2 + sd^2)^{\frac{1}{2}} = sB$, the distance; and $\sin. \frac{BD}{BS}$ the angle at

$B = \angle PSB$, north-westerly of the place the ball was projected from.

A general and correct expression, or theorem, may be readily drawn from these remarks, and the process observed by *Merones* in his own solution.

If the ball be let fall from a given height (as in prob. 198, *Phys. Prob. Emerson's Algebra*) instead of being projected from the surface of the earth, then will the centrifugal force arising from the earth's rotation on its axis be evidently greater at H than at L , and consequently in the time of the ball's descent (as will appear from the gene-

in the first *second* of time ; n the angle of the earth's rotation in the same time, at the rate of $360^\circ \div .99727$ in a day ; and θ , the co-latitude of the place.

This theorem adapted to the spheroidal figure of the earth, will be conveniently expressed for computation in this form,

$$\sqrt{\left\{ \left(\frac{2N \times HL}{3WC + HL} \times LK \times \sqrt{\frac{LH}{16.083}} \right)^2 + BD^2 \right\}} = SB, \text{ in}$$

which N = the earth's diurnal angular motion in a *second* of time, viz. $360^\circ \div (86400'' \times .99727)$.

Hence are known, the distance of the deviation SB , and the *bearing* of B , the ball's place of descent, from S , (or from whence demitted), $\sin \angle \frac{DS}{SB}$ = the $\angle PSB$, north-easterly.

Remark. It may not be improper finally to add to this retrospect of the question under consideration, that, though what is herein offered as a method of obtaining more accurate results in this enquiry, than appear to have been hitherto attempted, yet, it must certainly be acknowledged, that many assumptions therein, are, by no means, compatible with strict scientific truth ; and the only plea that can be offered in behalf of this procedure is, that in the diminution of quantities of any description, the errors or deviations from rigorous accuracy, become less in proportion to the degree, or mode of the diminution applied ; as the *powers* of indefinitely small quantities may become infinitely less than any assignable quantity, and ultimately evanescently *nothing*. The assumptions necessarily adopted in this mode of investigation, in common with many others, are,

1. The uniformity and equality of gravitation at all distances from the earth's surface, an inaccuracy however, that can scarcely occasion any sensible error, as the experiments on this subject are applicable only within the very narrow limits of human access. 2, The non-resistance of the air to the projected or falling body. 3, The homogeneity of the earth, and thence the *central* direction of gravity. 4, The parallelism of the direction of gravitation. 5, And thence the assuming of the parabolic curve for the elliptical, in ascertaining the areas of their sectors, and amplitude of projection. And 6. The considering of curved-lined figures as rectilineal ; and the equality of arcs of similar parallels of latitude, when not far distant from each other.

All these deviations from the rigour of mathematical investigation, find, notwithstanding, an apology, (perhaps, however, too frequently in *approximating theorems*, when not restricted within proper limitations) under the licence of the mathematical doctrine of "*Les petites Quantités*."

Postscript. In addition to what has been now premised, it has been thought, that it might be satisfactory to many readers, not only by way of elucidation ; but also as a confirmation of the *legitimacy* of the theorems that are now exhibited on this subject in the productions of the first scientific characters of the age, by the remarkable almost exact coincidence of their respective conclusions : a circumstance of which, however, we have many instances, even upon the most abstruse points of physical philosophy ; and which fully proves, that when true reasoning is founded upon just and established principles, the results must necessarily be essentially the same :

The late eminent mathematician, Mr. William Emerson, in his treatise on algebra (published upwards of half a century ago), has given us the following general *formula* and its investigation, for determining the deviation of the line of descent of a heavy body (being freely let fall from a given height) from a perpendicular to the horizon of the given place ; and which appears, on examination, to be precisely the same, in almost every respect, with that of M. de La Place, as given in the "*Bulletin des Sciences*," No. 75, for the same purpose, but under a different literal designation.

Mr. Emerson's *Theorem, Algebra, Prob. 198, is thus expressed* :

$$\frac{2dmp}{3r + m} = D.$$

Which determines the deviation (D) of a heavy body falling from quiescence, from the line perpendicular to the earth's surface ; the earth being supposed a perfect sphere. In this general expression, r denotes the earth's semidiameter, and p the cosine of the latitude of the place over which the ball was freely liberated ; m the height fallen

from ; t , 24 hours ; c , 3.1416 ; f , 16.1 feet ; and hence $d = \frac{2rc}{t} \sqrt{\frac{m}{f}}$.

The above theorem may very obviously be transformed to

$$\frac{2}{3r + m} \times \frac{2rc}{t} \times m \times p \sqrt{\frac{m}{f}} = D,$$

in which, if we eliminate the comparatively small quantity m in the denominator of the first factor, as supposed to occasion no *sensible* difference in this case, in the numerical result, and take r (as radius) = 1, it becomes

$$\frac{2}{3} \times \frac{2c}{t} \times m \times p \sqrt{\frac{m}{f}} = D,$$

we shall evidently have, by writing herein the following equal values

respectively, viz, h for m ; $\sin \theta$ for cosine p ; $\sqrt{\frac{2h}{g}}$ for $\sqrt{\frac{m}{f}}$;

and n for $\left(\frac{2c}{t}\right) \frac{360^\circ}{t}$, the earth's rotatory motion in one second of time, (t the sidereal day)

$$\frac{2}{3}nh \times \sin \theta \sqrt{\frac{2h}{g}} = \Delta,$$

the express *ultimate* theorem of M. de La Place, and manifestly on the principle of the earth's sphericity; the only difference (and which is ultimately almost immaterial) being in the value of t , which in the theorem of Emerson, is taken as 24 hours, but which undoubtedly ought to have been the sidereal day of 23 hours 56 minutes 4.1 seconds, the time the earth makes a complete rotation on its axis; or, which is the same thing, retaining the value of t (24 hours), and for 360° writing $360^\circ 59' 8''$ nearly, the earth's angular motion in 24

hours, that is, $360^\circ \div 86164.1'' = \frac{360^\circ \times 1.002723}{86400''} = 360^\circ \div (.99727 \times 86400'') = 15.041062''$ of a degree, being the correct diurnal angular motion of the earth in one second of time, and is, therefore M. La Place's value of n ; which exceeds $\frac{2c}{t}$ (expressed in motion) by .04 of a second.

This small deviation from rigorous accuracy, it is imagined, cannot, by any reader who is acquainted with the writings of the late Mr. Emerson, be imputed to oversight or inadvertence, but to what he most certainly conceived at the time to be sufficiently exact for the purpose of exhibiting the principles and method of application in investigations of this nature, and which, no doubt, he considered of little importance in point of real utility: Emerson's *Comment on the Principia of Newton*, need only be mentioned to authorize this assertion. And it is not partiality, but justice to add, that the

theorem in its original form $\frac{2dmp}{3r+m}$, that is $\frac{2}{3r+m} \times \frac{2rc}{t} \times mp \sqrt{\frac{m}{f}}$, is obviously much more conveniently adapted for immediate

computation than the modern one; as in the former (taking t = the sidereal day) by retaining the value of r , the earth's numerical semi-diameter, as also m (h) in the denominator of the above form, we have a plain and genuinely scientific correct process (that is, within the well-known limits of possible experiment),

devoid of any mutilation whatever; the quantity $\frac{2rc}{t}$ expressing immediately the equatorial arc of rotation in one second of time, and

therefore, $\frac{2rcp}{t}$ the same on the parallel of the given latitude, and in the same denomination of measure as that of r , the earth's semi-diameter; which, multiplied by the time of the ball's descent $\left(\sqrt{\frac{m}{f}}\right)$, gives the distance LS in fig. 2, considered here as a sphere.

And now, having said thus much, it is presumed, there will be no impropriety in observing farther, that, as a *general theorem* on this subject, that is, wherein no actual dimensions of the earth are given, which is the case in M. de La Place's form,

$$\frac{2}{3} \times nh \times \sin \theta \sqrt{\frac{h}{16.1}} = \Delta,$$

it is certainly necessary that some literal designation should be preserved for the earth's semidiameter (r) in the denominator $\frac{2}{3r + m}$, though it should be taken as *unity* in the values of $\frac{2rc}{t}$ and p , that is in n and

θ ; otherwise, in computation, when the numerical value of the earth's semi-diameter (r) is connected with that of the height of the ball's descent $m(h)$, and both necessarily in the same denomination of measure, the expression, considered as a general one, would evidently in such case become incongruous and unscientific; as numerical *data* might be so proposed, then when $m(h)$ was rejected in the above divisor, as to bring out the place of the ball's descent precisely in the *same* place, or even to the *westward* of that from whence it had been demitted; a conclusion totally inconsistent with the established laws of motion in the case under consideration.

Hence, it is hoped, the intelligent mathematician will not forego his farther attention to this subject:

"Verbum sat sapienti."

M.

QUESTION 211.

Additional Solution.

Put $a =$ the cot AD , $c =$ cot AB ,
 $b =$ the sine, $\beta =$ the cosine, and $d =$ the tan Ac ,
 $m =$ the sine and $n =$ the cosine of $\angle DAB$,
 $\phi =$ the $\angle DAC$, $\psi =$ the $\angle BAC$.

Then, by trigonometry,

$$\cot \angle DCA = \frac{ab - \beta \cos \phi}{\sin \phi},$$

$$\cot \angle BCA = \frac{cb - \beta \cos \psi}{\sin \psi}.$$

Therefore, since $\angle DCA = \angle BCA$, and $d = \frac{b}{\beta}$.

$$\frac{ad - \cos \phi}{\sin \phi} = \frac{cd - \cos \psi}{\sin \psi}.$$

But since $\phi + \psi = \angle DAB$

$$\sin \psi = m \cos \phi - n \sin \phi$$

$$\cos \psi = n \cos \phi + m \sin \phi,$$

or if x be put $= \sin \phi$, and $y = \cos \phi$,

$$\sin \psi = my - nx$$

$$\cos \psi = ny + mx,$$

therefore by substitution,

$$\frac{ad - y}{x} = \frac{cd - (ny + mx)}{my - nx},$$

$$\text{or } (ad - y)(my - nx) = (cd - ny - mx)x,$$

whence

$$mx^2 + (2ny - adn - cd)x + admy - m = y^2 \quad (1)$$

$$\text{But } x^2 + y^2 - 1 = 0 \dots \dots \dots (2)$$

and subtracting m times the second equation from the first,

$$(2ny - adn - cd)x + admy - 2my^2 + m = 0,$$

$$\text{whence } x = \frac{2my^2 - admy - m}{2ny - adn - cd};$$

and this value substituted for x in equation (2) gives

$$\left(\frac{2my^2 - admy - m}{2ny - adn - cd} \right)^2 = 1 - y^2$$

and by reduction,

$$\left. \begin{aligned} 4m^2y^4 - 4adm^2y^3 + a^2d^2m^2y^2 + 2adm^2y + m^2 \\ 4n^2 - 4adn + 2acd^2n + 4adn - c^2d^2 \\ - 4cdn + a^2d^2n^2 + 4cdn - 2ad^2cn \\ + c^2d^2 - a^2d^2n^2 \\ - 4m^2 \\ - 4n^2 \end{aligned} \right\} = 0.$$

The roots of this biquadratic will give values of y that will answer

the question ; but, in making the numerical calculation, it will be much easier to find at once approximate values of ϕ and ψ from the equation $\frac{ad - \cos \phi}{\sin \phi} = \frac{cd - \cos \psi}{\sin \psi}$ by the method of trial and error.

QUESTION 215.

The construction by Merones (Mr. Emerson) is correct for the positions of the points A and B as exhibited on the hexagon. The perpendiculars Aa, Bf from the given points are drawn at right angles to two sides (As, Ax) of the hexagon which, include one of its angles, this, and the remainder of the construction will be general for any polygon when the problem is possible : but no notice is taken of any case in which the construction would fail ; for example, if the point B was near the side tv, and in the prolongation of fb, the point f would fall beyond xi or wx when produced, and the line cd would not intersect the sides vt, vw ; in that case, however, the track may be determined by drawing the perpendiculars from the points at right angles to the adjacent sides ts, tv.

If the point B was at p in the perpendicular Aa, the perpendicular from A must be at right angles to the side xz, (produced in the present instance), and that from p the other point, must be considered as terminating at p ; and the ball from A will strike xz, xw, &c. in succession.

Should the point B be removed to A, the track may be described by drawing the two perpendiculars from that point to the sides ts, tv ; and the ball, after rebounding from all the sides will arrive again at A.

QUESTION 222.

Extract from the 37th volume of the Philosophical Magazine.

A Method for ascertaining the Latitude and Time by the means of two known Stars, by M. Delambre. Translated from the "Connoissance des Temps" by Mr. Firminger, many years Assistant Astronomer at the Royal Observatory, Greenwich.

This problem is well known to astronomers, who resolve it by the plain rules of spherical trigonometry, applied to three connected triangles. Several geometers have thought it worth their while to take it into their consideration, and M. Gauss has just made it the subject of a dissertation, an extract of which has appeared in the

Journal of Baron de Zach. Our readers will see with pleasure the new researches of the learned professor who holds so distinguished a rank among geometers and astronomers.

Besides endeavouring to render the solution of this useful problem more useful and commodious, M. Gauss's object has been to prove that, by a well understood use of analysis, the same results may always be obtained, as are more commonly sought for by purely geometrical considerations.

I was ever persuaded of the truth of this assertion; but I have also generally found that the demonstrations are longer and more painful; that a considerable degree of address is sometimes required to attain the end; and that often it is useful to know beforehand, and by another process, what is sought for, as otherwise we may be led into analytical combinations that produce nothing convenient: but though it may thus happen at times that trials are unproductive, they will also sometimes lead to unexpected theorems, of which much simpler synthetic demonstrations will afterwards be found.

The best way, in my opinion, would be not to exclude any thing; to mix analysis with synthesis, to set down the equations of the problem in an easy construction, to avail ourselves of all the formulas that an immediate inspection of the figure may furnish, and to endeavour afterwards, by analytical processes, to give those equations a form the most advantageous, and most applicable to the use of logarithms.

In order to set a proper value on the advantages of the analytical solution of a known problem, it is necessary first to compare it with the vulgar solution; but this last should be previously reduced to general formulas, whereby no other attention is required in the calculator but that of the algebraic signs. The two solutions are then applied to the same example, after which we are enabled to draw certain consequences.

This is what we are going to do:—we shall reduce the trigonometrical solution to general formulas: we shall then compare these formulas with M. Gauss's, and shall calculate the example he has chosen.

The difference of the two stars in right ascension, reduced by an angle proportionate to the diurnal motion during the interval of the observations, gives first an angle to the pole between two circles of declination passing through the points that were occupied by the stars at the moment of the observations: the two declinations and this angle being known, it is easy to resolve the triangle.

Let δ and δ' be the two declinations, and θ the angle; the formula

$$\cot \delta' \cos \theta = \tan x \dots \dots \dots (1)$$

gives the first segment of one side equal to $(90 - \delta)$,

the second segment will then be $90 - \delta - x = 90 - (\delta + x)$.

Suppose now that v is the angle to the first star, you will have

$$\tan v = \frac{\tan \theta \sin x}{\cos (\delta + x)} \dots \dots \dots (2).$$

The arc D , which joins the two points observed, would be found by the formula

$$\cos D = \frac{\sin \delta' \sin (\delta + x)}{\cos x} \dots \dots \dots (3):$$

and the solution of this first triangle would require eleven different logarithms, but you may leave out one by making

$$\cot D = \cos v \tan (\delta + x) \dots \dots \dots (3^*).$$

Thus this first triangle requires ten or eleven logarithms, according to the choice you may make. If in these formulas δ is changed into δ' , and δ' into δ , you will have the angle v' to the second star, D being always the same distance.

Then the three sides of the triangle between the zenith and the two points observed will be known. Let w be the angle opposite to the second distance to the zenith,

$$\sin \frac{1}{2} w = \pm \left(\frac{\cos \left(\frac{h+h'+D}{2} \right) \sin \left(\frac{h+h'+D}{2} - h' \right)}{\cos h \sin D} \right)^{\frac{1}{2}} \dots (4).$$

h and h' being the two observed altitudes.

w may be either positive or negative. The circumstances will almost always indicate the choice that is to be made.

By changing h into h' , and reciprocally, we would obtain the angle w' opposite to the first distance to the zenith. In all cases the solution may thus be obtained in two different manners, which prove each other mutually.

This analogy requires five logarithms, which being united to the ten or eleven preceding will make fifteen or sixteen at will.

Let $u = v \mp w$, u will then be the angle to the first star, between the vertical and the circle of declination. In the same manner we should have $u' = v \mp w'$ for the second star.

Knowing the angle u , you will endeavour to find the horary angle λ and the latitude ϕ by means of the following formulas, which are all similar to the formulas (1), (2) and (3).

$$\text{Tang } z = \cos u \cot h; \text{ tang } \lambda = \frac{\text{tang } u \sin z}{\cos (\delta + z)}; \text{ tang } \phi = \cos \lambda \text{ tang}$$

$(\delta + z)$, (5), (6) and (7). By changing δ into δ' , h into h' , and λ into λ' , and reciprocally, you will obtain the horary angle of the second star, by means of the second altitude.

Instead of the last formula, we might have

$$\sin \phi = \frac{\sin h \sin (\delta + z)}{\cos z} \dots \dots \dots (7^*).$$

This third triangle therefore requires ten or eleven more logarithms.

The entire solution thus requires five-and-twenty or seven-and-twenty logarithms, but the method requiring only five-and-twenty may always be preferred. Of these logarithms none is used more than once.

These formulas are easy to establish, and require no effort of memory. We might obtain the angle w by the formula

$$\cos w = \frac{\sin h' - \sin h \cos D}{\cos h \sin D} = \tan h \cot D \left(\frac{\sin h'}{\sin h \cos D} - 1 \right) \dots (4^*),$$

but it would require seven logarithms instead of five: in the case $h = h'$ or of two equal heights, the formulas are on the contrary reduced to

$$\cos w = \frac{\sin h (1 - \cos D)}{\cos h \sin D} = \frac{\tan h \cdot 2 \sin^2 \frac{1}{2} D}{2 \sin \frac{1}{2} D \cos \frac{1}{2} D} = \tan h \tan \frac{1}{2} D,$$

which only requires three logarithms instead of five.

$\cos w$ may belong to a negative as well as to a positive arc. It is this ambiguity which has already been remarked on the subject of $\sin^2 \frac{1}{2} w$. It is common to all methods.

It is not likely that a more easy or shorter solution can be found than the one contained in the foregoing formulas.

Let us now examine M. Gauss's, giving to the analytical calculations a more direct and elementary form.

M. Gauss first draws from spherical trigonometry the two fundamental equations;

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos \lambda \dots (1);$$

$$\sin h' = \sin \delta' \sin \phi + \cos \delta' \cos \phi \cos (\lambda - \theta) \dots (2).$$

He only applies two triangles, while we have made use of three, which have an intimate connection and necessary relations. M. Gauss has thus proposed a much harder problem; but his analysis has set aside all difficulties in a very happy manner.

From the equation (1) it is easy to obtain

$$\begin{aligned} \cos^2 h &= 1 - \sin^2 h = 1 - (\sin \delta \sin \phi + \cos \delta \cos \phi \cos \lambda)^2 \\ &= 1 - \sin^2 \delta \sin^2 \phi - \cos^2 \delta \cos^2 \phi \cos^2 \lambda \\ &\quad - 2 \sin \delta \cos \delta \sin \phi \cos \phi \cos \lambda \\ &= 1 - \sin^2 \delta + \sin^2 \delta \cos^2 \phi - \cos^2 \delta \cos^2 \phi \cos^2 \lambda \\ &\quad - 2 \sin \delta \cos \delta \sin \phi \cos \phi \cos \lambda \\ &= \cos^2 \delta + \sin^2 \delta \cos^2 \phi - \cos^2 \delta \cos^2 \phi + \cos^2 \delta \cos^2 \phi \sin^2 \lambda \\ &\quad - 2 \sin \delta \cos \delta \sin \phi \cos \phi \cos \lambda \\ &= \cos^2 \delta \sin^2 \phi + \sin^2 \delta \cos^2 \phi + \cos^2 \phi \sin^2 \lambda - \sin^2 \delta \cos^2 \phi \sin^2 \lambda \\ &\quad - 2 \sin \delta \cos \delta \sin \phi \cos \phi \cos \lambda \\ &= \cos^2 \phi \sin^2 \lambda + \cos^2 \delta \sin^2 \phi + \sin^2 \delta \cos^2 \phi \cos^2 \lambda \\ &\quad - 2 \sin \delta \cos \delta \sin \phi \cos \phi \cos \lambda \\ &= (\cos \phi \sin \lambda)^2 + (\cos \delta \sin \phi - \sin \delta \cos \phi \cos \lambda)^2; \end{aligned}$$

and therefore, dividing the whole by $\cos^2 h$

$$1 = \left(\frac{\cos \phi \sin \lambda}{\cos h} \right)^2 + \left(\frac{\cos \delta \sin \phi - \sin \delta \cos \phi \cos \lambda}{\cos h} \right)^2 \dots (A).$$

Surely nothing can be easier than all those successive modifications which have led to the equation (A); but it is not easy to conceive why in the process, $\sin^2 \phi$, for example, has been left out, and $(1 - \cos^2 \phi)$ substituted, instead of leaving out $\sin^2 \delta$, and substituting $1 - \cos^2 \delta$; and even there is no very conclusive reason why the value of $\cos^2 h$ should be sought for by means of that of $\sin h$, given directly by the problem; and in this generally consists the inconveniency of analytical demonstrations.

It is easy to perceive that $\left(\frac{\cos \phi \sin \lambda}{\cos h} \right)$ is the sine of the angle

to the star in the triangle ZPA to the pole, the zenith, and the first star. Thus

$$\sin Z A : \sin Z P :: \sin P : \sin A = \frac{\sin P Z \sin P}{\sin Z A} = \frac{\cos \phi \sin \lambda}{\cos h}.$$

The equation (A) becomes, therefore,

$$1 = \sin^2 A + \left(\frac{\cos \delta \sin \phi - \sin \delta \cos \phi \cos \lambda}{\cos h} \right)^2 = \sin^2 A + \cos^2 A ;$$

from which it is easy to conclude that

$$\cos A = \left(\frac{\cos \delta \sin \phi - \sin \delta \cos \phi \cos \lambda}{\cos h} \right).$$

A will then be in the first triangle the angle to the star. $M. Gauss$ called it u , without any other explanation, and makes

$$\sin u = \frac{\cos \phi \sin \lambda}{\cos h}, \text{ which is the formula.} \dots\dots\dots (3);$$

$$\cos u = \left(\frac{\cos \delta \sin \phi - \sin \delta \cos \phi \cos \lambda}{\cos h} \right), \text{ which is the formula.} \dots\dots\dots (4).$$

It might be concluded that

$$\text{tang. } u = \frac{\sin u}{\cos u} = \frac{\cos \delta \text{ tang } \phi - \sin \delta \cos \lambda}{\sin \lambda},$$

but as we have already found the value of this angle we may make use of it to obtain other useful equations. We have, to be sure, the analytical expression of the sine u , and of the cosine u , but they contain the two unknown quantities ϕ and λ ; they cannot, therefore, serve but as preparatives.

Previous to using them, let us draw from formula (A) general consequences which may be of service. Let A be the pole, B the zenith, and C the angle to the star; we shall have

$$\sin C = \frac{\sin AB \sin A}{\sin BC} = \left(\frac{\cos \phi \sin \lambda}{\cos h} \right) \dots\dots\dots B.$$

$$\cos C = \left(\frac{\sin AC \cos AB - \cos AC \sin AB \cos A}{\sin BC} \right) \dots\dots\dots C.$$

This last formula is general; it belongs to all spherical triangles, and expresses the relation between the three sides of any triangle whatever; and two of its angles. Generally, sine 1st side cosine 2d side—cosine 1st side sine 2d side cosine angle comprised = sine 3rd side cosine angle opposed to second side $\dots\dots\dots (D)$; and $\frac{\sin 2d \text{ side into sin the comprised } \angle}{\text{sine third side}} = \sin \angle \text{op. to 2d side} \dots\dots\dots (E).$

This formula, which we have just found by analysis, is the fourth of eighteen of the same kind I have found by synthesis, which I have explained in my *Courses at the College de France*: they are not of service in the common calculations of spherical triangles, because they require the knowledge of four of the parts of the triangles, instead of three, which are sufficient in all cases; but they are useful in analytical researches, and we shall have occasion to use them again in the course of this memoir.

Supposing the angle to the first star or angle u is known, we shall conclude that $\sin \phi = \sin h \sin \delta + \cos h \cos \delta \cos u \dots \dots (5)$; h and δ are the complements of the sides comprising the angle u , ϕ is the complement of the side opposite to angle u ; here is then a mean to ascertain ϕ , as soon as we shall have determined the value of the third unknown quantity that has been introduced, that is to say, of angle u .

M. Gauss finds analytically the equation (5) by the combination of his formulas (1) and (3); but we endeavour to reduce all analytical operations to known rules of spherical trigonometry, in order to render the relation or rather the identity of the two methods more evident.

The equation (3) $\sin u = \frac{\cos \phi \sin \lambda}{\cos h}$, by simply reversing the quantities, gives

$$\sin \lambda = \frac{\cos h \sin u}{\sin \phi} = \frac{\sin 2d \text{ side} \sin \text{angle comprised}}{\text{sine third side}} \quad (\text{formula } \pi):$$

thus $\cosine \lambda = \text{cosine of angle opposite to second side}$

$$= \frac{\text{sine 1st side} \cos 2d \text{ side} - \cos 1st \text{ side} \sin 2d \text{ side} \cos \angle \text{comprised}}{\text{sine third side}};$$

according to the general formula (D).

$$\text{Or else, } \cos \lambda = \frac{\cos \delta \sin h - \sin \delta \cos h \cos u}{\cos \phi} \dots \dots (6).$$

Such in fact is the formula (6) of M. Gauss, who finds it, as well as formula (5), by combining the equations (1) and (3).

The formulas (3), (4), (5), and (6), contain, as we perceive, three unknown quantities instead of two; but they serve to find the useful equations.

By developing the formula (2), we shall have

$$\sin h' = \sin \delta' \sin \phi + \cos \theta \cos \delta' \cos \phi \cos \lambda + \sin \theta \cos \delta' \cos \phi \sin \lambda.$$

Let us substitute in this formula the values of $\sin \phi$ (equation 5), of $\cos \phi \cos \lambda$ (equation 6), of $\cos \phi \sin \lambda$ (equation 3); we shall have $\sin h' - \sin h \sin \delta' \sin \delta - \sin h \cos \theta \cos \delta \cos \delta' - \cos u \cos h \cos \delta \sin \delta' + \cos u \cos h \cos \theta \sin \delta \cos \delta' - \sin u \cos h \sin \theta \cos \delta' = 0 \dots \dots \dots F.$

$$\text{Or, } \sin h' - \sin h \sin \delta \sin \delta' - \sin h \cos \theta \cos \delta \cos \delta' - \cos u \cos h (\cos \delta \cos \delta' - \cos \theta \sin \delta \cos \delta') - \sin u \cos h \sin \theta \cos \delta' = 0.$$

$$\text{Or, } \sin h' - \sin h \sin \delta \sin \delta' - \sin h \cos \theta \cos \delta \cos \delta' -$$

$$\cos h \sin \theta \cos \delta' \left[\left(\frac{\cos \delta \sin \delta' - \cos \theta \sin \delta \cos \delta'}{\sin \theta \cos \delta'} \right) \cos u + \sin u \right] = 0.$$

Suppose, for the sake of shortening,

$$\frac{(\cos \delta \sin \delta' - \cos \theta \sin \delta \cos \delta')}{\sin \theta \cos \delta'} = \cotang v \dots \dots \dots (7).$$

The equation will be reduced to

$$\sin h' - \sin h \sin \delta \sin \delta' - \sin h \cos \theta \cos \delta \cos \delta' - \cos h \sin \theta \cos \delta' (\cot v \cos u + \sin u) = 0.$$

$$\text{Or, } \sin h' - \sin h \sin \delta \sin \delta' - \sin h \cos \theta \cos \delta \cos \delta' -$$

$$\cos h \sin \theta \cos \delta' \left(\frac{\cos v \cos u + \sin v \sin u}{\sin v} \right) = 0.$$

$$\frac{\text{Or, } \sin h' - \sin h \sin \delta \sin \delta' - \sin h \cos \theta \cos \delta \cos \delta' - \cos h \sin \theta \cos \delta' \cos (v - u)}{\sin v} = 0.$$

(v) is known by means of the equation (7), therefore the last equation contains only the unknown quantity ($v - u$) or ($u - v$), for it may be the one as well as the other: we shall then have

$$\cos w = \cos (v - u) = \frac{\sin v}{\cos h \sin \theta \cos \delta'} (\sin h' - \sin h \sin \delta \sin \delta' - \sin h \cos \theta \cos \delta \cos \delta') \dots \dots \dots 8:$$

and $u = v - (v - u)$, or $u = (u - v) + v$.

It is not easy to perceive, at first sight, how M. Gauss has been led to seek the auxiliary angle v , by means of formula (7); but the processes of the calculations we have given, show how the author may have been led to it.

It remains to know whether the angle v might not be one of the angles of the two triangles we have made use of. I see first that the equation (7), which gives the value of it, may be put under a simpler form:

$$\cot v = \frac{\cos \delta \tan \delta'}{\sin \theta} - \sin \delta' \cot \theta.$$

$$\text{Or, } \tan v = \frac{\sin \theta}{\cos \delta \tan \delta' - \cos \theta \sin \delta'}.$$

Therefore I now recognise the formula of spherical trigonometry, which gives the angle to the first star in the triangle of which ($90 - \delta$) ($90 - \delta'$) are the sides, and θ the angle comprised; and which thus has one of its summits at the pole, and the other two at the points of the two stars.

The angle v is thus that which is determined by the two first analogies in the vulgar method.

The difference of the angles w and v , of the formulas 7 and 8, serves then to show the value of $u = \pm w \mp v$, or the angle to the first star in the triangle which has given rise to the equation (1).

In this triangle we know the sides ($90 - h$), ($90 - \delta$), and the angle comprised u : we shall get the angle to the pole of the first observation by the formula

$$\tan \lambda = \frac{\cos h \sin u}{\cos \delta \sin h - \sin \delta \cos h \sin u} \dots \dots \dots (9)$$

which could be set down,

$$\tan \lambda = \frac{\sin u}{\cos \delta \tan h - \sin \delta \cos u}.$$

λ is the horary angle of the first star at the time it has been observed; if it was then east, R first star $-\lambda$ = right ascension to the middle of the sky, from which you may conclude the time of the first observation, and the correction of the clock.

If the star was to westward, R first star $+\lambda$ = right ascension to the middle of the sky, of which you will make the same use.

Lastly, the combination of equations (4) and (5) will give you
 $\text{tang } \phi = \sin \lambda \left(\frac{\sin \delta \sin h + \cos \delta \cos h \cos u}{\cos h \sin u} \right) \dots \dots \dots (10).$

Such, in fact, is M. Gauss's tenth formula.

It might be written, $\text{tang } \phi = \frac{\sin \lambda}{\sin u} (\sin \delta \text{ tang } h + \cos \delta \cos u).$

The trigonometrical method has given us above the simplest formula,

$$\sin \phi = \frac{\sin h \sin (\delta + z)}{\cos z}.$$

Or, $\sin \phi = \sin \delta \sin h + \cos \delta \cos h \cos u.$

The problem is then completely resolved. It may be simplified ; but let us previously make a useful remark, in order to distinguish, between the two solutions of the problem, that which is applicable to the observations.

The triangle to the pole between the two spots where the stars have been remarked, gives

$\sin v : \cos \delta' :: \sin \theta : \sin D = \sin \text{ third side of the triangle.}$

The triangle to the zenith gives

$\cos h' : \sin w :: \sin D : \sin (A - A') = \sin \text{ of azimuth difference.}$

$$\sin (A - A') = \frac{\sin D \sin w}{\cos h'} = \frac{\sin \theta \cos \delta'}{\sin v} \cdot \frac{\sin w}{\cos h'}.$$

$$\text{Or, } \sin w = \frac{\sin (A - A') \sin v \cos h'}{\sin \theta \cos \delta'};$$

and $\frac{\cos h'}{\cos \delta'}$, is a quantity essentially positive.

$\sin (A' - A)$ will always be positive, if you reckon the azimuths from one star to the other by the shortest distance.

Therefore $\sin w$ will have the same sign affixed to it as $\frac{\sin v}{\sin \theta}$;

you will therefore know whether w is a positive or a negative angle.

Which is the equation given by M. Gauss in order to set all doubt aside :—there is no need of calculating it. It will also be easy without

it to know the sign of $\left(\frac{\sin v}{\sin \theta} \right)$, which is sufficient.

If the two observations are both to eastward, or both to westward, $u = w - v$; if one is to eastward and the other to westward, $u = v - w$, unless (dR — motion of rotation) be negative.

In order to render the calculation of the $\angle y$ easier, M. Gauss makes

$$(11) \text{ tang } F = \frac{\text{tang } \delta'}{\cos \theta} = \frac{1}{\cos \theta \cot \delta'}, \text{ or, } \cot F = \cos \theta \cot \delta';$$

in which we perceive that his angle F is the complement of the subsidiary angle which I have named x .

$$(12) \text{ tang } v = \frac{\cos F \text{ tang } \theta}{\sin (F - \delta)} = \frac{\text{tang. } \theta \cos x}{\cos (\delta + x)}.$$

So for the angle v we may say that our formulas are identical.

$$(13) \cos w = \frac{\cos v \tan h}{\tan (v - \delta)} \frac{\sin h' \sin v}{\sin h \sin \delta \cos (v - \delta)} - 1).$$

$$\begin{aligned} \text{I make } \cos w &= \tan h \cos v \left(\frac{\sin h'}{\sin h \cos v} - 1 \right) \\ &= \tan h \cos v \tan (\delta + x) \left(\frac{\sin h \cos x}{\sin h \sin \delta' \sin (\delta + x)} - 1 \right). \end{aligned}$$

Our equations are thus again identical, notwithstanding the apparent difference. I have sought for the arch v separately, but it is clear I could have done without: my calculation is somewhat the shortest.

If $v' = h$, as in the example chosen by M. Gauss, $\frac{\sin h'}{\sin h} = 1$, two logarithms are saved in each of the two methods; but the trigonometrical method shows a more considerable simplification.

My formula becomes

$$\begin{aligned} \cos w &= \tan h \cot v \left(\frac{1}{\cos v} - 1 \right) = \frac{\tan h \cot v}{\cos v} (1 - \cos v) \\ &= \frac{\tan h \cdot 2 \sin^2 \frac{1}{2} v}{\sin v} = \frac{\tan h \cdot 2 \sin^2 \frac{1}{2} v}{2 \sin \frac{1}{2} v \cos \frac{1}{2} v} = \tan h \tan \frac{1}{2} v : \end{aligned}$$

a formula which the inspection of the triangle to the zenith, which in that case is isosceles gives immediately.

I draw the same simplification from M. Gauss's formula; but it was not evident at first sight: his formula (8) then becomes

$$\begin{aligned} \cos w &= \frac{\tan h \sin v}{\sin \theta \cos \delta'} (1 - \sin \delta \sin \delta' - \cos \delta \cos \delta' \cos \theta) \\ &= \frac{\tan h \sin v}{\sin \theta \cos \delta'} (1 - \cos v). \end{aligned}$$

By making $\sin \delta \sin \delta' + \cos \delta \cos \delta' \cos \theta = \cos v$. Then

$$\begin{aligned} \cos w &= \frac{\tan h \sin v \cdot 2 \sin^2 \frac{1}{2} v}{\sin \theta \cos \delta'} = \frac{\tan h \cdot 2 \sin^2 \frac{1}{2} v}{\frac{\sin \theta \cos \delta'}{\sin v}} \\ &= \frac{\tan h \cdot 2 \sin^2 \frac{1}{2} v}{\sin v} = \tan h \tan \frac{1}{2} v, \text{ as heretofore,} \end{aligned}$$

If we continue the comparison:

$$(14) \cot \alpha = \frac{\tan h}{\cos u} = \frac{1}{\cos u \cot h} = \cot z, \text{ then my arc } z = 90 - \alpha.$$

$$(15) \tan \lambda = \frac{\cos \alpha \tan u}{\sin (\alpha - \delta)} = \frac{\sin z \tan u}{\sin (d + z)}; \text{ our equations are identical.}$$

$$(16) \tan \phi = \cos \lambda \cos (\alpha - \delta) = \cos \lambda \tan (\delta + z).$$

Thus the formulas (11), (12), (13), (14), (15), and (16) which contain the practical solution of the problem in M. Gauss's method,

and to which he has attained by a very dexterous analysis, are the same as are taken at sight in the three triangles that have been resolved at all times by astronomers.

So that it is here demonstrated, *ipso facto*, that analysis can by its own strength lead to the same result as geometrical consideration. But it is not the less clear, that these last have the advantage of facility and conciseness; and we may add that it is always pleasing for the calculator to understand what he goes through; he sees that x and z are the first segments of the two bases, $90 - \delta - x$, and $90 - \delta - z$, the second segments of the same bases; w , v , and u angles comprised between known sides. These notions would suffice for him to find all the formulas he wants, without having recourse to any book; while it is, as I may say, impossible to grave in our memory the six analytical formulas; so that we are obliged to follow them blindly, because we cannot divine what are the subsidiary arcs, r , w , v , u and g , nor $(r - \delta)$ and $(g - \delta)$ in this problem.

From the demonstrated identity of the two methods, it is easy to conceive that in point of conciseness the difference cannot be great. In the angle w alone, to ascertain which the processes are not quite the same, the trigonometrical method is somewhat shorter, as will be seen by the calculated example.

M. Gauss lastly enquires, by differentiation of fundamental formulas, what may be the influence of the errors dh and dh' of the two altitudes observed on the latitude and horary angle λ . I have expressed this same effect by different formulas, with which I shall begin.

In the first place: It is clear that the errors dh and dh' can have no influence on the first three formulas which gave x , v , and D , for those quantities only depend on the two stars.

The triangle to the zenith gives the equation

$$\cos w \cos h \sin D = \sin h' - \sin h \cos D.$$

Whence

$-dw \sin w \cos h \sin D - dh \sin h \cos w \sin D = dh' \cos h' - dh \cos h \cos D$,
for we suppose D constant, then

$$\begin{aligned} -dw &= dh \left(\frac{\sin h \cos w \sin D - \cos h \cos D}{\sin w \cos h \sin D} \right) + \frac{dh' \cos h'}{\cos h \sin D \sin w} \\ &= \frac{dh' \cos h'}{\cot h \sin D \sin w} - \frac{dh \cos h \cos D}{\sin w \sin D \cos h} + dh \tan h \cos w \\ &= -dh \left(\frac{\cos h \cos D - \sin h \sin D \cos w}{\cos h \sin D \sin w} \right) + \frac{dh' \cos h'}{\cos h \sin D \sin w} \\ + dw &= +dh \left(\frac{\cot D}{\sin w} - \tan h \cot w \right) - \frac{dh' \cos h'}{\cos h \sin D \sin w}, \dots (1). \end{aligned}$$

This equation may be put in another form: for,

Let $(90 - h)$ be the first side of the triangle, D the second, w the angle comprised: we shall have

$$\begin{aligned} -dw &= -dh \left(\frac{\sin 1^{\text{st}} \text{ side} \cos 2^{\text{d}} \text{ side} - \cos 1^{\text{st}} \text{ side} \sin 2^{\text{d}} \text{ side} \cos \text{cont. } \angle}{\sin 1^{\text{st}} \text{ side} \sin 2^{\text{d}} \text{ side} \sin \text{contained angle}} \right) \\ &\quad + \frac{dh' \sin \text{third side}}{\sin 1^{\text{st}} \text{ side} \sin 2^{\text{d}} \text{ side} \sin \text{contained angle}} \end{aligned}$$

$$\begin{aligned}
 &= -dh \left(\frac{\sin \text{third side} \cos \text{angle opposite second side}}{\sin \text{first side} \sin \text{second side} \sin \text{contained angle}} \right) \\
 &\quad + \frac{dh'}{\sin \text{first side} \sin \text{angle opposite second side}} \\
 &= -dh \left(\frac{\cos \angle \text{opposite 2d side}}{\sin \text{1st side} \sin \angle \text{opposite 2d side}} \right) \\
 &\quad + \frac{dh'}{\sin \text{1st side} \sin \angle \text{op. 2d side}} \\
 &= -\frac{dh \cos (\Lambda - \Lambda')}{\cos h \sin (\Lambda - \Lambda')} + \frac{dh'}{\cos h \sin (\Lambda - \Lambda')}, \Lambda \text{ and } \Lambda' \text{ being} \\
 &\text{the azimuths in the two observations,} \\
 -dw &= \frac{dh' - dh \cos (\Lambda - \Lambda')}{\cos h \sin (\Lambda - \Lambda')} = \frac{dh' - dh + 2dh \sin^2 \frac{1}{2}(\Lambda - \Lambda')}{\cos h \sin (\Lambda - \Lambda')}; \text{ and} \\
 dw &= \frac{dh \cos (\Lambda - \Lambda') - dh'}{\cos h \sin (\Lambda - \Lambda')} \\
 &= \frac{dh \cot (\Lambda - \Lambda')}{\cos h} - \frac{dh'}{\cos h \sin (\Lambda - \Lambda')} \dots \dots \dots (2).
 \end{aligned}$$

$w - v = u$, then $dw = du$ when the two stars have been observed on the same side of the meridian; $u = v - w$, if they have been observed on different sides, then $du = -dw$.

In the second formula, it is supposed we are acquainted with Λ and Λ' , or at least $(\Lambda - \Lambda')$.

But as $\cos h' : \sin w :: \sin D : \sin (\Lambda - \Lambda') = \frac{\sin w \sin D}{\cos h'}$; it must, besides, be known whether $(\Lambda - \Lambda')$ be more or less than 90° , so that no great advantage is obtained by preferring the second formula to the first.

We have found above,

$$\sin \phi = \sin h \sin \delta + \cos h \cos \delta \cos u$$

$$d\phi = \frac{dh \cos h \sin \delta - dh \sin h \cos \delta \cos u}{\cos \phi} - \frac{du \sin u \cos h \cos \delta}{\cos \phi} \dots (3).$$

$$= dh \left(\frac{\sin \text{1st side} \cos \text{2d side} - \cos \text{1st side} \sin \text{2d side} \cos \angle \text{comp.}}{\sin \text{3d side}} \right)$$

$$- \frac{du \sin u \cos h \cos \delta}{\cos \phi}$$

$$= dh \frac{\sin \text{3d side} \cos \text{angle opposite 2d side}}{\sin \text{3d side}} - du \cos h \sin \Lambda$$

$$= dh \cos \Lambda - du \cos h \sin \Lambda \dots \dots \dots (3*).$$

$$= dh \cos \Lambda - \left(\frac{dh \cos (\Lambda - \Lambda') - dh'}{\cos h \sin (\Lambda - \Lambda')} \right) \cos h \sin \Lambda$$

$$= dh \cos \Lambda - \frac{(dh \cos (\Lambda - \Lambda') - dh') \sin \Lambda}{\sin (\Lambda - \Lambda')}$$

$$= \frac{dh \cos \Lambda \sin (\Lambda - \Lambda') - dh \sin \Lambda \cos (\Lambda - \Lambda') + dh' \sin \Lambda}{\sin (\Lambda - \Lambda')}$$

$$= \frac{dh \sin (\Lambda - \Lambda' - \Lambda) + dh' \sin \Lambda}{\sin (\Lambda - \Lambda')} = \frac{-dh \sin \Lambda' + dh' \sin \Lambda}{\sin (\Lambda - \Lambda')} \dots (4).$$

This is M. Gauss's formula, who computing the azimuths outwardly on the triangle, writes $(A' - A)$ which comes to the same.

This formula requires that A and A' be known.

$$\text{Now, } \cos \phi : \sin u :: \cos \delta : \sin A = \frac{\cos \delta \sin u}{\cos \phi};$$

$$\text{or, } \cos A = \frac{\sin \delta - \sin h \sin \phi}{\cos h \cos \phi}.$$

The second value is longer to compute, but it leaves no doubt on the sort of the angle A which will be obtuse. If $\sin \delta < \sin h \sin \phi$;

$$\text{we shall likewise have } \cos A = \frac{\sin \delta' - \sin h' \sin \phi}{\cos h' \cos \phi}.$$

After having calculated A and A' , we shall have $(A - A')$, and we may determine $d\phi$ without going through du .

We have, lastly;

$$\begin{aligned} \sin u \cot \lambda &= \tan h \cos \delta - \sin \delta \cos u \\ du \cos u \cot \lambda - \frac{d\lambda \sin u}{\sin^2 \lambda} &= \frac{dh \cos \delta}{\cos^2 h} + du \sin u \sin \delta, \\ du \cos u \cot \lambda - \frac{dh \cos \delta}{\cos^2 h} - du \sin u \sin \delta &= \frac{d\lambda \sin u}{\sin^2 \lambda} \\ d\lambda &= \frac{du \cot u \cos \lambda \sin \lambda}{\sin u} - \frac{dh \cos \delta \sin^2 \lambda}{\cos^2 h \sin u} - \frac{du \sin \delta \sin^2 \lambda}{1} \\ &= du \sin \lambda (\cot u \cos \lambda - \sin \delta \sin \lambda) - \frac{dh \cos \delta \sin \lambda \sin \lambda}{\cos h \cos h \sin u} \dots (5) \\ &= du \sin \lambda \left(\frac{\cos u \cos \lambda - \sin \delta \sin \lambda \sin u}{\sin u} \right) - \frac{dh \sin \lambda \sin \lambda}{\cos h \sin u} \\ &= - \frac{du \sin \lambda}{\sin u} (\sin \lambda \sin u \sin \delta - \cos \lambda \cos u) - \frac{dh \sin \lambda}{\cos \phi} \\ &= - \frac{du \cos h}{\cos \phi} (\cos A) - \frac{dh \sin A}{\cos \phi} \dots (5^a) \\ &= - \frac{\cos h \cos A}{\cos \phi} \left(\frac{dh \cos (A - A') - dh'}{\cos h \sin (A - A')} \right) - \frac{dh \sin A}{\cos \phi} \\ &= - \left(\frac{dh \cos A \cos (A - A') - dh' \cos A}{\cos \phi \sin (A - A')} \right) - \frac{dh \sin A}{\cos \phi} \\ &= - \frac{dh \cos A \cos (A - A')}{\cos \phi \sin (A - A')} + \frac{dh' \cos A}{\cos \phi \sin (A - A')} \\ &\quad - \frac{dh \sin A \sin (A - A')}{\cos \phi \sin (A - A')} \\ &= - \frac{dh [\cos A \cos (A - A') + \sin A \sin (A - A')] - dh' \cos A}{\cos \phi \sin (A - A')} \\ &= - \frac{dh \cos (A - A + A') - dh' \cos A}{\cos \phi \sin (A - A')} = \frac{-dh \cos A' + dh' \cos A}{\cos \phi \sin (A - A')} \dots (6) \end{aligned}$$

If we take the A 's outwardly on the triangle, the formula will be the same as M. Gauss's, and will become

$$d\lambda = \frac{+ dh \cos A' - dh' \cos A}{\cos \phi \sin (A' - A)}.$$

Thus, my formulas 1, 3, 5, can be brought back to M. Gauss's 4 and 6, who has found them by other means.

The formulas 4 and 6 having for denominator the sine of $(\Lambda - \Lambda')$ M. Gauss concludes that if $\Lambda = \Lambda'$; that is to say, if the two stars have been observed in the same vertical, or in different verticals of 180° , the errors dh and dh' will have a prodigious influence.

The same consequence may be drawn from my first formula, for it bears for its denominator the sine w ; and in the same case $w = 180^\circ$, and $\sin w = 0$ like sine $(\Lambda - \Lambda')$.

This consequence, however, is subject to some remarkable exceptions,

For, suppose that $\Lambda = \Lambda'$, that is to say, that the two stars have been observed in the same vertical line, a single altitude will suffice, for $\Lambda - \Lambda' = 0$, $u = 180^\circ - v$, $u + v = 180^\circ = w$.

α will then be known without error, the formula (3*) $d\phi = dh \cos \Lambda - du \cos h \sin \Lambda$ is reduced thereby to the term $dh \cos \Lambda$.

The formula (5) is reduced to the term $-\frac{dh \sin \Lambda}{\cos \phi} = d\lambda$, so that the case which appeared most unfavourable is, on the contrary, the easiest, and in which the error may be a *minimum*.

Let us suppose, lastly, that $\Lambda = \Lambda' = 90^\circ$, or that the two stars have been observed at the first vertical: in this case the error will be nought.—In fact,

$$\begin{aligned} \cot \phi &= \cot \delta \cos \lambda = \cot \delta' \cos (\lambda - \theta) \\ &= \cot \delta' \cos \theta \cos \lambda + \cot \delta' \sin \theta \sin \lambda; \\ \text{or } \tan \delta' \cot \delta \cos \lambda &= \cos \theta \cos \lambda + \sin \theta \sin \lambda, \\ \tan \delta' \cot \delta &= \cos \theta + \tan \lambda \sin \theta, \\ \tan \lambda &= \frac{\tan \delta' \cot \delta}{\sin \theta} - \cot \theta, \text{ and } \cot \phi = \cot \delta \cos \lambda. \end{aligned}$$

It is evident that the errors dh and dh' are nought, since h and h' need not be observed.

This case is, therefore, the truest of all, and the one that would be found most favourable, if it was easy to observe in the vertical. An *instrument des passages* (transit instrument) would be required for that purpose, directed east and west.

Let us see, lastly, the case wherein Λ' , though not equal to Λ , differs from it a slight quantity: in this case x being equal to $\Lambda - \Lambda'$, α will be very small.

$$\begin{aligned} \text{The formula (4) becomes } & \frac{-dh \sin (\Lambda - x) + dh' \sin \Lambda}{\sin x} = \\ & \frac{-dh \sin \Lambda \times \cos x + dh \cos \Lambda \times \sin x + dh' \sin \Lambda}{\sin x} = \\ & \frac{dh' \sin \Lambda - dh \sin \Lambda + 2dh \sin \Lambda \sin^2 \frac{1}{2}x + dh \cos \Lambda \sin x}{\sin x} = \\ & \frac{(dh' - dh) \sin \Lambda}{\sin x} + dh \sin \Lambda \tan \frac{1}{2}x + dh \cos \Lambda. \end{aligned}$$

$dh \cos \Lambda$ is the same as was obtained by supposing $x = 0$; the term $dh \sin \Lambda \tan \frac{1}{2}x$ will be very small. As to the term $\frac{(dh' - dh) \sin \Lambda}{\sin x}$, it will be reduced to nought, or to very little.

To nought, if $dh' = dh$; and the two altitudes being measured by the same instrument, we shall have $dh' - dh = 0$, when the errors will be those of the line of collimation, which will happen in most cases.

It will be reduced at least to very little. Suppose $dh' - dh = 10''$; $x = 1^\circ$; $\left(\frac{dh' - dh}{\sin 1^\circ}\right) \sin A = 57'' \sin A = 28''$, by supposing $A = 30^\circ$ for a mean between the extreme values of sine A :

If h alone has been observed, because $A = A'$, we shall then have $h' = h + d$; thus h' is supposed to be equal to $h + d$ and dh' to dh , therefore $dh' - dh = 0$; in which case likewise $\frac{1}{2} \text{ tang. } x = 0$, there only remains $dh \cos A$, which will never exceed dh .

The errors $d\lambda$ become in the same manner:

$$\begin{aligned} \frac{-dh \cos(A-x) + dh' \cos A}{\cos \phi \sin x} &= \frac{-dh \cos A \cos x - dh \sin A \sin x + dh' \cos A}{\sin x \cos \phi} \\ &= \frac{(dh' - dh) \cos A + 2dh \cos A \sin^2 \frac{1}{2}x}{\sin x \cos \phi} - \frac{dh \sin A}{\cos \phi} \\ &= \left(\frac{dh' - dh}{\sin x}\right) \frac{\cos A}{\cos \phi} + \frac{dh \cos A \text{ tang } \frac{1}{2}x}{\cos \phi} - \frac{dh \sin A}{\cos \phi}. \end{aligned}$$

At first glance, the error on ϕ and λ should increase in proportion as the sine x lessens, and should be infinite when $x = 0$; still if

$x = 0$, the errors are reduced to $dh \cos A$ and $\frac{dh \sin A}{\cos \phi}$.

It might, therefore, be thought that such is the error when arrived to a maximum; but it appears more natural to say that, whenever x is very small, the differential formulæ cannot give the exact effects produced by dh and dh' :

I have to give an example of all these calculations.

Let us suppose, with M. Gauss, that $\delta = 8^\circ 22' 43''$, 1 ; $\delta' = 28^\circ 2' 13''$, 4 ; $\theta = 62^\circ 33' 26''$, 9 ; $h = h' = 45^\circ 44' 52''$, 6 .

Calculation of the subsidiary Angle v according to M. Gauss.

Tang δ ..9.7263516	tang θ 0.2845877
	cos F 9.8158318
c cos θ ..0.3364323	c sin $F - \delta$ 0.1852598
0.0627839	tang $v = 62^\circ 36' 58''$, 79
	0.2856793
tang $F = 49^\circ 7' 37''$, 70	
- δ 8 22 43, 10	
$F - \delta = 40 44 54$, 60	

Trigonometrical Method.

Cot δ 0.2736484	tang θ 0.2845877
Cos θ 9.6635677	sin u 9.8158318
Tan $x = 40^\circ 52' 22''$, 30 = 9.9372161	
$\delta = 8 22 43$, 10	c cos $(\delta + x)$ 0.1852598
$\delta + x = 49 15 5$, 4	tan $v = 62^\circ 36' 58''$, 79

Both methods require seven logarithms, which are either the same as in the second operation, or arithmetic complements of one another, as in the first. r and x , $r - \delta$ and $\delta + x$ are arcs complements of each other.

On the contrary, the operations required in order to find out the third auxiliary angle w are quite different.

Calculation of the angle w according to M. Gauss.

Sin r 9.8786157	$n - 1 = 1.123433$ 0.0505666
c sin δ' 0.3278629	cos v 9.6627075
c cos $(r - \delta)$ 0.1205703	tang h 0.0113399
	cot $(r - \delta)$ 0.0646895
Sin h	cos $w = 52^\circ 0' 14'', 25$ 9.7893035
c sin h'	$v = 62 \ 36 \ 58, 79$
$n = 2.123433$ 0.3270489	$v - w = u = 10 \ 36 \ 44, 54$

This operation requires nine logarithms in the case of $h = h'$; it would require eleven, if the altitudes were unequal.

In the case of both being equal, the trigonometrical method would have much the advantage. I neglect it in order to give to the calculation the disposition requisite for all cases.

Trigonometrical Calculation.

Cos v 9.6627075	$n - 1 = 1.123433$.. 0.0505666
Tang $(\delta + \lambda)$ 0.0646895	Tang h 0.0113399
Cot $D = 61^\circ 54' 20'', 5$ 9.7273970	Cot D 9.7273970
c cos D 0.3270490	cos $w = 52^\circ 0' 14'', 25$ 9.7893035
c sin h	
Sin h'	
$n = 2.123433 = 0.3270490$	

I must first find $\cot D = \cos v \tan (\delta + x)$. The logarithms of $\cos v$ and $\tan (\delta + x) = \cos (r - \delta)$ are common to both methods. The arc D is the third side of the triangle which has given v .

In the case of both being equal, the cosine of D is the number n of the analytical method. Commonly $n = \frac{\cos D \sin h'}{\sin h}$; $n - 1$ is common to both methods, they assign a same value to the angle w .

The trigonometrical method only requires eight logarithms when $h = h'$, and ten when the altitudes are unequal.

In the case of two equal heights, we would have $\cos w = \tan h \tan \frac{1}{2} D$, and this angle would not require more than six logarithms instead of nine.

Here is the calculation of it:

Tang h 0.0113399	$w = 52^\circ 0' 14'', 25$
Tang $\frac{1}{2} D = 30^\circ 57' 10'', 25$ 9.7779636	$v = 62 \ 36 \ 58, 79$
cos $w = 52 \ 0 \ 14, 25$ 9.7893035	$u = 10 \ 36 \ 44, 54$

The same simplification could be obtained from the analytical for-

mula; but it could not be so easily perceived. Thus analysis, handled as it should be, conduces to the same solution as spherical trigonometry; but it is not always easy to choose such modifications as would reduce to the least terms the calculation of a complicated formula.

Analytical Calculation of the Horary Angle.

$c \cos u$	0.0074926	$\text{Tang } u$	9.2726964
$\text{Tang } h$	0.0113399	$\cos g$	9.8398647
$\text{Tang } g = 46^\circ 14' 30'', 77$	0.0188325	$c \sin (g - \delta)$	0.2119881
$\delta = 8 \ 22 \ 43, 1$		$\text{Tan } \lambda = 11^\circ 55' 18'', 31$	9.3245492
$g - \delta = 37 \ 51 \ 47, 67$			

Trigonometrical Calculation.

$\cos u$	9.9925074	$\text{Tang } u$	9.2726964
$\text{Cot } h$	9.9886601	$\sin z$	9.8398647
$\text{Tang } z = 43^\circ 45' 29'', 23$	9.9811675	$c \cos (\delta + z)$	0.2119881
$\delta = 8 \ 22 \ 43, 10$		$\text{Tang } \lambda = 11^\circ 55' 18'', 31$	9.3245492
$\delta + z = 52 \ 8 \ 12, 33$			

Which shows that both operations are identical, which is also the case of the following

Calculation of the Latitude.

Analytic Method.

Trigonometric Method.

$\cos \lambda$	9.9905300	$\cos \lambda$	9.9905300
$\text{Cot } (g - \delta)$	0.1093281	$\text{Tang } (\delta + z)$	0.1093281
$\text{Tan } \phi = 51^\circ 31' 47'', 19$	0.0998581	$\text{Tan } \phi = 51^\circ 31' 47'', 19$	0.0998581

The analytical method requires twenty-five logarithms in the case of $h = h'$, and twenty-seven in the more general case.

The trigonometrical method would require twenty-four logarithms in the case of $h = h'$, setting aside the facility offered by the isosceles triangle. It requires twenty-six in the more common case. In fact, it only requires twenty-two in the case of two equal altitudes, which, therefore, has the advantage by three logarithms. It is true that the analytical method may be reduced to the same conciseness; in which case, both methods would be perfectly identical: but the trigonometrical method will always have this advantage,—that it gives the formula at sight, while analysis cannot attain to them but by going a great round about, whereby one is apt to be misled.

The differential formulæ remain to be compared; but without these formula, leaving h and h' indeterminate, the errors $d\lambda$ and $d\phi$ that will result from dh and dh' may be ascertained while calculating λ and ϕ .

In the first place it is evident that the errors dh and dh' can have no effect on x , v , nor D ; they only begin to have any influence on w .

By calculating w , I see that the logarithm of z will increase 20.5

parts for each second of error in dh' , and will lessen as much for each second of error in dh .

Let us then suppose that h and h' are too great, the logarithm of n will be too great by $-20,5 dh + 20,5 dh'$, n will be too great of itself by $-0,00001dh + 0,00001dh'$; for by an uncommon coincidence n increases $0,00001$ in the same manner as h increases $1''$, when the logarithm is increased $20,5$ parts; $n - 1$ will, therefore, be too great by $-0,00001dh + 0,00001dh'$.

The logarithm ($n - 1$) will, therefore, be too great by $-38,7dh + 38,7dh'$.

The logarithm of tang h will be too great by $42,1dh$.

The logarithm of cos w will be too great by $+3,4dh + 38,7dh'$;

but w decreased $26,9$ part for $1''$. w then is too small by $\frac{3,4}{26,9} dh$

$+ \frac{38,7}{26,9} dh'$, or by $0,1264dh + 1,4387dh'$; u is then too great by the

same quantity, since $u = v - w$.

Thus, by supposing that $dh = dh' = 10''$, u will be too great by $15'',65 = 1,264 + 14,387$.

Cos u will be too little by.... $0,5056dh + 5,7548dh'$.

Cot h will be too little by.... $42,1 dh$

Tang z too little by..... $42,6056dh + 5,7548dh'$.

z too little by..... $1,0096dh + 0,1364dh'$.

$\delta + z$ too little likewise by the same amount.

Tang u will be too great by $+14,700dh + 167,320dh'$.

Sin z too great by..... $-22,212dh - 2,864dh'$.

c cos ($\delta + z$) too great by $-27,361dh - 3,696dh'$.

Tang λ too great by $-34,873dh + 160,760dh'$.

λ too great by $-0,3350dh + 1,5444dh'$.

Thus, supposing $dh = dh' = 10''$, $d\lambda$ will be $+12'',094$, the horary angle too great by $12''$.

Cos λ too great by..... $+ 1,474dh - 6,7954dh'$.

Tang ($\delta + z$) too great by.... $-43,919dh - 5,9322dh'$.

Tang ϕ too great by..... $-42,445dh - 12,7276dh'$.

ϕ too great by..... $-0,9803dh - 0,2939dh'$.

And, supposing $dh = dh' = 10''$, the latitude will be too little by $12'',78$; by recommencing the calculation and using the two altitudes as increased $10''$, I had found $d\lambda = +12''$ and $d\phi = -13''$.

This method of logarithmic differentials is general; it is more concise than the direct calculation with increased altitudes, but it requires a very minute attention. The formulæ are more commodious. I begin with mine.

$$\begin{array}{rclclcl} \text{Cot } D & 9,7274 & - \text{Tang } h & 0,0113 & - \text{c sin } D & 0,0544 \\ \text{c sin } w & 0,1035 & \text{Cot } w & 9,8927 & \text{c sin } w & 0,1035 \\ + 0,6775 & 9,8309 & - 0,8017 & 9,9040 & - 1,438dh' & 0,1570 \\ - 0,8017 & & & & & \end{array}$$

$$- 0,1242dh - 1,438dh' = - du.$$

Let us suppose $dh = dh' = 10''$, we shall have $du = + 0,1242dh' + 1,438dh = 15'',622$: the logarithmic variations have given $15'',65$.

Cot u	0.7273	— sin δ	— 9.1635	— cos δ	— 9.9953
Cot λ	9.9905	sin λ	9.3151	sin ² λ	8.6302
+ 5.2216	0.7178	— 0.0301	8.4786	c cos ² h	0.3126
— 0.0301				c sin u	0.7348
+ 5.1915	0.7153		0.7153		— 9.6729
Sin λ	9.3151		9.3151		
0.1242	9.0941		1.438	0.1579	— 0.4709 dh'
+ 0.1332 dh	9.1245	+ 1.5427 dh'	0.1883		+ 1.5427 dh'
		$d\lambda = + 0.1332 dh + 1.0718 dh'$			+ 1.0718 dh' .

Suppose $dh = du = 10''$, we shall have $d\lambda = + 12''.05$ the logarithmic variations have given $12''.09$.

c cos ϕ	0.2061		0.2061		— 0.2061
Cos h	9.8437	— sin h	— 9.8552	cos h	9.8437
Sin δ	9.1635		cos δ 9.9953	cos δ	9.9953
			cos u 9.9925	sin u	9.2652
+ 0.1634	9.2133	— 1.1197	0.0491	— 0.2042	— 9.3103
		+ 0.1634		0.1242	9.0941
		— 0.9563 dh		— 0.0254	8.4044
		— 0.0254 dh			
		— 0.9817 dh		— 0.2042	— 9.3103
$d\phi = \{$		— 0.2939 dh'		1.438	0.1579
				— 0.2939	— 9.4682

Supposing $dh = dh' = 10''$, $d\phi = - 12''.756$. The logarithmic differentials have given $- 12''.78$. A more satisfactory coincidence cannot be wished for. These formulæ appear to require more calculation than the logarithmic differences, but they do not require so minute an attention.

M. Gauss's formulæ are simpler, but they suppose that the two azimuths are determined, in order to which it is sometimes necessary to make use of the longest formulæ, to avoid all uncertainty on the description of the angle.

For the greater uniformity and generality, I chose the following formula :

$$\text{Cot } A = \frac{\text{tang } \delta \cos \phi - \sin \phi \cos \lambda}{\sin \lambda}$$

Tang δ	9.1683	— sin ϕ	— 9.8937
Cos ϕ	9.7939	cos λ	9.9905
+ 0.09166	8.9622	— 0.76594	— 9.8842
— 0.76594			
— 0.67428	— 9.8288	negative denominator.	
c sin λ	+ 0.6849	positive numerator.	

Cot $A = 162^\circ 57' 50'' 0.5137$; A must be in the second quadrant.

We shall obtain the other azimuth A' by a similar formula.

Tang δ'	9.7263	— sin ϕ	— 9.8937
Co ϕ	9.7939	cos $(\lambda - \theta)$	9.8023
+ 0.33115	9.5202	— 0.49660	— 9.6960
— 0.49660			
— 0.16515		— 9.2187	denominator negative.
c sin $(\lambda - \theta)$		— 0.1118	numerator negative.
Cot $\Delta' = 257^\circ 55' 10'' +$	9.3305	Δ' will be in the third quadrant.	
$\Delta = 162^\circ 57' 50''$			
$\Delta' - \Delta = 94^\circ 57' 20''$	0.0016	c sin $(\Delta' - \Delta) \dots$	0.0016
	0.2601	c cos $\phi \dots \dots \dots$	0.2601
+ cos Δ'	— 9.3208	— cos $\Delta + \dots$	9.9805
— 0.3824dh	9.5825	+ 1.7467dh'	0.2422

$d\lambda = -0.3824dh + 1.7467dh' = +13''.643$, in supposing $dh = dh' = 10''$.

	0.0016	c sin $(\Delta' - \Delta) \dots$	0.0016
+ sin Δ'	— 9.9903	— sin $\Delta \dots \dots \dots$	— 9.4668
— 0.9815dh	— 9.9919	— 0.2940dh'	9.4684

$d\phi = 0.9815dh - 0.2940dh' = -12''.7555$, by supposing $dh = dh' = 10''$.

The azimuths are computed from the inferior meridian, which I suppose to be 0 to 360° , eastward. The λ are positive on the east side, and negative on the west. They are of 180° at the lower meridian, 360° or 0 at the upper.

$$\begin{aligned} dh &= d\phi \cos \Delta - d\lambda \cos \phi \sin \Delta \\ dh' &= d\phi \cos \Delta' - d\lambda \cos \phi \sin \Delta', \end{aligned}$$

which M. Gauss has given, and may be easily deduced from my formulæ, might be still more easily demonstrated by the figure which has supplied the fundamental equation of the problem; they show evidently that if $\Delta' = \Delta$, dh will necessarily be $= dh'$; for the second members being equal, the first must be so likewise. If $\Delta' = 180^\circ + \Delta$, dh' will be equal to $-dh$. In fact, in the first case, $h' = h + v$, and in the second case $h + h' + v = 180$. Thus, if the first altitude h had been observed, it would only be required to set down the time when h' would be equal to $h + v$, or where $h' = 180 - h - v$; but it would be necessary to be assured that the second star would pass by the same vertical as the first: but if two complete observations of altitudes and times are made, it is possible that dh' and dh will be such that the altitudes h and h' will no longer agree with the time elapsed, the two declinations and with the distance n . Thus, the suppositions it is necessary to admit no longer agree with what passes in the heavens; and in order that the differential formulæ could be established, either the declinations or the angle θ should have been made to vary.

If Δ' differs not much from Δ , the two equations will not differ much from each other. And as it is from their differences that the values of $d\phi$ and $d\lambda$ are computed, in order properly to determine

those two quantities, an extreme precision is requisite in ascertaining the values of Δ' and Δ , a precision which can be expected neither from the observation nor the calculation. These differential formulæ are therefore only approximations, on which, dependance should not always be placed, which, however, does not imply that $d\phi$ and $d\lambda$ will in that case be very considerable. If the cosines of the azimuths are small, it may be expected that the latitude will not be well determined; if the sines of the azimuths are small, the hour will not be well known; but if one of the observations is made towards the first vertical, and the other towards the meridian, all the precision will be obtained that can be expected from this method, which may be of great value, when no other is feasible, but which can never be compared with those that separately determine the latitude and the time.

Mr. Van Beck Calkoen, director of the observatory royal at Utrecht, has just given a different solution of this problem, with a slight change, which was not even indispensable. It consists in reducing to the same moment the observations of the two altitudes. For this purpose Mr. Calkoen commences by observing one of the two stars immediately after he observes the other, and after that he again observes the first:—few minutes are sufficient for these three observations. During this interval, it may be supposed that the first star has ascended by an uniform motion; and by means of simple proportional parts, the altitude of the first star is reduced to the moment when the second was observed.

This being laid down, we shall, in order to save a figure, express in general formulæ, Mr. Calkoen's solution.

Let D be the distance between the two stars, we shall have

$$\cos D = \cos (AR - AR') \cos \delta \cos \delta' + \sin \delta \sin \delta' \dots \dots (1).$$

$$\text{Make } \cot x = \frac{\sin \delta' - \sin \delta \cos D}{\sin \delta \sin D} = \left(\frac{\sin \delta'}{\sin \delta} - \cos D \right) \frac{1}{\sin D} \dots \dots (2).$$

x will be the continuation of the distance D till it reaches the equator (line.)

$$\sin A = \frac{\sin \delta}{\sin x} \dots \dots \dots (3).$$

A will be the angle by which the continuation x intersects the equator.

$$\text{Make } \cot F = \frac{\sin h' - \sin h \cos D}{\sin h \sin D} = \left(\frac{\sin h'}{\sin h} - \cos D \right) \frac{1}{\sin D} \dots \dots (4).$$

F will be the continuation of the distance D to the horizon.

$$\sin B = \frac{\sin h}{\sin F} \dots \dots \dots (5).$$

B will be the angle by which the continuation F will intersect the horizon. x and F will be in the same plane.

$$\text{Make } \tan x = \cos (x - F) \tan B \dots \dots \dots (6).$$

$$\sin \phi = \frac{\cos B \cos (A - x)}{\cos x} \dots \dots \dots (7).$$

Then $(90^\circ - \phi)$, $(90 - \delta)$, and $(90 - h)$ will be the three sides

of a spherical triangle, and the angle opposite to $(90 - h)$ will be the horary angle of the star whose declination is δ .

Thus eight formulæ give the solution of the problem, which is founded on a simple and ingenious construction. The demonstration of the formulæ is easy to find, it is perceived on the simple inspection of the figure. The solution requires 33 logarithms actually different; or, with tables of natural and logarithmic sines, 30 only; but in all cases it is longer than the preceding solution.

The advantage that could be found by it is, that for two given stars the arc of distance p is almost constant, as well as its continuation z , and the angle λ on the equator. These three quantities vary but slowly by the precession, aberration, and nutation, which are a little different in the two stars.

Tables of them could, therefore, be made which would save about 16 logarithms, or half the work. These tables could, without sensible error, be used for some weeks; but this advantage would be lost, if the problem was restored to its original form, by not taking the first altitude at the same moment as the second. Then instead of $(R - R')$, $(R - R' - m)$ must be employed.

The distance p would no longer be the real distance between the two stars. The remainder of the solution would not be altered; but as it is simple and natural, it would be better to substitute our formulæ.

For ascertaining the value of the error that is to be apprehended in the latitude ϕ , Mr. Calkoen gives the same formula as M. Gauss; he does not give the one for the horary angle.

M. Gauss's method was no more than the vulgar method expressed and demonstrated by analysis simply: Mr. Calkoen's is solely founded on a geometrical construction, which we have reduced to general formulæ. Dr. Mollweide has just given in Mr. De Zach's Journal of June, 1809, page 545, a purely analytical method, which has no relation with the trigonometrical solution.

Like M. Gauss, he commences by the equation $\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos \lambda$; and $\sin h' =$, &c.

He makes $z = \tan(45^\circ - \frac{1}{2}\phi) \sin \phi = \frac{1 - z^2}{1 + z^2}$, $\cos \phi = \frac{2z}{1 + z^2}$; he substitutes those values, and uniting all the known

quantities, which he designs by m, n, m', n' , he derives the two equations $m + Nz^2 = z \cos \lambda$

$m' + N'z^2 = z \cos(\lambda - \theta)$; he cuts off z^2 , and finds

$$z = \frac{m'N - mN'}{N \sin \theta \sin \lambda - (N' - N \cos \theta) \cos \lambda};$$

after which, by substitutions too extensive to detail here, he obtains the following formulæ:

$$\text{Tang}(45^\circ + \lambda) = \frac{\sin \frac{1}{2}(h' + \delta') \cos \frac{1}{2}(h' - \delta') \cos \delta}{\sin \frac{1}{2}(h + \delta) \cos \frac{1}{2}(h - \delta) \cos \delta};$$

$$\text{Tang } (45^\circ + A') = \frac{\cos \frac{1}{2}(h' + \delta') \sin \frac{1}{2}(h' - \delta') \cos \delta}{\cos \frac{1}{2}(h + \delta) \sin \frac{1}{2}(h - \delta) \cos \delta'};$$

$$\text{Tang } B = \text{tang } A \cot \frac{1}{2}\theta. \quad \text{Tang } B' = \text{tang } A' \cot \frac{1}{2}\theta.$$

$$c = (B' + B + \theta) \quad \text{and} \quad d = (B' - B).$$

$$\text{Tang } E = \frac{\sin (h + \delta) \sin (h - \delta) \sin d}{2 \cos^2 \delta \cos \frac{1}{2}(c + d) \cos \frac{1}{2}(c - d)};$$

$$\cos (2\lambda - c) = \frac{\cos (d - E)}{\cos E} = \cos (c - 2\lambda).$$

$$\text{Tang } (\lambda - \frac{1}{2}c) = \pm \sqrt{[\text{tang } \frac{1}{2}d \text{ tang } \frac{1}{2}(d - E).]}$$

$$\text{Tang } (45^\circ - \frac{1}{2}\phi) = \frac{\cos \frac{1}{2}(h + \delta) \sin \frac{1}{2}(h - \delta) \sin d}{\cos \delta \sin (\lambda - \frac{c - d}{2}) \cos \frac{1}{2}(c + d)};$$

Of the two values of λ , the one is chosen which makes $(45^\circ - \frac{1}{2}\phi) < 45^\circ$; that is to say, ϕ positive. These formulæ are purely analytical, at least it is difficult to perceive in what they could assimilate to the ordinary solution. They are elegant; they indicate, in a more direct manner than any of the preceding methods, the choice to be made between the two roots of the equation, but at the same time require too long a preparation, more logarithms, and have, besides that great inconveniency in a method of common use, that it cannot be retained by the best memory. This solution requires 29 logarithms absolutely different; besides that, ten of these logarithms are to be used twice; so that with all the composition of half sums and half differences, the operation is longer by half than that of our formulæ: thus, notwithstanding its elegance and general application, I do not expect it will be preferred in practice.

QUESTION 241.

Additional Solutions from Turner's Exercises.

The degree of heat being supposed as any given power of the sun's altitude into any given power of the time of his continuance above the horizon. And in this form the question was proposed by Stone, in the preface to his fluxions.

PROBLEM. *By Mr. W. Toft.*

To find the hottest time of the day in any given latitude; supposing the heat to be conjunctly as any given power of the sine of the sun's altitude, and any given power of the time of his continuance above the horizon.

Answered by Mr. R——son.

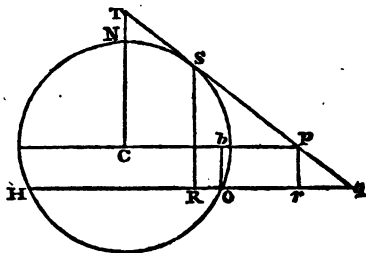
Suppose the degree of heat to be as the m th power of the sun's altitude into the n th power of the time of his continuance above the horizon, and let a and b be the sine and cosine of the given latitude, also c and d the sine and cosine of the given declination to radius 1; put s = the semi-diurnal arc, Λ = the arc from noon to the time when this Phenomenon happens, and x and y for the sine and cosine thereof: Then $bd + acy$ will express the sine of the sun's altitude; and, it being certain that the greatest heat cannot happen before noon, $(bd + acy)^m \times (s + \Lambda)^n$ must be a maximum, and its fluxion $macy \times (s + \Lambda) + n\Lambda \times (bd + acy) = 0$. But, by the property of the circle, $-\frac{y}{x}$ is $= \dot{\Lambda}$, and consequently $(s + \Lambda) \times x - \frac{ny}{m} = \frac{n}{m} \times \frac{bd}{ac}$; from whence by the method of trial-and-error, or by infinite series, the value of x or y may be determined.

The same answered by the proposer, Mr. Toft.

Let the circle $HNRO$ be the parallel of the sun's declination on the the proposed day, and let HO be the common section of the plane thereof with the plane of the horizon; also let HN be the semi-diurnal arc of the parallel, and NS the arc thereof corresponding to the time from noon when the phenomenon happens: Let the tangent TSQ be drawn, and thro' the centre C draw CP parallel to HO ; also draw SN , OR and PR perpendicular to HO ; then, if the arc HN be denoted by a , the perpendicular PR by d , the arc HS by z , the line SN by y , and the tangent TS by x , we shall, as SN has a constant ratio to the sine of the sun's altitude, have $z^m \times y^n$ a maximum, therefore its fluxion $mz^{m-1} \times \dot{z} \times y^n + ny^{n-1} \times \dot{y} \times z^m = 0$.

Whence $mzy = -nyz$, and $\frac{m}{n} \times \frac{zy}{-y} = z$. But as sq is denot-

ed by $\frac{zy}{-y}$, it appears that, in the required circumstance, $m : n ::$ the arc HS : tangent sq ; therefore, as the arch NS is expressed by $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}$, &c. and sq by $\frac{1 + d\sqrt{1+x^2}}{x}$, we



shall have $m : n :: a + x - \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7}, \&c. : \frac{1 + d\sqrt{1+x^2}}{x}$;
from whence the value of x may be determined.

QUESTIONS 185, 264, &c.

Additional Solution.

Let Λ be the arc, and v the area of the curve required; z the similar arc and u the corresponding area of a similar curve, having a given parameter, suppose in the present case = 1.

Then the areas of similar curves being as the squares of their like parts, $z^2 : \Lambda^2 :: u : v$; therefore $v = \frac{u \times \Lambda^2}{z^2}$, which is to be a maximum.

Taking fluxions, $\dot{z} \times z^2 - 2uz\dot{z} = 0$,

$$\text{or } z = \frac{\dot{z}}{u} \times 2u.$$

Put x for the abscissa, and y for the ordinate corresponding to the arc z , then $z = x\sqrt{1 + \frac{y^2}{x^2}}$, $u = fyx$ and $\dot{u} = y\dot{x}$; these substituted in the expression for z , give

$$z = \sqrt{1 + \frac{y^2}{x^2}} \times \frac{2fyx}{y}.$$

Now, when the parameter is 1, the equation of the curve is $x^2 = y^2$;

then $\frac{\dot{y}}{x} = \frac{4x^2}{9y^4} = \frac{4}{9y}$, $\frac{2fyx}{y} = \frac{6x}{5} = \frac{6y}{5}\sqrt{y}$; therefore,

$$z = \sqrt{1 + \frac{4}{9y}} \times \frac{6y}{5}\sqrt{y} = \sqrt{y + \frac{4}{9}} \times \frac{6y}{5}.$$

But $z = \frac{(4 + 9y)^{\frac{3}{2}}}{27} - \frac{8}{27}$, Simpson's Fluxions, Art. 137,

$$\text{therefore } \frac{(4 + 9y)^{\frac{3}{2}}}{27} - \frac{8}{27} = \sqrt{y + \frac{4}{9}} \times \frac{6y}{5},$$

$$\text{or } (4 + 9y)^{\frac{3}{2}} - 8 = 9\sqrt{y + \frac{4}{9}} \times \frac{6y}{5}.$$

Put $w = \sqrt{4 + 9y}$, then $y = \frac{w^2 - 4}{9}$, and the equation becomes $w^3 - 8 = w \times \frac{6}{5} (w^2 - 4)$, or $w^3 - 24w + 40 = 0$;
the root of which is $w = 3.5826$.

Then y is $= .98166$, $x = .97237$ and $z = \sqrt{(y + \frac{4}{9})} \times \frac{6y}{5} = 1.40676$; whence, by similar figures,

$$\frac{50}{z} = a = 35.54,$$

$$\frac{50}{z} \times y = 34.89 \text{ the ordinate}$$

$$\frac{50}{z} \times x = 34.56 \text{ the abscissa.}$$

J. L.

QUESTION 268.

General Solution from Clarke's Laws of Chance.

PROBLEM.

Let there be a solid die having (m) similar and equal faces, and those faces numbered progressively 1, 2, 3, &c. it is required to find the probability of throwing precisely (p) faces of one sort, (q) of another, (r) of a third, (s) of a fourth, &c. in one throw with a given number (n) of such dice.

Solution.

Put n for the number of sorts of faces, r for as many of those sorts as are denoted by unequal quantities. Now, by supposing the dice thrown singly, we have $\frac{m}{m^p} \times \frac{m-1}{m^q} \times \frac{m-2}{m^r} \times \frac{m-3}{m^s} \&c..(n)$

for the probability of throwing precisely (p) faces of one sort, (q) of another sort, (r) of a third, (s) of a fourth, &c. but in such manner, that the assigned number of faces of each sort may come up successively, without the interposition of any other sort, until that number be completed; and as this probability is not any ways influenced by the order in which they may happen to arise, it follows that the

quotient of $\frac{m}{m^p} \times \frac{m-1}{m^q} \times \frac{m-2}{m^r} \times \frac{m-3}{m^s} \&c....(n)$, divided

by 1, 2, 3, 4, &c. (n) will be the probability of throwing the assigned number of each sort of faces in some determinate order, as suppose that in which they are proposed, viz. (p) first, (q) next, &c. Then we have

$$\frac{1.2.3.\&c.(q) \text{ into } 1.2.3.\&c.(r) \text{ into } 1.2.3.\&c.(s) \&c.}{s \times (s-1) \times (s-2) \times (s-3) \times (s-4) \&c....(s-p)}$$

for the probability of their being thrown in that order, consequently

$$\frac{s \times (s-1) \times (s-2) \times (s-3) \times (s-4) \&c \dots (s-p)}{1.2.3. \&c. (q) \text{ into } 1.2.3. \&c. (r) \text{ into } 1.2.3. \&c. (s) \&c.}$$

expresses the number of variations that can be made with the quantities p, q, r, s , &c. whence

$$\frac{\frac{m}{m^p} \times \frac{m-1}{m^q} \times \frac{m-2}{m^r} \times \frac{m-3}{m^s} \&c \dots (R)}{1.2.3.4.5. \&c \dots (R)} \text{ multiplied by } \frac{s \times (s-1) \times (s-2) \times (s-3) \times (s-4) \&c \dots (s-p)}{1.2.3. \&c. (q) \text{ into } 1.2.3. \&c. (r) \text{ into } 1.2.3. \&c. (s)}$$

is the probability of succeeding under this restriction, viz. that the faces of each are constantly thrown with the same dice, but as the several parcels of dice, p, q, r, s , &c. (where the number of each is unequal) may mutually change among themselves, it follows that the last mentioned probability must be repeated as often as there are changes to be made by the unequal quantities p, q, r, s , &c. that is,

$R \times (R-1) \times (R-2) \times (R-3) \&c \dots (P)$ to give the probability required equal to

$$\frac{\frac{m}{m^p} \times \frac{m-1}{m^q} \times \frac{m-2}{m^r} \times \frac{m-3}{m^s} \times \&c \dots (R)}{1.2.3.4.5. \&c \dots (R)} \times \frac{s \times (s-1) \times (s-2) \times (s-3) \times (s-4) \&c \dots (s-p)}{1.2.3. \&c. (q) \text{ into } 1.2.3. \&c. (r) \text{ into } 1.2.3. \&c. (s) \&c.} \times R \times (R-1) \times (R-2) \times (R-3) \&c \dots (P).$$

Example 1. Required the probability of throwing precisely two faces of one sort, three faces of another sort, in one throw with seven common dice.

Here $m = 6$, $s = 7$, $p = 2$, $q = 3$, $r = 1$, $s = 1$, $R = 4$, and $P = 2$; whence the required probability is $\frac{6}{6^2} \times \frac{5}{6^3} \times \frac{4}{6} \times \frac{3}{6} \times \frac{7.6.5.4.3}{1.2.1.1.3} \times \frac{4.3}{1.2.3.4} = \frac{75600}{279936}$ or $\frac{35}{1296}$.

Example 2. What is the probability of throwing precisely four faces of some one sort, in one throw with six common dice?

Here $m = 6$, $s = 6$, $p = 4$, $q = 1$, $r = 1$, $R = 3$, $P = 1$, and the probability required = $\frac{1800}{46656}$ or $\frac{50}{1296}$.

Example 3. It is required to find the probability of throwing three, or a greater number of faces, of some one sort, in five throws with a single die.

The probability of bringing up a different face each throw is $\frac{5}{54}$,

which subtracted from unity, leaves $\frac{49}{54}$, that of bringing up two faces or more of some one sort; and again, subtracting the probability of throwing precisely two faces of one sort, viz. $\frac{25}{54}$, the remainder $\frac{24}{54}$ or $\frac{4}{9}$ is the probability required.

Example 4, which is the same as the Diary question.

To find the probability of bringing up four faces of one sort, and the other three of different sorts in one throw with seven common dice.

Here we have $m = 6$, $p = 4$, $q = 1$, $r = 1$, $s = 1$, $s = 7$, $p = 1$, and therefore $\frac{12600}{279936}$ for the probability required.

In the same manner we get the other probabilities for b , c and d equal to $\frac{15120}{279936}$, $\frac{37800}{279936}$ and $\frac{75600}{279936}$ respectively.

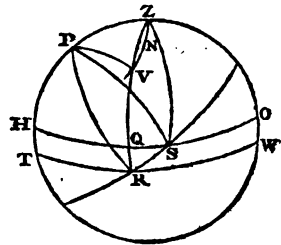
QUESTION 271.

Additional Solutions.

1. *By Mr. Wallace, R. M. College.*

PROBLEM. Having given the latitude of a place, required the sun's declination when the twilight is the shortest possible.

Let p be the pole, z the zenith, pzo the meridian, ho the horizon, tw the circle which terminates the twilight, and rs that part of the parallel described by the sun when the twilight is a minimum, which is intercepted between the horizon and twilight circle. Join pr , ps , zs by arches of great circles. About r as a pole describe a circle zrv that may pass through z ; make the angle zpv equal to rps , and join the points r and v by an arch of a great circle. The angle rps is equal to zpv , therefore, adding the common angle spv , the angle rpv is equal to spz , now rp and pv are equal to sp and pz respectively, therefore the triangles rpv , spz are in all respects equal, and hence rv is equal to sz , but zs is a quadrant, therefore rv is given in magnitude, and as by the nature of the problem, the angle rps or rvz is to be the least possible, the arch vnz , which is proportional to that angle, must also be the least possible; now the arch zrv is given by position on the sphere, and vu having been shewn to be given



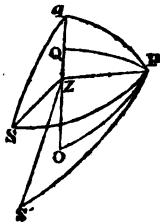
in magnitude, it is evident that the arch vnz will be the least possible when vn is perpendicular to the circle rw , and this being the case it follows that the arch rv , when produced, will pass through z , the pole of rw . Let the arch rvz meet the horizon in q , then rv and qz being each a quadrant, $rq \cong vz$, hence vz is given, and the three sides of the isosceles spherical triangle rvz being now known, every thing else may be determined.

11. *By β . Cygni, taken from the Eighth Number of the Mathematical Companion.*

PROBLEM. Given the latitude of a place, and two almicanterers or circles, parallel to the horizon; to find the declination of a star, which will pass from one to the other in the shortest time possible.

A simple case of this problem, viz. to find the day of shortest twilight in a given latitude, employed for several years the two brothers James and John Bernoulli without success. By treating it algebraically, they were led to an equation of the fourth order, in which they were embarrassed to separate the useful roots from those which ought to be rejected; but, afterwards, by employing the synthetic method, they separately obtained answers very convenient for astronomical computation. In the year 1780, Fontaine attempted a solution by algebraic analysis, which he asserts to be far preferable to the indirect and oblique method of synthesis, being ignorant, it would appear, that the antient geometers possessed, and frequently employed, an analytic method of their own, more elegant, though not so comprehensive, as the modern. In this manner he obtained an equation of the fourth order, which required twenty quarto pages to reduce and explain. This problem has been often examined by others, and many solutions have been given of it, but none, I believe, so simple as the following.

Let z be the zenith of the place, r the north pole of the equator, and s, s' the places of the star, when it transits the given almicanterers. Then, describing the great circles zp, zs, zs', rs, rs' , the arches zp, zs, zs' are given, the polar distances rs, rs' are equal, and the angle srs' is a minimum by the question. If we wanted an algebraic solution, without being at the trouble of drawing any more lines, we might suppose rs the unknown quantity, and thence find expressions for the cosines of the angles zrs, zrs' ; which is the method followed by Fontaine, and leads to an equation of the fourth order. But by drawing a few lines we shall see how the solution may be simplified.



Let the arch rq be described equal to pz , and making with it the spherical angle qrz equal to srs' ; draw also the arches zq, qs , and the

perpendicular pq on zq . Then will the angles zrq , qrp be equal, and the radius be to the sine of zr as the sine of zrq , or its equal the sine $\frac{1}{2}srs'$, to the sine of zq , or its equal qz . Consequently, when the arch zq is a minimum, the angle srs' will be a minimum. But since rs is equal to rs' , pq equal to pz , and spq equal to $s'pz$, qs is equal to zs' , and therefore given. We have, then, in the spherical triangle zsq the two sides zs , qs given, and the third zq a minimum, which is evidently the case when equal to the difference between qs , zs , because the sum of any two sides of a spherical triangle is greater than the third. Consequently, when the elapsed time is a minimum, qz is equal to the difference between the altitudes of the two almicanter above the horizon; and if qz be produced till $zo = zs$, and the arch ro be described, ro will be equal to the polar distance of the star, which was required. The calculation is manifest; for if the zenith distances of the almicanter be equal to A , A' respectively, and the latitude of the place $= \lambda$, zq will be $= \frac{A - A'}{2}$, $oq = \frac{A + A'}{2}$,

$$\text{and therefore the sine } zrq = \text{sine } \frac{1}{2}srs' = \frac{\sin \frac{1}{2}(A - A')}{\cos \lambda}, \text{ and } \cos ro \\ = \text{sine declination} = \frac{\cos oq}{\cos zq} \times \cos pz = \frac{\cos \frac{1}{2}(A + A')}{\cos \frac{1}{2}(A - A')} \times \sin \lambda.$$

In the problem of the shortest twilight it is required to find the day of the year in which the sun passes over the space comprehended between the horizon, and a lesser circle depressed eighteen degrees below it, in the shortest time. In this case A' is $= 90^\circ$, $A = 108^\circ$;

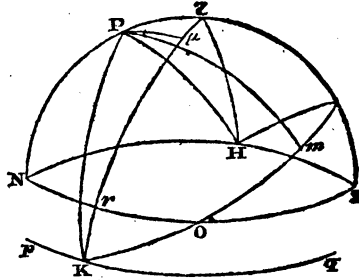
$$\sin \frac{1}{2}srs' = \frac{\sin 9^\circ}{\cos \lambda}, \text{ and sine } \odot \text{'s declination} = \frac{\cos 99^\circ}{\cos 9^\circ} \times \sin \lambda = \\ - \tan 9^\circ \sin \lambda, \text{ which shews that the declination is south in northern latitudes, and north in southern latitudes. It is also evident, from the formula, that the duration of twilight in latitude } 81^\circ \text{ north, is 12 hours sidereal time, and is least of all under the equator.}$$

III. By Astronomicus; taken from the First Vol. of the New Series of the Mathematical Repository. Art. XI. Part II.

PROBLEM I. To find the day when the twilight is shortest in a given latitude?

Let p be the elevated pole, z the zenith, and $npzs$ the meridian: let pq be part of the small circle where twilight ends (which is 18° below the horizon), and let u be the point of the horizon where the sun rises in the morning, and κ the point where he passes the circle pq , bounding twilight, in the evening: then it is obvious that the differences of the horary angles spu and $sp\kappa$ will be proportional to the duration of twilight.

Through the points H and K draw the great circle HOK : bisect the arc HK in m , and draw the great circle pm ; draw also the azimuth circles ZH , ZK .



Supposing the sun's declination to remain unchanged from the time of rising in the morning to the time of passing the small circle pq in the evening, the polar distances PH and PK will be equal to one another. In the triangle ZPH , we have (No. 2, Repository, Part 3, page 11)

$$\cos ZH = \cos PZ \cos PH + \sin PZ \sin PH \cos ZPH;$$

and in like manner, in the triangle KPZ , we have

$$\cos ZK = \cos PK \cos ZP + \sin PK \sin ZP \cos KPZ.$$

Take the difference of the equations just found, and

$\sin PZ \sin PK (\cos HPZ - \cos KPZ) = \cos ZH - \cos ZK$,
now $\cos ZH = 0$, because $ZH = 90^\circ$, and $-\cos ZK = -\cos 108^\circ = \sin$

18° : also $\cos HPZ - \cos KPZ = 2 \times \sin \frac{KPZ + HPZ}{2} \times \sin \frac{KPZ - HPZ}{2}$;

therefore we have $\sin PZ \sin PK \sin \frac{KPZ + HPZ}{2} \sin \frac{KPZ - HPZ}{2} = \frac{1}{2} \sin 18^\circ$.

Because $KP = PH$ and $Km = mH$, it is obvious that the great circle pm will cut the great circle HK at right angles, and further that the angle $KPM = \text{angle } MPH$. Therefore angle $KPM = \frac{KPZ + KPH}{2}$.

And, on account of the right-angled triangle KPM , $\sin KP \sin KPM = \sin Km = \sin \frac{KH}{2} = \sin KP \sin \frac{KPZ + HPZ}{2}$. Therefore we shall have

$$\sin PZ \sin \frac{HK}{2} \sin \frac{KPZ - HPZ}{2} = \frac{1}{2} \sin 18^\circ.$$

Now $HK = HO + OK$; and it is manifest that HO is a semi-circle;

therefore $HK = 180^\circ + OK$; and $\sin \frac{HK}{2} = \sin (90^\circ + \frac{OK}{2}) =$

$\cos \frac{OK}{2}$: therefore we have

$$\sin PZ \cos \frac{OK}{2} \sin \frac{KPZ - HPZ}{2} = \frac{1}{2} \sin 18^\circ.$$

The difference of the horary angles KPZ and HPZ (and consequently, the duration of twilight) will be a minimum when $\sin \frac{KPZ - HPZ}{2}$

is a minimum: but from the last equation $\sin \frac{KPZ - HPZ}{2}$ will be a

minimum when $\cos \frac{KO}{2}$ is a maximum; for all the other quantities in that equation are constant. Therefore, it is plain, that the duration of twilight is a minimum, when the arch KO is a minimum, that

is, when the great circle hok is perpendicular to the horizon; or when it passes through z , the pole of the horizon.

Thus the duration of twilight is a minimum in any latitude, on that day when the sun crosses the small circle bounding twilight, after sun-setting, on the same azimuth circle on which he rose in the horizon in the morning; or, which is the same thing, when the sun crosses the small circle bounding twilight, before sun-rising, on the same azimuth circle on which he will set in the horizon in the evening.

It is now easy to find the sun's declination (and, consequently, the day of the year) when the twilight is a minimum: for suppose kzu to be a great circle passing through the zenith and the points h and k , as has been shewn to be the case when the twilight is the least of all; and draw pm perpendicular to that great circle: then because $pk =$

ph , therefore $km = \mu h$. Therefore $\mu k = \frac{kzh}{z} = 99^\circ$ and $z\mu = 9^\circ$:

and in the triangle kzp , we have

$$\cos z\mu : \cos \mu k :: \cos pz : \cos pk :$$

$$\text{Therefore } \cos pk = \frac{\cos \mu k \cos pz}{\cos \mu z},$$

or, $\sin \odot$'s declin. = $-\tan 9^\circ \times \sin$ latitude, where the sign (—) shews that the declination is south when the latitude is north, and *vice versa*.

To find the length of the shortest twilight; in the right-angled triangle zpm , we have

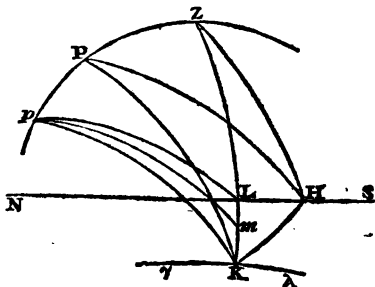
$$\sin zpm = \frac{\sin z\mu}{\sin pz} = \frac{\sin 9^\circ}{\cos \text{latitude}},$$

and the double of the angle zpm , being converted into time, will give the duration of the shortest twilight required.

The same result may be obtained by putting 18° (which is the least value of ko) for ko in the equation obtained above.

PROBLEM II. To find the latitude of that place where the twilight is a minimum on a given day.

Let z be the zenith, ns the horizon, and h the point in the horizon where the sun sets; also let k be the point where the sun, after setting, crosses the small circle γr bounding twilight: let p be the pole, and draw the circles of declination ph and pk and the azimuth circles zh and zk : lastly, let a great circle pass through h and k .



Then because the day of the year is given the sun's declination will be given: therefore the polar distance $ph = pk$, will be given: therefore in the triangle hpk , of which the two sides hp and pk are constant, the less the base hk is, the less will be the horary angle hpk , which measures,

the duration of twilight; and when the base HK is least of all, the angle HPK will likewise be least of all. But in the triangle KZH , the side HK can never be less than the difference of the two sides ZK and ZH , that is, HK can never be less than KL , or 18° . Therefore HK will be least of all when it is equal to KL , or 18° . Therefore the duration of twilight will be least of all at that place where the pole is so situated (as at p) that the point u may coincide with the point L , or so that PK and PL may be each equal to the distance of the sun from the elevated pole on the given day.

Thus the duration of twilight is a minimum on a given day at that place where the sun crosses the small circle 18° below the horizon; after sun setting, on the same azimuth circle on which he set in the horizon; or, which is the same thing, where the sun crosses the small circle 18° below the horizon, before sun rising, on the same azimuth circle on which he will rise in the horizon.

To find the latitude draw pm perpendicular to zk : then because $pk = PL$, therefore $km = Lm$: therefore $km = 9^\circ$ and $mz = 99^\circ$. In the triangle pzk ,

$$\cos mk : \cos mz :: \cos kp : \cos pz,$$

therefore $\cos pz = \frac{\cos 99^\circ}{\cos 9^\circ} \times \cos pk = -\tan 9^\circ \times \cos pk$, or $\sin \text{latitude} = -\tan 9^\circ \times \sin \odot$'s declination, where the sign (—) shews that the latitude is north when the sun's declination is south, and *vice versa*.

To find the length of twilight on a given day at that place where it is the least of all; we have

$$\sin xpm = \frac{\sin km}{\sin pk} = \frac{\sin 9^\circ}{\cos \odot \text{'s declination}},$$

and the double of the angle xpm , converted into time, will give the duration of the shortest twilight on the given day.

The very least twilight over all the earth, and for all the year, is at the equator, when the sun is in the equinoctial points.

Take $\sin q = \tan 9^\circ \times \sin 23^\circ 28'$; and describe two small circles on the earth, on opposite sides of the equator, and at the distance q from it: then at every place comprehended in the zone between the two circles, the twilight will, twice in a year, be less than at any other place on the surface of the earth.

IV. *By the Chevalier Delambre; being the fourteenth Chapter of the first vol. of his highly valuable treatise on Astronomy.**

Previous to the solution of the problem, Delambre makes the following observations on twilight.

The atmosphere produces refractions, which are the plague of astronomers; but it produces also twilights which lengthen sensibly the duration of day, and prevents a total obscurity from succeeding to the brightest light at the moment of the sun's disappearance.

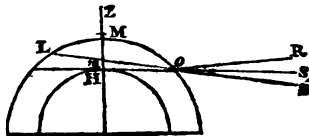
1. *Twilight* is that light which gradually decreases until the

* For the translation of this chapter we are indebted to Dr. Evans of Christ's Hospital.

sun has descended below the horizon by a quantity which we shall presently give the means of estimating.

The same name is given to that light, which in the morning continues to increase, as the sun rises towards the horizon, and prepares us to see his first appearance without being too much dazzled. The morning light is commonly called the *aurora*, but in Astronomy we only use the word *twilight*.

2. When the sun is at *s*, 33' below the horizon, it has been shown that the ray so bending at *o*, arrives at *u*, along the tangent *uo*, and that we see the sun at *z*; but when the sun is at *s'*, the ray *s'o* still experiencing a refraction of more than 33' passes on to *ol*, above our heads; we now no longer see the sun; but the section *oml* is enlightened: and all the particles of this part of the atmosphere reflect the light to us. The more the sun lowers, the more the spheric segment *oml* diminishes: and lastly a moment arrives, when we can no longer perceive any sensible light; the twilight then ends; and we say it is dark night.



3. The enlightened part of the atmosphere, at the instant the sun appears or disappears, has for its base the circle of the horizon: when the sun is lower, the curve, which is the limit of reflected light, becomes more and more oblique.

When the limit *l* passes by the zenith *m*, an observer who turns his back to the sun, no longer sees the light: this instant is that which Lambert calls the end of civil twilight, because in a house whose windows face the east, they no longer see sufficiently well to continue their ordinary occupations. In like manner in a house turned towards the west, they only begin to see a little at the moment when the crepuscular curve reaches the zenith. Lambert has found that the sun is at that moment 6° 24' below the horizon.

4. By following the motion of this curve attentively, during a fine night, we may fix, very nearly, the time when it disappears: and if we compute for this time, the sun's zenith distance, we can obtain his depression for the moment when twilight begins or ends.

5. Let *TA* be this depression, and calling this arc *2a*, we shall have in the triangle *ZPA*,

$$\cos ZPA = \frac{\cos(90^\circ + 2a)}{\cos H \cos D} - \tan H \tan D^*$$

but when the sun is in the horizon, we have

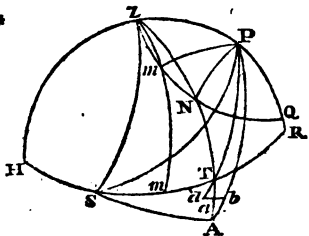
$$\cos ZPS = -\tan H \tan D,$$

therefore,

$$\cos ZPS - \cos ZPA = \frac{\sin 2a}{\cos H \cos D},$$

that is

$$2 \sin \frac{1}{2}(ZPA - ZPS) \sin \frac{1}{2}(ZPA + ZPS) = \frac{\sin 2a}{\cos H \cos D},$$



* *H* is the height of the pole, and *D* the declination.

$$\text{or, } \sin \frac{1}{2}(P' - P) = \frac{\sin 2a}{2 \sin \frac{1}{2}(P' + P) \cos H \cos D}.$$

By P we designate always the angles at the pole, or the horary angles.

The difference $P' - P$ of the horary angles, at rising, and the beginning of twilight, divided by 15, will give the duration of twilight.

6. The duration being known by observation, we may compute the depression $TA = 2a$; and the depression being found, we may then calculate the duration for any day and place: for it is evident the duration depends on the latitude and declination.

7. We know of but very few observations of this duration. Two are given by La Caille, who, in the torrid zone, found $\Delta T = 16^\circ$ and 17° . Lemonnier found it from 17° to 21° . The ancients supposed it to be 18° .

If we suppose $H = 0$ and $D = 0$, we have

$$\cos H \cos D = 1; \quad \tan H \tan D = 0 = \cos P.$$

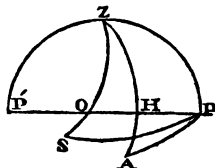
whence $P = 90^\circ = 6^h$; $\cos P = \cos P' = \cos 90^\circ = \cos P' = 0 = \cos P' = \sin 2a$; whence again $P' = 90^\circ + 2a$ and the duration of twilight $= \frac{2a}{15}$. This is the shortest of all the twilights; for the common

expression is $2 \sin \frac{1}{2}(P' - P) = \sin 2a \sec L \sec D \operatorname{cosec} \frac{1}{2}(P' + P)$; and the product of the three secants has unity for its *minimum*.

8. The twilights are therefore longer in high Latitudes.

Let us suppose the latitude and declination to be nothing, the two poles are then in the horizon, and the sun will describe the equator sz , passing by the zenith; the arc os of the sun's motion is then the measure of the crepuscular angle $ors = 18^\circ$: and

twilight lasts $\frac{18}{15} = 1^h 12^m$.



At another season, let PA be the distance of the sun from the pole; $ZA = 90^\circ + 2a$:

$HA = 2a$; HFA will be the duration of twilight; now the tri-

angle HFA , right-angled at H , gives $\sin HFA = \frac{\sin HA}{\sin PA} = \frac{\sin 2a}{\cos D} >$

$\sin 2a$, so that even at the equator, the duration increases with the declination.

Solution of the Problem.

9. The problem of the shortest twilight is one of mere curiosity; but it occupied the two Bernoullis five years: John resolved it at last by his method *de maximis et minimis*. He acknowledges that the calculation is prolix and intricate. See his works, vol. 1, page 64.

Another solution still more intricate, but synthotic, is given by

whence $PS = 180^\circ - PT$, but $PS = 90^\circ + D$, therefore $PT = 90^\circ - D$;

$$\text{but } \tan RPS = \frac{\tan R_i}{\sin PR} = - \frac{\tan RT}{\sin PR} = - \tan RPT,$$

$$\text{therefore } RPS + RPT = 180^\circ.$$

$$13. \quad \cos PZA = - \cos PZs,$$

$$\text{therefore } \tan H \tan 2a + \frac{\sin D}{\cos H \cos 2a} = - \frac{\sin D}{\cos H},$$

$$\sin H \sin 2a = - \sin D - \sin D \cos 2a = - \sin D (1 + \cos 2a),$$

$$2 \sin H \sin a \cos a = - 2 \cos^2 a \sin D;$$

$$\text{and } \sin D = - \sin H \tan a;$$

this is the formula that John Bernoulli has given without any demonstration.

$$\cos PZs = \frac{\sin D}{\cos H} = - \tan H \tan a \text{ and } \cos PZA = + \tan H \tan a$$

$$= \sin mT = - \sin ms \text{ being a new formula.}$$

14. This being done, the equation (10) $\cos AS = \cos AT \cos ST$ becomes $\cos AS = \cos 2a \cos 2mT$

$$1 - 2 \sin^2 \frac{1}{2} AS = (1 - 2 \sin^2 a) (1 - 2 \sin^2 mT)$$

$$= 1 - 2 \sin^2 a - 2 \sin^2 mT + 4 \sin^2 a \sin^2 mT$$

$$\sin^2 \frac{1}{2} AS = \sin^2 a + \tan^2 a \tan^2 H - 2 \sin^2 a \tan^2 a \tan^2 H$$

$$= \frac{\sin^2 a}{\cos^2 H} (\cos^2 H + \frac{\sin^2 H}{\cos^2 a} - 2 \tan^2 a \sin^2 H)$$

$$= \frac{\sin^2 a}{\cos^2 H} (\cos^2 H + \sin^2 H - \sin^2 H \tan^2 a).$$

$$= \frac{\sin^2 a}{\cos^2 H} (1 - \sin^2 D) = \frac{\sin^2 a \cos^2 D}{\cos^2 H},$$

$$\sin \frac{1}{2} AS = \frac{\sin a \cos D}{\cos H},$$

$$\text{but } \frac{\sin \frac{1}{2} AS}{\cos D} = \frac{\sin a}{\cos H} = \sin \frac{1}{2} \text{ the angle of duration} = \sin \frac{1}{2} (P' - P)$$

an equation for which, we are indebted to M. Cagnoli, and which those who had resolved the problem in a manner purely analytic, had not seen.

$$15. \quad \text{But } \sin \frac{1}{2} (P' + P) = \frac{\sin 2a}{2 \sin \frac{1}{2} (P' - P) \cos H \cos D} =$$

$$\frac{2 \sin a \cos a}{2 \sin a \cos H \cos D} = \frac{\cos a}{\cos D}, \text{ and the triangle } rsz \text{ gives}$$

$$\cos PZ = \cos s \sin Zs \sin Pz + \cos Zs \cos Pz,$$

$$\sin H = \cos s \cos D \text{ and}$$

$$\cos s = \cos A = \frac{\sin H}{\cos D}.$$

an expression which is given directly by the right-angled triangle PRS .

From the formulæ 13 and 15, we obtain

$$\cos pzs \cos psz = \frac{\sin D}{\cos H} \times \frac{\sin H}{\cos D} = \tan D \tan H = \cos zpz.$$

Thus in the rectilateral triangle pzs the cosine of the angle opposite to the side of 90° is equal to the product of the cosines of the two other angles.

16. From $pt = 180^\circ - ps = 180^\circ - pa = 180^\circ - 90^\circ - D = 90^\circ - D$, it is easy to see that the equator dab cuts TA into two

equal parts, and that $\frac{\sin D}{\sin a} = \sin$ angle formed by the vertical with

the equator $= \frac{\sin Ab}{\sin Aa}$.

Thus by simple analysis applied to two spheric triangles, and without having recourse to the differential calculus, we have obtained a much more complete solution than any of the authors who have treated on this subject. We now proceed by considerations totally different, to find the same formulæ, supposing the heavens to be immoveable, and the earth to turn round on its axis.

17. This new research will not only give us the same formulæ, but they will be demonstrated in a more exact manner. They serve to show all that can be obtained by a little attention from a very simple construction: they prove to us that the same phenomena may be explained by hypotheses that are altogether contrary: We have hitherto kept to first appearances, supposing the heavens to be a spheric vault which turns round the earth. Indeed all that we have hitherto observed is explained perfectly by this hypothesis; but the phenomena will be equally well explained by supposing the heavens to be immoveable, and by giving the earth a motion in a contrary direction of 360° in 24 hours of sidereal time round a line which traverses it, and proceeds till it cuts the celestial vault in two opposite points, that is to say, in those two points which we have found to be the poles of the diurnal motion.

We now proceed to show how, by this new hypothesis, we may resolve the problem of the shortest twilight, by considerations that are purely trigonometric.

Another Solution altogether synthetic of the Problem of the shortest twilight.

18. Let P be the pole of the world, z the zenith of the observer, A the sun; $zA = 108^\circ = 90^\circ + 18^\circ$, or more generally, $zA = 90^\circ + 2a$. The observer, whose zenith is z , will begin to see the twilight. By the diurnal motion, the circle of declination PA turning round the

point m rises or lowers ; the intercepted part zb will vary, and it may vary from 0, until $zq = 2rz$. Thus, the intercepted part may have all the values comprized between 0 and $2(90^\circ - h) = 180^\circ - 2h$: and consequently have the value $2a$. Thus, in the case where $zb = 2a$, and $\delta A = 90^\circ$, the shortest twilight will take place ; and its semi-duration is found by making

$$\sin ZPB = \frac{\sin a}{\cos H} \text{ and the whole duration} = \frac{2}{15} ZPB.$$

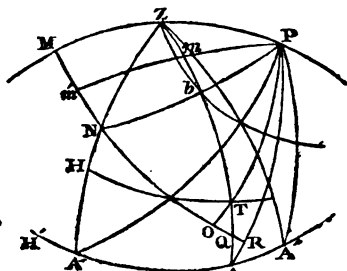
The problem is therefore now resolved in the most simple and direct manner : and we may still keep to this formula, for which we are indebted to M. Cagnoli, who found it by the differential calculus, aided by a very ingenious construction. The mathematicians who had treated on this question before him, had not discovered this formula, and contented themselves with giving the sun's declination on the day of the longest twilight.

We proceed to find this formula, and several other things, by a graphic construction, to which we shall then apply calculation.

20. On the arc $zb = 2a$, and with the complement of the latitude zP , from the isosceles triangle zPb ; let fall the perpendicular arc pm ; then zPb will be the angle that measures the duration.

Prolong zmb until $\delta A = 90^\circ$, and draw the arc PA , which will be the sun's polar-distance for the day of the sun's shortest twilight.

The triangle zPA gives us
 $\cos zm : \cos ma :: \cos pz : \cos PA,$
 or
 $\cos a : \cos(90 + a) :: \sin H : \sin D$
 equal to the sun's declination.



Whence $\cos a : -\sin a :: \sin H : \sin D = -\frac{\sin a}{\cos a} \sin H = -\tan a \sin H.$

This is the formula given by the mathematicians who have left astronomers the trouble of resolving the triangles APb , APZ to determine $zPb = zPA - \delta PA$.

21. The right-angled triangle rzp gives

$\tan z = \cos z \tan rz$ and $\cos z = \tan zm \cot rz = \tan a \tan H$; a formula which has not hitherto been noticed.

z is the sun's azimuth at the beginning of twilight, but $rzb = rPz = 180^\circ - P\delta A$; now $P\delta A$ is the sun's azimuth at the moment when its centre is in the true horizon.

Therefore the azimuths, at the beginning and at the end of twilight, are two angles which are supplements of one another ; and whose sum is 180° ; which is a remarkable property, and has not hitherto been observed.

Therefore, $\cos pba = -\tan a \tan H$, since $\cos pza = \tan a \tan H$.

22. The triangle bpa gives

$$\cos A = \frac{\cos pb - \cos ba \cos pa}{\sin ba \sin pa} = \frac{\cos pz}{\sin pa} = \frac{\sin H}{\cos D}.$$

This is the angle formed at the sun, by the vertical and declination circle; and this angle is the same both for the beginning and end of twilight, since it is common to the triangles bpa and zpa . This equality is the foundation of the analytic solution. It is here only an indifferent accessory. This and the preceding property take place even at the equator.

23. zpa is the horary angle at the beginning of twilight: we call it p' : bpa is the horary angle at the end of twilight; we call it p .

Thus $zpa - bpa = p' - p = zpb$: this angle measures the duration of twilight. We have therefore

$$\sin \frac{1}{2}(p' - p) = \frac{\sin a}{\cos H},$$

$$mpa = bpa + mpb = p + \frac{1}{2}(p' - p) = \frac{1}{2}(p' + p),$$

$$\text{now, } \sin am = \frac{\sin am}{\sin pa} = \frac{\sin(90^\circ + a)}{\cos D} = \frac{\cos a}{\cos D} = \sin \frac{1}{2}(p' + p)$$

with $\frac{1}{2}(p' + p)$ and $\frac{1}{2}(p' - p)$ we get p and p' , that is to say, the horary angles at the beginning and end. We have seen that

$$\sin D = -\tan a \sin H.$$

The solution therefore leaves us nothing further to desire; we even have,

$$\cos z = \tan a \tan H \text{ and } \cos A = \frac{\sin H}{\cos D}.$$

24. Let $at = 2a = zb$, and draw pt ; we have $zt = 90^\circ$. t is a point in the horizon for the moment when the zenith is at z . The triangle pzt gives

$$\begin{aligned} \cos pt &= \cos z \sin pz \sin zt + \cos pz \cos zt = \cos z \sin pz \\ &= \tan a \tan H \cos H = \tan a \sin H = \sin D; \end{aligned}$$

therefore,

$90^\circ - pt = D$, or $pt = 90^\circ - D$; but $pa = 90^\circ + D$: consequently

$$pt + pa = 180^\circ;$$

another remarkable property of the shortest twilight not before noticed.

25. The triangle apt gives, by means of the perpendicular pm ,

$$\sin am : \sin tm :: \tan t : \tan A,$$

$$\text{or, } \sin(90^\circ + a) : \sin(90^\circ - a) :: \tan t : \tan A;$$

therefore $t = A$, and we shall also have

$$\sin pt : \sin pa :: \sin A : \sin t = \frac{\sin A \sin pa}{\sin pt} = \frac{\sin A \sin(90^\circ + D)}{\sin(90^\circ - D)} = \sin A,$$

therefore $t = A$.

So that the angles formed by the vertical TA with the circle of declination drawn to the sun A , and to the horizon T , are likewise equal to each other.

26. Prolong rt to o , so that $po = 90^\circ$, then we have $ot = d$.

Draw $rqom$ perpendicular to po , then rqm will be the equator: and we get $ar = d$.

The triangles toq , arq have their three angles equal to each other, and the homologous side to , $ar =$, are perfectly equal; therefore, $rq = aq$ and $qr = oq$, therefore $rq = a = aq$, since $rq + aq = 2a$.

So that on the day of the shortest twilight the equator, divides into two equal parts the arc of depression $at = 2a = 108^\circ$.

27. The triangle tqo gives

$$\sin q = \frac{\sin to}{\sin tq} = \frac{\sin d}{\sin a} = \frac{\tan a \sin h}{\sin a} = \frac{\sin h}{\cos a} = \cos pm,$$

this is the angle which the vertical za makes with the equator.

Prolong pz and pb to the equator at m and n , then $pm = pn = 90^\circ$, and $zm = bn = h$.

$$\cos qn = \frac{\cos bq}{\cos bn} = \frac{\cos(90^\circ - a)}{\cos h} = \sin \frac{1}{2}(p' - p) = \sin(90^\circ - qn),$$

therefore,

$$90^\circ - qn = \frac{1}{2}(p' - p) \text{ and } qn = 90^\circ - \frac{1}{2}(p' - p):$$

Prolong pm to m' then $m'n = \frac{1}{2}(p' - p)$.

28. Therefore $qm' = 90^\circ$, which was evident in other respects, since the angles m and m' are right, q is the pole of mm' and the angle $q = mm' = 90^\circ - pm$.

$$\cos oq = \frac{\cos tq}{\cos to} = \frac{\cos a}{\cos d} = \sin \frac{1}{2}(p' + p) = \cos qr.$$

$$m'r = mq + qr = 90^\circ + qr = \frac{1}{2}(p' + p).$$

Draw th perpendicular to zt ; th will be the horizon at the beginning of twilight.

$\sin ht \tan thq = \tan tq$, or $\sin ht = \tan tq \cos thq = \tan a' \tan h = \cos pza = \sin$ of the amplitude of the point t in the horizon.

29. Draw ah' perpendicular to za , ah' will be the horizon at sunrise.

The horizon descends parallel along ta , at the same time that the zenith descends from z to b , as in the equator (*See fig. on page 321*) the day of the equinox, it descends from o to s ; thus the general property of the shortest twilight is when the zenith descends towards the sun remaining always in the same plane passing through the sun.

At the equator on the day of the solstice it actually does not quit this plane; in the oblique sphere the zenith moves in a parallel circle, which differs a little from this plane, but the chord of this arch of a small circle is entirely in the plane which passes through the sun; and the zenith itself returns again to this plane at the end of twilight the same as at the beginning.

30. The chord of this arc of a small circle is always $= 2 \sin a$, but the arc of a small circle has a greater curvature, because the radius is less; this chord subtends an arc of a greater number of degrees, and this is the reason why the twilight lasts a longer time.

All these details would be more curious than useful, did they not serve to shew, how many consequences may be derived with facility from a simple and well conceived construction, and which could only be obtained with difficulty from an analytic formula, but in which they are nevertheless all contained.

One more reflexion: $HT = 90^\circ - z$, whence $z = 90^\circ - HT$; but $zBA = 180^\circ - z = 180^\circ - 90^\circ + HT = 90^\circ + HT$: therefore on the day of the shortest twilight, the amplitudes of the vertical are equal, and of a contrary sign at the beginning and end of twilight.

RECAPITULATION.

31. *Formulae relative to the Shortest Twilight.*

1. $\sin D = -\tan a \sin H$: so that the declination is always of a contrary sign to the latitude of the place.

2. $\cos z = +\tan a \tan H$. z will be acute when H is positive.

$$3. \cos A = \frac{\sin H}{\cos D}.$$

$$4. \sin \frac{1}{2}(P' - P) = \frac{\sin a}{\cos H}.$$

$$5. \sin \frac{1}{2}(P' + P) = \frac{\cos a}{\cos D}.$$

$$6. z' = 180^\circ - z.$$

$$7. PT = 90^\circ - D.$$

$$8. PA = 90^\circ + D.$$

$$9. ms = mT.$$

10. The equator passes through the middle of the arc of depression,

11. Sine of the angle formed by the vertical and the equator

$$\frac{\sin D}{\sin A} = -\frac{\tan a \sin H}{\sin a} = -\frac{\sin H}{\cos a}.$$

This angle will be below the equator; but the angle above z has its sine $= +\frac{\sin H}{\cos a}$.

Suppose $H = 0$; then we have $D = 0$, and $mT = ms = 0$; also, $90^\circ = z = z' = A = s = PT = PA$; so that $180^\circ = z + z' = PT + PA$; $A = s$; $mT = ms$; are all properties appertaining to the shortest twilight.

32. According to the example of other authors we have neglected the refraction, and taken, for the duration of twilight, the time elapsed between the zenith distance $90^\circ + 2a$ and the distance 90° .

We have neglected the change of declination in the interval of 1h. 50m. for Paris; this change is more than a minute, but it is a quantity of little importance in a question of mere curiosity, and above all where there is an uncertainty of one or more degrees in the twilight depression.

33. At Paris, for example, when $H = 48^\circ 50'$, we have

$$\begin{aligned} \text{co. ar. cos. } H &= 0.1816080 \\ \text{log. sin. } 9^\circ &= 9.1943324 \\ \frac{1}{2} (P' - P) &= 13^\circ 44' 53'' = 9.3759404 \\ P' - P &= 27 \ 29 \ 46 = 1\text{h. } 50\text{m.} \\ \text{Log. sin. } H &= 9.8766785 \\ \text{Log. tan. } 9^\circ &= 9.1997125 \\ \text{Log. sin. } D &= 9.0763910 \\ D &= -6^\circ 10' 50'' \end{aligned}$$

Thus, the shortest twilight at Paris is 1h. 50m. and it takes place on the 11th of October and 5th of March of each year.

34. Since the twilight only ends when the sun has 18° of depression, it would not end if the sun did not descend 18° below the horizon. The greatest depression below the horizon is $90^\circ - H + 90^\circ - D - 90^\circ = 90^\circ - H - D$, therefore dark night will not take place when $90^\circ - H - D < 18^\circ$, or when $H + D > 72^\circ$; at Paris it suffices for the sun's declination to be above $23^\circ 10'$ to have $H + D > 72^\circ$: therefore there is no dark night at Paris about the time of the summer solstice.

If the twilight depression were 17° it would be requisite that $H + D > 73^\circ$, and then every night would be dark at Paris.



Since the preceding article was set up by the printer, we have met with another by Delambre on the same subject, in the *Connaissance des Temps* for 1818.

It is extracted from his *History of Astronomy*, in which all the known books, at least such as contain any thing useful or curious, are analyzed, expounded, in some cases improved, and always at least translated into modern language. The printing of this work was begun in 1815. As the historical information contained in this article is curious, and the solution of the problem of the twilights elegant, we shall extract from it as much as relates to the general problem, leaving out some other details and tables shewing the duration of the twilight calculated for the latitude of Paris.

Concerning Nonius and his formulae for the Twilights.

New and general Formulae for calculating the two declinations which correspond to a given duration of twilight. New demonstration of the shortest twilight.

By M. Delambre.

The problem of the twilights has been the object of the researches of many astronomers, and several geometers, rather on account of some difficulties which it presents, than its real importance. Nothing is easier than to calculate its duration for a given day; because, then, the declination being known, the problem is reducible to the solution of two spherical triangles. It is not so easy to determine the declination when the duration is known. But this is not what has given celebrity to the question, for no one, as far as I know, has considered it under this point of view, which makes one of the objects of these new researches. It is the determination of the shortest twilight, and the declination which produces it, that has excited the curiosity of astronomers, and particularly of geometers. The solution which I shall give resolves with equal facility the three parts of the problem, without requiring for the *minimum* any metaphysical reasoning, or any calculus either trigonometrical or differential.

Since the twilights depend on the same cause as refraction, we cannot expect to find them mentioned in the writings of the Greeks; Ptolemy himself in his Optics has been entirely silent on this subject; indeed in his Mathematical Syntaxis, he determines the depression of the sun which allows the stars and planets to be seen in the horizon, and this is at bottom the essential *datum* of the problem. But Ptolemy thought only of the possibility of seeing the stars; he did not trouble himself either with the twilight or its duration; it never came into his mind to calculate the time that should elapse between sun set and the instant when the stars should be visible.

It does not appear that Aristotle, or any ancient philosopher, ever sought the cause of the twilight. It is not even quite certain that the Greeks had in their language a word which expressed exactly what we now understand by the astronomical twilight. The word ἠώς, *aurora* signifies little more than the instant which immediately precedes sunrise; πρῶτι, means only the *morning*; κνέφας, *obscurity and darkness*; δειλῆ, the *afternoon*, or *evening*; λυκαυγής, the end of morning twilight; ἀμφιλυκὴ νύξ, in Homer, means only a night mixed with uncertain light. All these notions are extremely vague, and refer less to the astronomical twilight, than to that other twilight which allows to labour and to read, which was first mentioned by Lambert, and named by him *civil twilight*.

The Arab, Alhazen, is the first who has treated expressly of the twilights, his work is very superficial; he treats almost solely of the method of determining the height of the atmospheric stratum, that reflects the first and last rays of the sun. Vitellon, says yet less, and limits himself to the same problem. Maginus, in his *Primum Mobile*; Gemma Frisius, in his Supplement to the Astrolabe; Barocius, in his *Cosmographia*; Blancanus, in his book of the Sphere, merely resolve the two spherical triangles, or repeat what they had found in the works of their predecessors. Tycho in his *Progymnasmata*, speaks of little more than the manner of finding the duration by observation, and thence deducing the sun's depression. Cardan, Stevin, Roggembach, Longomontanus, Gassendi, even Kepler and Riccioli, teach us nothing more. Glorioso in his Treatise of Comets, only repeats the same method of Stevins, for calculating the heights of the atmosphere.

We have not mentioned in their place, either Nonius or Clavius, who deserve a more detailed notice. Nonius was the first that truly handled the subject; he was the first that proposed the problem of the shortest twilight, which he resolved in such a way as to leave nothing to be done by his successors, except to demonstrate otherwise his formulæ, or rather rules of calculation.

Bailly speaking of this solution merely announces it as worthy of notice, he gives no notion of it. Perhaps he had not taken the trouble to study it sufficiently, and he may have been frightened by the length of the demonstrations and the want of order in the ideas. One is tempted to believe that Nonius did not wish to render it clearer. Although he has demonstrated every thing, even to the most common theorems, yet he has given, merely by the way, and without demonstration, the fundamental principle of his method. On account of this confusion, this prolixity, mixed with obscurity, it has happened that some curious propositions have neither been observed nor understood. Nonius himself has not perceived their importance; he has not seen that the rules he had found for a particular case might be generalized, and might furnish simple and direct rules for determining the two declinations which answer to any duration whatever of twilight. He did not see that the duration of the shortest twilight and the declination which belongs to it might be deduced from his construction, at sight, and without any calculation, and therefore, he has only found them by intricate reasonings which produce obscurity.

Clavius alone seems to have studied Nonius' book. He found that it contained knowledge, elegance, address, and notions which would have appeared paradoxical, if they had not been supported by strict demonstrations. He was so well pleased with the work, that he gave a considerably long extract from it, but changed some demonstrations and added illustrations which seemed necessary. His extract consists of twenty-four propositions, by far too many for so trivial a subject.

All these propositions are not alike useful; the most part follow naturally as corollaries, from the formulæ we are to establish. The

real solution begins at the 19th. He there seeks the declination which produces a twilight equal to that of the equinoctial day. He remarks, that in the frigid and torrid zones, one of Nonius' propositions is subject to some restrictions. In the twentieth he changes Nonius' demonstration ; instead of the fundamental principle indicated or supposed by Nonius ; he substitutes the theory of circles, tangents to the same parallels, explained at length by Theodosius in the 2nd book of his Spherics. It is possible that Clavius, not entering completely into the ideas of Nonius, has wished, for greater certainty to arrive at the same results by a different method. In other respects he finds no new theorem, and if the doctrine of Nonius has gained any thing in his hands it is not on the score of brevity, although he often refers to Theodosius.

He concludes his treatise by a table furnished by one of his scholars, shewing the length of twilight for every degree of the ecliptic, and for all heights of the pole from 35° to 61° . Keill, in his *Astronomical Lectures*, has improved the doctrine of Clavius, by adding a shorter and more direct method of finding the declination which produces the shortest twilight. The solution of Nonius was still more complete, but Keill's rule abridges it a little, and saves the analogy for finding eastern amplitude, which he shows how to avoid.

Authors differ sensibly on the depression of the sun at the beginning of the twilight.

Ptolemy, and after him Clavius, Maginus, Gemma Frisius, Blancanus, Stevin, Gassendi, and Kepler suppose it	18°
Alhazen, Vitellon, Barocius, Cardan, Roggembach ..	19
Tycho and Cabaeus from 16 to	17
Rothmann	24
Longomontanus	20
Bodinus	
Nonius	16 2'
Ricciolus, on the equinoctial day, in the morning ...	16
in the evening	23 3
at the summer solstice in the morning	21 25
at the winter solstice in the morning	17 0
Lemonnier, ...	17 to 21
La Caille, in the torrid zone, ...	16 to 17
In this uncertainty, it is usually reckoned	18°.

The problem of the shortest twilight drew the attention of the two Bernoullis, and they were occupied with it a long time, without being able to find a solution. Perhaps they sought one purely analytical. At last, after five years, John resolved it by his method *de Maximis et Minimis*. He confessed that his demonstration was prolix and difficult ; he suppressed it, (see his work, page 64) and he only gave the formulæ of the declination. It is true, the remainder is only a common trigonometrical calculation, but it is much too long, because it is useless, when we have the far more elegant formula of Nonius. Besides, the duration is the proper object in question ;

it may be supposed that Bernoulli's analysis did not directly lead him to it.

L'Hopital, in his *Traité des Infiniment Petits*, has resolved the problem by means of the synthesis of Keill, and the differential calculus, but like Bernoulli, he stopped at the declination, and it does not appear why he neglects the duration.

Maupertuis, in his *Nautical Astronomy*, has treated the same question, and found Keill's formula for the declination; but his analysis leaving some difficulties, which required longer details; he suppressed it in a second edition.

D'Alembert in the *Encyclopedie methodique*, has taken up the problem to free it from all difficulties, his analysis, sufficiently simple, led to an equation of the fourth degree.

Let π be the height of the pole, $2a$ the crepuscular depression, δ the declination, the four roots are

$$\sin \delta = +1, \sin \delta = -1, \sin \delta = -\frac{\sin \pi}{\sin 2a} \pm \frac{\sin \pi}{\sin 2a} (1 - \sin^2 2a)^{\frac{1}{2}},$$

the two first ought to be rejected because they give $\delta = \pm 90^\circ$ and the greatest declination cannot exceed $23^\circ, 28'$.

The third is

$$\sin \delta = -\frac{\sin \pi}{\sin 2a} (1 + \cos 2a) = -\frac{2 \sin \pi \cos^2 a}{2 \sin a \cos a} \\ = -\sin \pi \cot a = -\sin \pi \tan 81^\circ,$$

it gives the declinations much too great, even when the latitudes are small, and if in reality it belonged to the problem, it would for the most part be impossible, and would wholly apply only to such countries as are close to the equator, which would be a sufficient reason for rejecting it: but in fact it does not belong to the problem.

The fourth is

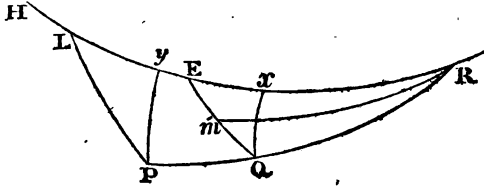
$$\delta = -\frac{\sin \pi}{\sin 2a} (1 - \cos 2a) = -\frac{2 \sin \pi \sin^2 a}{2 \sin a \cos a} = -\tan a \sin \pi.$$

This is Keill's formula, and equivalent to that of Nonius; D'Alembert employs four pages of analytical reasoning to prove that the third root is inadmissible, and belongs to another problem; it contradicts demonstrated propositions. In the same volume, as a sequel to D'Alembert's article, we find one much shorter by Cagnoli, who by a very simple construction and two well known differential analogies of spherical triangles arrives without difficulty at the two formulæ of Nonius and Keill.

In the fourteenth chapter of my *Astronomy* I have demonstrated these formulæ in three different ways, simply by Trigonometry, and I had believed the subject to be exhausted. But in again reading Nonius, with more attention for the exposition of his discoveries prepared for my *History of Astronomy*, I have observed that by simplifying his construction, and reducing it almost to a single triangle,

I could demonstrate his formulae and those of Keill, and also the general formulae of which they are a particular case.

Let the indefinite arc HR represent the horizon, and LP the arc of the parallel which the sun describes during the twilight, so that L the intersection of the parallel and horizon is the point of sun rise, and P , the lower extremity of the arc, is the point where the sun is found at the beginning of the twilight; let EQ be an arc of the equator containing the same number of degrees as the arc of the parallel, and which therefore measures the duration of the twilight; the point E , which is at once in the equator and the horizon, is the point of equinoctial rising; the point Q , the other extremity, is depressed below the horizon; and the arc QE makes with the horizon an angle equal to the height of the equator which is $(90^\circ - H)$.



Through the lower points P and Q of the parallel and the arc of the equator, draw an arc of a great circle; this will cut the horizon in R , and make with it an angle easy to be determined.

If the arc $EQ = c$ the measure of the twilight be supposed given, the triangle EQR will be entirely known; the angle $REQ = 90^\circ - H$; the angle $RQE = 90^\circ + H$; the sides $RQ + RE = 180^\circ$; let $RQ = 90^\circ - m$, then $RE = 90^\circ + m$, m being a variable arc to be determined. In the triangle REQ we have

$$\cos R = \cos EQ \sin RQE \sin REQ - \cos RQE \cos REQ,$$

$$\text{or } \cos R = \cos c \cos^2 H + \sin^2 H = \sin^2 H + \cos^2 H - 2 \cos H \sin^2 \frac{1}{2}c,$$

$$\text{that is, } 1 - 2 \sin^2 \frac{1}{2}R = 1 - 2 \sin^2 \frac{1}{2}c \cos^2 H,$$

$$\text{Hence } \sin \frac{1}{2}R = \sin \frac{1}{2}c \cos H.$$

Neither Nonius, Clavius nor Keill have attended to this triangle, which is the key of the whole.

The same triangle gives $\sin R : \sin EQ :: \sin E : \sin RQ$,

$$\text{or } \sin R : \sin c :: \cos H : \sin RQ = \cos m,$$

$$\begin{aligned} \text{hence } \cos m &= \frac{\sin c \cos H}{\sin R} = \frac{2 \sin \frac{1}{2}c \cos \frac{1}{2}c \cos H}{2 \sin \frac{1}{2}R \cos \frac{1}{2}R} \\ &= \frac{2 \sin \frac{1}{2}c \cos \frac{1}{2}c \cos H}{2 \sin \frac{1}{2}c \cos H} = \frac{\cos \frac{1}{2}c}{\cos \frac{1}{2}R}; \end{aligned}$$

we have said that $RQ = 90^\circ - m$, and $RE = 90^\circ + m$, we therefore know every thing relating to the triangle.

Now the arc drawn from the vertex R upon the middle of the base EQ bisects the angle at the vertex; the whole triangle is thus divided into two partial triangles, of which the computation is easy.

Let m' be the middle of rq so that $m'q = m'e = \frac{1}{2}c$; the arc $m'r$ drawn to the vertex will be 90° , and we shall have

$$\tan m = \cot RQ = \tan \frac{1}{2}c \sin H.$$

This formula gives more precision than the former.

The angle $rm'q$ is found by the formula

$$\tan rm'q = \cos \frac{1}{2}c \cot H;$$

but it is of no use in our enquiry, it is always acute.

In the other partial triangle, the angle

$$rm'e = 180^\circ - rm'q; \tan rm'e = -\cos \frac{1}{2}c \cot H.$$

This angle is always obtuse, it is of no use.

From p , the position of the sun at the beginning of the twilight, draw py perpendicular to the horizon, we have

$$py = 2a = \text{depression of the sun.}$$

The triangle pay , right angled at y , gives,

$$\sin PR = \frac{\sin 2a}{\sin R} = \frac{2 \sin a \cos a}{2 \sin \frac{1}{2}R \cos \frac{1}{2}R} = \frac{\sin a \cos a}{\sin \frac{1}{2}R \cos \frac{1}{2}R} = \cos n.$$

The arc PR found by its sine has two values, the one ($90^\circ - n$), the other ($90^\circ + n$); these two values will now be known.

The arc $pqr = (90^\circ \pm n)$ consists of two parts,

$$pq + qr = (90^\circ \pm n);$$

$rq = (90^\circ \pm n) - qr = (90^\circ \pm n) - (90^\circ - m) = \pm n + m = m \pm n$; rq is the eastern amplitude.

From p let fall a perpendicular on the equator, this will be the declination (d) which, like the hypotenuse pq , will have two values, viz.

$$\sin d = \sin (m \pm n) \cos H$$

The two declinations will be north, provided that n negative does not exceed m .

Therefore, c being supposed known, the problem is reduced to the following equations

$$\sin \frac{1}{2}R = \sin \frac{1}{2}c \cos H \dots \dots \dots (1)$$

$$\cos m = \frac{\sin c \cos H}{\sin R} = \frac{\cos \frac{1}{2}c}{\cos \frac{1}{2}R},$$

but we shall have greater precision by making

$$\tan m = \tan \frac{1}{2}c \sin H \dots \dots \dots (2)$$

$$\cos n = \frac{\sin 2a}{\sin R} = \frac{2 \sin a \cos a}{2 \sin \frac{1}{2}R \cos \frac{1}{2}R} = \frac{\sin a \cos a}{\sin \frac{1}{2}R \cos \frac{1}{2}R} \dots \dots \dots (3)$$

$$\left. \begin{array}{l} \sin d = + \sin (n - m) \cos H \dots (4) \\ \sin d' = - \sin (n + m) \cos H \dots (5) \end{array} \right\} \begin{array}{l} \text{observing the rule for} \\ \text{the signs,} \end{array}$$

The known formula

$$\sin \frac{1}{2}c = \sin (p' - p) = \frac{\sin 2a}{2 \cos H \cos d \sin \frac{1}{2}(p' + p)} \dots \dots \dots (6)$$

from which we deduce

$$\cos D = \frac{\sin 2a}{2 \sin \frac{1}{2}(P' - P) \sin \frac{1}{2}(P' + P) \cos H} \dots \dots \dots (7)$$

is indirect in both cases, because it always contains $\frac{1}{2}(P' + P)$ which is unknown; but the two preceding formulae give two declinations which correspond to every value of c .

When the declination is given we compute the two formulae.

$$\cos P' = -\frac{2 \sin 2a}{\cos H \cos D} - \tan H \tan D, \quad \cos P = -\tan H \tan D;$$

$$\text{The duration} = \frac{1}{15}(P' - P) \dots \dots \dots (8)$$

Here then is the general solution deduced from the fundamental idea of Nonius.

It may happen that the two declinations are reducible to one, and that the arc PR has only one value, which will be the case when $PR = 90^\circ$; then

$$n = 0, \quad \cos n = \frac{\sin 2a}{\sin R} = 1, \quad 2a = R, \quad a = \frac{1}{2}R.$$

And (1)

$$\sin \frac{1}{2}c = \frac{\sin \frac{1}{2}R}{\cos H} = \frac{\sin a}{\cos H} \dots \dots \dots (9)$$

which is precisely the formula of Nonius for the shortest twilight. For since the angle R is always opposite to the perpendicular $2a$, R is a *minimum* when $R = 2a$; if R be a *minimum*, $\sin \frac{1}{2}c = \frac{\sin \frac{1}{2}R}{\cos H} =$

$\frac{\sin a}{\cos H}$ will be a *minimum*; we shall have now the shortest twilight which can have but one value: thus the solution is disengaged from all embarrassment about the roots. It remains that we find the declination.

Formula (4) becomes

$$\sin D = \sin(n - m) \cos H = \sin(0 - m) \cos H = -\sin m \cos H \\ = -\cos PR \cos H.$$

This is Nonius' formula and is equivalent to Keill's; for

$$\cos m = \frac{\cos \frac{1}{2}c}{\cos \frac{1}{2}R};$$

$$\sin m = (1 - \cos^2 m)^{\frac{1}{2}} = (1 - \frac{\cos^2 \frac{1}{2}c}{\cos^2 \frac{1}{2}R})^{\frac{1}{2}} \\ = (\frac{\cos^2 \frac{1}{2}R - \cos^2 \frac{1}{2}c}{\cos^2 \frac{1}{2}R})^{\frac{1}{2}} = (\frac{\sin^2 \frac{1}{2}c - \sin^2 \frac{1}{2}R}{\cos^2 \frac{1}{2}R}), \\ \sin m = \left(\frac{\frac{\sin^2 \frac{1}{2}R}{\cos^2 H} - \sin^2 \frac{1}{2}R}{\cos^2 \frac{1}{2}R} \right)^{\frac{1}{2}} = \left(\frac{\sin^2 \frac{1}{2}R - \sin^2 \frac{1}{2}R \cos^2 H}{\cos^2 H \cos^2 \frac{1}{2}R} \right)^{\frac{1}{2}}$$

$$= \left(\frac{\sin^2 \frac{1}{2} R \sin^2 H}{\cos^2 H \cos^2 \frac{1}{2} R} \right)^{\frac{1}{2}} = \tan \frac{1}{2} R \tan H = \tan a \tan H,$$

$$\sin D = -\sin m \cos H = -\tan a \tan H \cos H \\ = -\tan a \sin H \dots \dots \dots (10)$$

This is the formula of Keill, Bernoulli, L'Hopital, Maupertuis, Maduit, and Cagnoli; $\tan a \tan H$ is the sine of the amplitude, which is an additional richness of Nonius' solution, but which however is merely a thing of curiosity.

If the declination be nothing, the arc LP of the parallel, coincides with the arc EQ of the equator. The perpendicular Qx becomes then $2a$;

$$\sin Qx = \sin 2a = \sin RQ \sin R = \cos m \sin R = \sin c \cos H \dots (11)$$

$$\text{and hence, } \sin c = \frac{\sin 2a}{\cos H};$$

c measures the duration of the twilight on the equinoctial day; this is also a formula of Nonius. But to this same duration there ought to correspond a second declination. Our general formulae become

$$\sin \frac{1}{2} R' = \sin \frac{1}{2} c' \cos H \dots \dots \dots (12)$$

$$\cos m' = \frac{\cos \frac{1}{2} c'}{\cos \frac{1}{2} R'} = \frac{\sin c' \cos H}{\sin R'} \dots \dots \dots (13)$$

$$\cos n' = \frac{\sin 2a}{\sin R'} = \frac{\sin a \cos a}{\sin \frac{1}{2} R' \cos \frac{1}{2} R'} \dots \dots \dots (14)$$

$$\sin D = -\sin (n' + m') \cos H = -\sin 2 m' \cos H \dots \dots (15)$$

$$\text{or } \tan m' = \tan \frac{1}{2} c' \sin H \dots \dots \dots (16) \\ m' = \frac{1}{2} \text{ amplitude.}$$

These formulae (12) (13) (14) belong to Nonius, (13) and (14) may with advantage be replaced by our formula (15) which for this particular case may be deduced from the general formula of the semi-duration,

$$\sin \frac{1}{2} c = \frac{\sin 2a}{2 \cos H \cos c \sin (P + \frac{1}{2} c)} \dots \dots \dots (17)$$

$$2 \sin \frac{1}{2} c \sin (P + \frac{1}{2} c) = \frac{\sin 2a}{\cos H \cos D}.$$

If $D = 0^\circ$, then $P = 90^\circ$,

$$2 \sin \frac{1}{2} c \sin (90^\circ + \frac{1}{2} c) = 2 \sin \frac{1}{2} c \cos \frac{1}{2} c = \sin c = \frac{\sin 2a}{\cos H}.$$

This is also a formula of Nonius for the twilight of the equinox.

Let c' be this particular value, and let us introduce it into the general formula which will become

$$\sin c \sin P + 2 \sin^2 \frac{1}{2} c \cos P = \frac{\sin c'}{\cos D} \dots \dots \dots (18)$$

If we wish that $c = c'$, we have

$$\sin c' + 2 \sin^2 \frac{1}{2} c' \cot P = \frac{\sin c'}{\cos D \sin P};$$

$$\sin c' \sin P + 2 \sin^2 \frac{1}{2} c' \cos P = \sin c' \sec D,$$

$$\sin P + \frac{2 \sin^2 \frac{1}{2} c'}{2 \sin \frac{1}{2} c' \cos \frac{1}{2} c} \cos P = \sec D$$

$$\sin P + \tan \frac{1}{2} c' \cos P = \sec D.$$

Now let A be the amplitude, and we shall have

$$\sin P = \frac{\cos A}{\cos D}; \cos P = -\tan H \tan D; \text{ hence}$$

$$\frac{\cos A}{\cos D} - \tan \frac{1}{2} c' \tan H \tan D = \sec D.$$

$$\cos A - \tan \frac{1}{2} c' \tan H \sin D = 1,$$

$$\cos A - \tan \frac{1}{2} c' \sin H \frac{\sin D}{\cos H} = 1,$$

$$\cos A - \tan \frac{1}{2} c' \sin H \sin A = 1,$$

$$- \tan \frac{1}{2} c' \sin H - 2 \sin \frac{1}{2} A \cos \frac{1}{2} A = 1 - \cos A = 2 \sin^2 \frac{1}{2} A$$

$$- \tan \frac{1}{2} c' \sin H = \tan \frac{1}{2} A,$$

and this amplitude will be negative.

Delambre proceeds to demonstrate some remarks by Nonius on the twilights in general, but that we may not carry this extract too far, we shall conclude with his defence of Nonius against a reflection cast on his solution by Montucla.

Montucla in his History of the Mathematics, where he gives a very brief account of the works of Nonius, cites the problem of the shortest twilight, and adds, that his solution is less elegant than that of Bernoulli. One cannot see upon what grounds the historian gave this opinion. Bernoulli confesses that his solution was long and troublesome; these are also the only faults which can be alleged against that of Nonius, which in other respects is much more instructive and more complete. Bernoulli has found only one formula, and it does not give the semiduration, Nonius has found it, and his formula is very simple and elegant. The advantage is therefore incontestably on the side of Nonius, more especially if the time be considered; he has succeeded much better with fewer means.

PRIZE QUESTION 1746.

This question has been formed from the following elegant Geometrical Theorem.

If a circle be inscribed in a triangle and another be described about it; the distance between their centres is a mean proportional between the radius of the circumscribing circle, and its excess above twice the radius of the inscribed circle.

The solution to the diary question could hardly be considered as

a demonstration of this theorem: An elegant analytical investigation was however afterwards given by the very ingenious Mr. Landen in his *Mathematical Lucubrations*, part 1, printed in 1755. We also observe that two *Porisms* formed from it, have been proposed for demonstration in the second number of *Annales des Mathematiques*, a periodical work begun at Nismes, in 1810, by *Gergonne* and *Lavernède*, two professors in the Lyceum of that city. The porisms are these :

1. A circle and any point within it being given by position ; another circle, and only one, can be described on that point as a centre, so that a triangle may be inscribed in the first circle, and at the same time be circumscribed about the second.

2. A circle and a point being both given by position, another circle, and only one, can be described on that point as a centre, so that a triangle may be described about the first of the circles, and also be inscribed in the second.

The editors have given various demonstrations of the theorem by themselves, and by L'Huillier and Garnier : but they do not notice, probably because they did not know, that the theorem had been long before known in Britain. We shall give an analytical and also a geometrical demonstration.

1. *The Analytical Solution, by Mr. Wallace, of the Royal Military College.*

Let the angles of the triangle be indicated by the letters A, B, C : Let r be the radius of the inscribed, and R that of the circumscribed circle ; and let d be the distance of their centres : Then from the nature of the figure, lines drawn from the centres of the inscribed and circumscribed circles to B, one of the angles of the triangle,

will be $\frac{r}{\sin \frac{1}{2}B}$ and R respectively, and the angle contained by the two lines will be $90^\circ - (C + \frac{1}{2}B)$: Hence, by a well known theorem,

$$\frac{r^2}{\sin^2 \frac{1}{2}B} + R^2 - 2Rr \frac{\sin (C + \frac{1}{2}B)}{\sin \frac{1}{2}B} = d^2.$$

And therefore,

$$r^2 + R^2 \sin^2 \frac{1}{2}B - 2Rr \sin (C + \frac{1}{2}B) \sin \frac{1}{2}B = d^2 \sin^2 \frac{1}{2}B;$$

But $\sin^2 \frac{1}{2}B = \frac{1}{2}(1 - \cos B)$

$$\text{and } \sin (C + \frac{1}{2}B) \sin \frac{1}{2}B = \frac{1}{2} \cos C - \frac{1}{2} \cos (C + B) \\ = \frac{1}{2} \cos C + \frac{1}{2} \cos A.$$

Therefore,

$$2r^2 + R^2 (1 - \cos B) - 2Rr (\cos C + \cos A) = d^2 (1 - \cos B) \dots (1).$$

From the manner in which this equation has been formed, it will manifestly hold true, if we substitute the angle c instead of B , and *vice versa* : hence also we get

$$2r^2 + R^2 (1 - \cos c) - 2Rr (\cos B + \cos A) = d^2 (1 - \cos c) \dots (2).$$

Take now the difference of the corresponding members of these equations, and we have

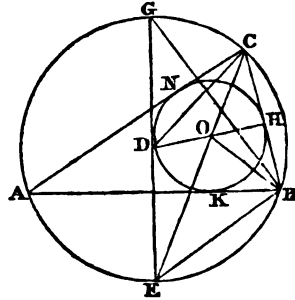
$R^2 (\cos B - \cos C) - 2Rr (\cos B - \cos C) = d^2 (\cos B - \cos C)$;
and leaving the common factor out of all the terms, we get

$$R^2 - 2Rr = d^2,$$

the analytical expression of the theorem to be demonstrated.

11. *The Geometrical Demonstration, by Mr. J. Lowry, of the Royal Military College.*

Let ACB be a triangle described about the circle nHX , and inscribed in the circle $AGCBE$, through the centre O of the inscribed circle, draw CO to meet the circumscribing circle in E . Also draw the diameter EDG and the radius DC : join BO , DO , EB and GB , and draw the radius OH perpendicular to CB ; the



$\angle EOB = \angle OCB + \angle CBO$
and $\angle EBO = \angle EBA + \angle ABO$,
but $\angle OCB = \angle EBA$ and $\angle CBO = \angle ABO$;
therefore $\angle EOB = \angle EBO$, and
consequently $EB = EO$.

Because the triangle CDE is isosceles,

$$DC^2 = DO^2 + OC \times OE,$$

but $OC \times OE = OC \times EB$, and by similar triangles,

$$OC : OH :: GE = 2DC : EB,$$

$$\text{therefore } OC \times EB = 2DC \times OH,$$

$$\text{and } DC^2 = DO^2 + 2DC \times OH,$$

$$\text{or } DO^2 = DC^2 - 2DC \times OH.$$

Whence, if R be put for the radius of the circumscribing circle, r for that of the inscribed one, and d for the distance of their centres: then,

$$d = \sqrt{R^2 - 2rR}.$$

Hence if two circles whose radii are R and r , be so posited that the distance between their centres is $= \sqrt{R^2 - 2rR}$, any triangle whatever described about the circle whose radius is r , so that one of its sides may terminate in the other circle, will be inscribed in that circle.

The triangle will evidently be a maximum or minimum when it is isosceles as in the original solution. When $R^2 = 2rR$, or $R = 2r$, d is $= 0$, and the circles are concentric. The triangle is then equilateral and the greatest possible.

It may be shewn in the same manner, that if r' , r'' and r''' be the radii of circles touching one side of a triangle, and the continuation of the other two sides, the distance of their respective centres from that of the circumscribing circle, will be expressed by

$$\sqrt{R^2 + 2r'R}, \sqrt{R^2 + 2r''R}, \text{ and } \sqrt{R^2 + 2r'''R}.$$

QUESTION 305. (Vol. II.)

The following table, containing 63 pair of amicable numbers, is taken from a Tract, by Euler, published in 1750; in which the subject is treated at very considerable length.

In this we are informed that Stifelius was the first who took notice of such numbers.

Mr. John Gough has lately investigated some properties of these numbers, especially of the Cartesian form. See the 7th number of the New Series of the Mathematical Repository.

Table of Amicable Numbers.

I	$\left\{ \begin{array}{l} 2^2. 5.11 \\ 2^1. 71 \end{array} \right\}$	II	$\left\{ \begin{array}{l} 2^1.23.47 \\ 2^1.1151 \end{array} \right\}$	XVII	$\left\{ \begin{array}{l} 2^1.23.1367 \\ 2^1.53.607. \end{array} \right\}$
III	$\left\{ \begin{array}{l} 2^7.191.383 \\ 2^7.73727 \end{array} \right\}$			XVIII	$\left\{ \begin{array}{l} 2^1.47.89 \\ 2^1.53.79 \end{array} \right\}$
IV	$\left\{ \begin{array}{l} 2^2.23.5.137 \\ 2^2.23.827 \end{array} \right\}$			XIX	$\left\{ \begin{array}{l} 2^1.23.479 \\ 2^1.89.127 \end{array} \right\}$
V	$\left\{ \begin{array}{l} 3^2.7.13.5.17 \\ 3^1.7.13.107 \end{array} \right\}$			XX	$\left\{ \begin{array}{l} 2^1.23.467 \\ 2^1.103.107 \end{array} \right\}$
VI	$\left\{ \begin{array}{l} 3^2.5.13.11.19 \\ 3^2.5.13.239 \end{array} \right\}$			XXI	$\left\{ \begin{array}{l} 2^1.17.5119 \\ 2^1.239.383 \end{array} \right\}$
VII	$\left\{ \begin{array}{l} 3^1.7^2.13.5.41 \\ 3^1.7^2.13.251 \end{array} \right\}$			XXII	$\left\{ \begin{array}{l} 2^1.17.10303 \\ 2^1.167.1103 \end{array} \right\}$
VIII	$\left\{ \begin{array}{l} 3^2.5.7.53.1889 \\ 3^2.5.7.102059 \end{array} \right\}$			XXIII	$\left\{ \begin{array}{l} 2^1.19.1439 \\ 2^1.149.191 \end{array} \right\}$
IX	$\left\{ \begin{array}{l} 2^2.13.17.389.509 \\ 2^1.13.17.198899 \end{array} \right\}$			XXIV	$\left\{ \begin{array}{l} 2^5.59.1103 \\ 2^5.79.827 \end{array} \right\}$
X	$\left\{ \begin{array}{l} 3^2.5.19.37.7.887 \\ 3^2.5.19.37.7103 \end{array} \right\}$			XXV	$\left\{ \begin{array}{l} 2^5.37.12671 \\ 2^5.227.2111 \end{array} \right\}$
XI	$\left\{ \begin{array}{l} 3^4.5.11.29.89 \\ 3^4.5.11.2699 \end{array} \right\}$			XXVI	$\left\{ \begin{array}{l} 2^5.53.10559 \\ 2^5.79.7127 \end{array} \right\}$
XII	$\left\{ \begin{array}{l} 3^2.7^2.11.13.41.461 \\ 3^2.7^2.11.13.19403 \end{array} \right\}$			XXVII	$\left\{ \begin{array}{l} 2^6.79.11087 \\ 2^6.383.2309 \end{array} \right\}$
XIII	$\left\{ \begin{array}{l} 3^2.5.13.19.29.569 \\ 3^2.5.13.19.17099 \end{array} \right\}$			XXVIII	$\left\{ \begin{array}{l} 2^2.383.9203 \\ 2^2.1151.3067 \end{array} \right\}$
XIV	$\left\{ \begin{array}{l} 3^2.7^2.13.97.5.193 \\ 3^2.7^2.13.97.1163 \end{array} \right\}$			XXIX	$\left\{ \begin{array}{l} 2^2.11.17.263 \\ 2^2.11.43.107 \end{array} \right\}$
XV	$\left\{ \begin{array}{l} 3^2.7.13.41.163.5.977 \\ 3^2.7.13.41.163.5867 \end{array} \right\}$			XXX	$\left\{ \begin{array}{l} 3^3.5.7.71 \\ 3^3.5.17.31 \end{array} \right\}$
XVI	$\left\{ \begin{array}{l} 2^3.17.79 \\ 2^3.23.59 \end{array} \right\}$			XXXI	$\left\{ \begin{array}{l} 3^2.5.13.29.79 \\ 3^2.5.13.11.199 \end{array} \right\}$

XXXII	{ 3 ⁵ .5.13.19.47 3 ⁵ .5.13.29.31	XLVIII	{ 2 ⁵ .29.47.59 2 ⁵ .17.4799
XXXIII	{ 3 ⁵ .5.13.19.37.1583 3 ⁵ .5.13.19.227.263	XLIX	{ 2 ⁵ .17.167.13679 2 ⁵ .809.51071
XXXIV	{ 3 ⁵ .7 ² .13.19.11.220499 3 ⁵ .7 ² .13.19.89.29399	L	{ 2 ⁵ .23.47.9767 2 ⁵ .1583.7103
XXXV	{ 3 ⁵ .5.19.37.47 3 ⁵ .5.19.7.227	LI	{ 2 ⁵ .5.13.1187 2 ⁵ .43.2267
XXXVI	{ 2 ⁵ .67.37.2411 2 ⁵ .67.227.401	LII	{ 3 ⁵ .7.13.5.17.1187 3 ⁵ .7.13.131.971
XXXVII	{ 3 ⁵ .5.7.11.29 3 ⁵ .5.31.89	LIII	{ 3 ⁵ .7 ² .13.53.11.211 3 ⁵ .7 ² .13.53.2543
XXXVIII	{ 2.5.23.29.673 2.5.7.60659	LIV	{ 3 ⁵ .5 ² .11.59.179 3 ⁵ .5 ² .17.19.359
XXXIX	{ 2.5.7.19.107 2.5.47.359	LV	{ 3 ⁵ .5.17.23.397 3 ⁵ .5.7.21491
XL	{ 2 ⁵ .11.163.191 2 ⁵ .31.11807	LVI	{ 3 ⁵ .7.11 ² .19.47.7019 3 ⁵ .7.11 ² .19.389.863
XLI	{ 3 ⁵ .7.13.23.11.19.367 3 ⁵ .7.13.23.79.1103	LVII	{ 3 ⁵ .7.11 ² .19.53.6959 3 ⁵ .7.11 ² .19.179.2087
XLII	{ 3 ⁵ .5.23.11.19.367 3 ⁵ .5.23.79.1103	LVIII	{ 3 ⁵ .7 ² .13.19.47.7019 3 ⁵ .7 ² .13.19.389.863
XLIII	{ 2 ⁵ .11.59.173 2 ⁵ .57.2609	LIX	{ 3 ⁵ .7 ² .13.19.53.6959 3 ⁵ .7 ² .13.19.179.2087
XLIV	{ 2 ⁵ .11.23.2543 2 ⁵ .383.1907	LX	{ 2 ⁵ .19.41 2 ⁵ .199
XLV	{ 2 ⁵ .11.23.1871 2 ⁵ .467.1151	LXI	{ 2 ⁵ .41.467 2 ⁵ .19.233
XLVI	{ 2 ⁵ .11.23.1619 2 ⁵ .719.647	LXII	{ 2 ⁵ .5.131 2 ⁵ .17.43
XLVII	{ 2 ⁵ .11.29.239 2 ⁵ .191.449	LXIII	{ 2 ⁵ .13.107 2 ⁵ .5.251

QUESTION 311. (Vol. II.)

Additional Solutions.

1. *By Mr. Cunliffe, of the Royal Military College.*

To find three such numbers that the sum or difference of any two of them shall be a square number.

Before we proceed to the solution of the proposed question, we shall put down a solution to the following problem; as the resulting expressions will be found of very essential use in the solution of the question.

PROBLEM. To divide a quantity, or number composed of two squares, into two other squares.

Let $a^2 + b^2$ denote the quantity, composed of the two squares a^2, b^2 , which is to be divided into two other squares c^2, d^2 , so that $a^2 + b^2 = c^2 + d^2$.

Solution. Put $c = a + sv$, and $d = b - rv$, and then $a^2 + b^2 = c^2 + d^2 = (a + sv)^2 + (b - rv)^2 = a^2 + b^2 + 2sav - 2rbv + v^2 \times (r^2 + s^2)$, which gives $v = \frac{2rb - 2sa}{r^2 + s^2}$; hence

$$c = a + sv = a + \frac{2rsb - 2s^2a}{r^2 + s^2} = \frac{(r^2 - s^2)a + 2rsb}{r^2 + s^2}; \text{ and}$$

$$d = b - rv = b - \frac{2r^2b - 2rsa}{r^2 + s^2} = \frac{2rsa - (r^2 - s^2)b}{r^2 + s^2}.$$

Solution of the Question.

Let the numbers be denoted by

$$a^2c^2 + b^2d^2, a^2d^2 + b^2c^2 \text{ and } 2abcd.$$

For the sum or difference of the first and third of these is evidently a square. Also the sum or difference of the second and third is a square. It remains therefore to make the sum and difference of the first and second, each equal to a square.

The sum of the first and second is

$$a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 = (a^2 + b^2) \times (c^2 + d^2)$$

which is to be a square, by the question; now the expression $(a^2 + b^2) \times (c^2 + d^2)$, will be made a square by putting

$$c = (r^2 - s^2)a + 2rsb, \text{ and } d = 2rsa - (r^2 - s^2)b.$$

The fitness and propriety of this substitution is easily perceived from the preceding problem.

By means of the foregoing substitution,

$$(a^2 + b^2) \times (c^2 + d^2) = (a^2 + b^2) \times (a^2 + b^2) \times (r^2 + s^2)^2 = (a^2 + b^2)^2 \times (r^2 + s^2)^2$$

which is a square.

We have now only to make the difference of the first and second expressions a square.

The difference of the first and second expressions (supposing the first the greater) is

$$a^2c^2 + b^2d^2 - a^2d^2 - b^2c^2 = (a^2 - b^2) \times (c^2 - d^2) =$$

$$(a^2 - b^2) \times \left\{ (a^2 - b^2) \times (r^2 - s^2)^2 + 8abrs \times (r^2 - s^2) - (a^2 - b^2) \times 4r^2s^2 \right\} \\ = (a^2 - b^2)^2 \times (r^4 - 6r^2s^2 + s^4) + 8ab \times (a - b^2) \times rs \times (r^2 - s^2)$$

which is to be a square.

Dividing by $(a^2 - b^2)^2$, and putting $t = \frac{ab}{a^2 - b^2}$, the last expression will become

$$r^4 - 6r^2s^2 + s^4 + 8trs \times (r^2 - s^2) = r^4 + 8tr^3s - 6r^2s^2 - 8trs^3 + s^4,$$

which must be a square.

Assume $r^2 - 4trs + s^2$ for its root, that is, put

$$\begin{aligned} r^4 + 8tr^3s - 6r^2s^2 - 8trs^3 + s^4 &= (r^2 - 4trs + s^2)^2 = \\ r^4 - 8tr^3s + 16t^2r^2s^2 + 2r^2s^2 - 8trs^3 + s^4, \end{aligned}$$

whence, and by restoring the value of t ,

$$\frac{r}{s} = \frac{2t^2 + 1}{2t} = \frac{a^4 + b^4}{2ab \times (a^2 - b^2)}.$$

From whence it appears that a and b may be any two rational numbers at pleasure; and r, s , must be two numbers in the ratio of

$$a^4 + b^4 \text{ to } 2ab \times (a^2 - b^2).$$

Example 1. Take $a=2, b=1$, then $\frac{r}{s} = \frac{a^4 + b^4}{2ab \times (a^2 - b^2)} = \frac{17}{12}$; therefore we may take $r = 17$, and $s = 12$;

$$\text{whence } c = (r^2 - s^2)a + 2rsb = 698,$$

$$d = 2rsa - (r^2 - s^2)b = 671;$$

$$\text{and hence, } a^2c^2 + b^2d^2 = 2399057,$$

$$a^2d^2 + b^2c^2 = 2288168,$$

$$\text{and } 2abcd = 1873432,$$

which three numbers will answer the question; viz. the sum and difference of every two is a rational square.

Example 2. Take $a = 3$ and $b = 1$:

$$\frac{r}{s} = \frac{a^4 + b^4}{2ab \times (a^2 - b^2)} = \frac{82}{45} = \frac{41}{24}.$$

Take $r = 41$, and $s = 24$,

$$\text{then } c = (r^2 - s^2)a + 2rsb = 5283,$$

$$d = 2rsa - (r^2 - s^2)b = 4799,$$

$$a^2c^2 + b^2d^2 = 274221202,$$

$$a^2d^2 + b^2c^2 = 235183698,$$

$$\text{and } 2abcd = 152118702,$$

and these are another set of numbers that will answer.

We shall now proceed to shew some other ways of managing the problem, the results of which give answers in smaller numbers than the foregoing.

Suppose the second of the assumed expressions, viz. $a^2d^2 + b^2c^2$ to be

the greater : then, the difference between the first and second will be

$$a^2d^2 + b^2c^2 - a^2c^2 - b^2d^2 = (a^2 - b^2) \times (d^2 - c^2),$$

which is to be a square, by the question.

$$\text{We have } d = 2rsa - (r^2 - s^2)b,$$

$$c = (r^2 - s^2)a + 2rsb;$$

$$\text{whence } d + c = (a - b) \times (r^2 - s^2) + (a + b) \times 2rs,$$

$$d - c = (a + b) \times (s^2 - r^2) + (a - b) \times 2rs,$$

$$\frac{d + c}{a - b} = r^2 - s^2 + \frac{a + b}{a - b} \times 2rs = r^2 + 2rst - s^2,$$

$$\text{putting } t = \frac{a + b}{a - b};$$

$$\frac{d - c}{a + b} = s^2 - r^2 + \frac{a - b}{a + b} \times 2rs = s^2 + \frac{2rs}{t} - r^2 = \frac{(s^2 - r^2)t^2 + 2rst}{t^2}.$$

Now, if we make each of the two preceding expressions a square; then, their product,

$$\frac{(d + c)}{(a - b)} \times \frac{(d - c)}{(a + b)} = \frac{d^2 - c^2}{a^2 - b^2} = \frac{(a^2 - b^2) \times (d^2 - c^2)}{(a^2 - b^2)^2},$$

must necessarily be a square; and consequently $(a^2 - b^2) \times (d^2 - c^2)$ will also be a square.

$$\text{Put the first, } r^2 + 2rst - s^2 = (ps - r)^2 = p^2s^2 - 2prs + r^2,$$

$$\text{whence } \frac{r}{s} = \frac{p^2 + 1}{2 \times (t + p)}; \text{ take } r = p^2 + 1, \text{ and then } s = 2 \times (t + p).$$

By means of these values of r and s , the remaining expression to be made a square, viz. $(s^2 - r^2)t^2 + 2rst$, becomes

$$4t^4 + 8t^3p - t^2 \times (p^2 - 1)^2 + 4t \times (t + p) \times (p^2 + 1) =$$

$$4t^4 + 8t^3p + t^2 \times (3 + 6p^2 - p^4) + 4tp \times (p^2 + 1).$$

Assume $2t^2 + 2tp$ for its root, that is, put

$$4t^4 + 8t^3p + t^2 \times (3 + 6p^2 - p^4) + 4tp \times (p^2 + 1) = (2t^2 + 2tp)^2 = 4t^4 + 8t^3p + 4t^2p^2;$$

$$\text{Whence } t = \frac{4p \times (p^2 + 1)}{p^4 - 2p^2 - 3} = \frac{4p}{p^2 - 3} = \frac{a + b}{a - b},$$

$$\text{hence } \frac{a}{b} = \frac{p^2 - 4p - 3}{3 + 4p - p^2}, \text{ and } \frac{r}{s} = \frac{p^2 + 1}{2 \times (t + p)} = \frac{p^2 - 3}{2p},$$

where p may be expounded by any rational number at pleasure.

$$\text{Example 1. Take } p = 2, \text{ then } \frac{a}{b} = \frac{9}{7}, \frac{r}{s} = \frac{1}{4},$$

therefore we may take $a = 9, b = 7, r = 1$, and $s = 4$, then

$$d = 2rsa - (r^2 - s^2)b = 177,$$

$$c = (r^2 - s^2)a + 2rsb = 79,$$

$$\begin{aligned}\text{whence } a^2c^2 + b^2d^2 &= 2040642, \\ a^2d^2 + b^2c^2 &= 2843458, \\ \text{and } 2abcd &= 1761858,\end{aligned}$$

which three numbers will answer the question.

Example 2. Take $p = 6$, then $\frac{a}{b} = \frac{19}{-3}$, $\frac{r}{s} = \frac{11}{4}$,
therefore we may take $a = 19$, $b = -3$, $r = 11$, and $s = 4$:

$$\begin{aligned}\text{then } d &= 2rsa - (r^2 - s^2)b = 1987, \\ c &= (r^2 - s^2)a + 2rsb = 1731; \\ \text{and hence } a^2c^2 + b^2d^2 &= 1117219842, \\ a^2d^2 + b^2c^2 &= 1452256258, \\ \text{and } 2abcd &= 392102658,\end{aligned}$$

which are another set of numbers that will answer.

Another expression for t , &c. may be found as follows.

The general expression to be made a square is

$$4t^4 + 8t^3p + t^2 \times (3 + 6p^2 - p^4) + 4tp \times (p^2 + 1);$$

take $2t^2 + 2tp - g$ for its root, that is, put

$$\begin{aligned}4t^4 + 8t^3p + t^2 \times (3 + 6p^2 - p^4) + 4tp \times (p^2 + 1) &= (2t^2 + 2tp - g)^2 = \\ 4t^4 + 8t^3p + t^2 \times (4p^2 - 4g) - 4ptg + g^2.\end{aligned}$$

$$\text{Put } 3 + 6p^2 - p^4 = 4p^2 - 4g, \text{ or } g = \frac{p^4 - 2p^2 - 3}{4} =$$

$\frac{(p^2 + 1) \times (p^2 - 3)}{4}$, to take away the three leading terms on each side of the equation: then from the equality of the remaining terms, viz.

$$4tp \times (p^2 + 1) = g^2 - 4tpg,$$

$$\text{we shall get } t = \frac{a + b}{a - b} = \frac{g^2}{4p \times (g + p^2 + 1)} = \frac{(p^2 - 3)^2}{16p};$$

$$\text{whence, } \frac{a}{b} = \frac{(p^2 - 3)^2 + 16p}{(p^2 - 3)^2 - 16p}, \text{ also } \frac{r}{s} = \frac{p^2 - 1}{2 \times (p + t)} = \frac{8p}{p^2 + 9};$$

where p may be taken at pleasure.

Example 1. Take $p = 1$,

$$\text{then } \frac{a}{b} = \frac{20}{-12} = \frac{5}{-3}, \text{ and } \frac{r}{s} = \frac{4}{5};$$

therefore we may take $a = 5$, $b = -3$, $r = 4$, and $s = 5$.

$$\text{From whence } d = 2rsa - (r^2 - s^2)b = 173,$$

$$c = (r^2 - s^2)a + 2rsb = 165,$$

$$\text{and hence } a^2c^2 + b^2d^2 = 949986,$$

$$a^2d^2 + b^2c^2 = 993250,$$

$$\text{and } 2abcd = 856350,$$

which are three numbers that will answer; that is, the sum and difference of every two of them will be a rational square.

Example 2. Take $p = 2$, then $\frac{a}{b} = \frac{33}{-31}$, $\frac{r}{s} = \frac{16}{13}$;

therefore we may take $a = 33$, $b = -31$, $r = 16$, and $s = 13$.

From whence $d = 2rsa - (r^2 - s^2)b = 16425 = 25 \times 657$

and $c = (r^2 - s^2)a + 2rsb = -10025 = 25 \times -401$,
rejecting the square factor 25, we may take

$$d = 657, \text{ and } c = -401 \text{ or } c = 401,$$

whence $a^2c^2 + b^2d^2 = 589926978$,

$$a^2d^2 + b^2c^2 = 624595522,$$

$$2abcd = 539033022,$$

and these are another set of numbers that will answer the question.

Thus we have found six different sets of numbers that will answer the conditions of the question; and none of these sets of numbers, when the nature of the question is considered, can be deemed remarkably large. And it is manifest, that innumerable sets of such numbers might be found from any of the formulæ which have been obtained in the foregoing solution.

Before we leave the question, we shall put down another method of proceeding with the solution.

And in this case we shall begin with making $(a^2 - b^2) \times (c^2 - d^2)$ a square.

$(a^2 - b^2) \times (c^2 - d^2)$ will become a square, by putting

$$c = (m^2 + n^2)a - 2mn b,$$

$$\text{and } d = 2mna - (m^2 + n^2)b;$$

which expressions for c and d , are derived in a manner similar to those obtained in the foregoing part of the solution; viz. by dividing a number or quantity consisting of the difference of two squares, into the difference of two other squares.

By means of the preceding values of c and d ,

$$(a^2 - b^2) \times (c^2 - d^2) = (a^2 - b^2)^2 \times (m^2 - n^2)^2$$

which is a square.

It therefore only remains to make $(a^2 + b^2) \times (c^2 + d^2) =$ a square.

Again, by the above values of c and d ,

$$c^2 + d^2 = (a^2 + b^2) \times (m^4 + 6m^2n^2 + n^4) - 8abmn \times (m^2 + n^2);$$

and hence

$$(a^2 + b^2) \times (c^2 + d^2) = (a^2 + b^2)^2 \times (m^4 + 6m^2n^2 + n^4) \\ - 8ab \times (a^2 + b^2) \times mn \times (m^2 + n^2),$$

which is to be a square by the question.

Dividing by $(a^2 + b^2)^2$, and putting $q = \frac{2ab}{a^2 + b^2}$ the last expression becomes

$$m^4 + 6m^2n^2 + n^4 - \frac{8ab}{a^2 + b^2} \times mn \times (m^2 + n^2) =$$

$$m^4 - 4qm^2n + 6m^2n^2 - 4qmn^3 + n^4,$$

which must also be a square.

Assume $m^2 - 2qmn - n^2$ for its root, that is, put

$$m^4 - 4qm^2n + 6m^2n^2 - 4qmn^3 + n^4 = (m^2 - 2qmn - n^2)^2 =$$

$$m^4 - 4qm^2n + 4q^2m^2n^2 - 2m^2n^2 + 4qmn^3 + n^4;$$

whence $\frac{m}{n} = \frac{2q}{2 - q^2} = \frac{2ab \times (a^2 + b^2)}{a^4 + b^4}$, where a and b may be taken at pleasure; and m and n must be numbers in the ratio of $2ab \times (a^2 + b^2)$ to $a^4 + b^4$.

Example 1. Take $a = 2$ and $b = 1$, then $\frac{m}{n} = \frac{20}{17}$;

therefore we may take $m = 20$ and $n = 17$.

Hence $c = (m^2 + n^2)a - 2mnb = 698$,

$$d = 2mna - (m^2 + n^2)b = 671,$$

which are the very same values of c and d , which were found in the first example annexed to our first method of solution.

Example 2. Take $a = 3$, $b = 1$, then $\frac{m}{n} = \frac{30}{41}$;

therefore we may take $m = 30$ and $n = 41$.

And hence $c = 5283$,

and $d = 4799$,

which are the same values of c and d as we found in the second example annexed to our first method of solution.

II. By the late Colonel Watson, taken from Burrow's Diaries, for 1776.

Assume $x^2z^2 + y^2u^2$, $2xyzu$, and $x^2u^2 + y^2z^2$ for the three required numbers, which assumption evidently satisfies four of the six required conditions; it is therefore only necessary to find values of x, y, z and u , so related that the sum and difference of the first and third assumed numbers may each be a square; or,

$$x^2z^2 + y^2u^2 + x^2u^2 + y^2z^2 = \square, \text{ or } (x^2 + y^2)(z^2 + u^2) = \square,$$

$$\text{and } x^2z^2 + y^2u^2 - x^2u^2 - y^2z^2 = \square, \text{ or } (x^2 - y^2)(z^2 - u^2) = \square.$$

Again assume $x = r + s$, $y = r - s$, $z = p + q$, $u = p - q$, which being substituted for their respective values, we shall then have $(2r^2 + 2s^2)(2p^2 + 2q^2) = \square$, and therefore $(r^2 + s^2)(p^2 + q^2) = \square$: also $4rs \times 4pq = \square$, and therefore $rspq = \square$.

If rs be supposed equal to pq , then will $rspq$ be a square number ; and there will remain $(r^2 + s^2)(p^2 + q^2)$, or its equal $(r^2 + \frac{p^2 q^2}{r^2}) \times (p^2 + q^2)$ to be made a square number ; but if p and q be supposed any two numbers, the sum of whose squares is a square number, we shall then have only to determine the values of p , q , and r , so that $r^2 + \frac{p^2 q^2}{r^2}$, or $r^4 + p^2 q^2$ may be a square number : assume now its root $= b - r^2$ or $r^2 - b$, then will $r^4 + p^2 q^2 = b^2 - 2br^2 + r^4$, hence $r^2 = \frac{b^2 - p^2 q^2}{2b}$: Now, in order to make the denominator $2b$ a square, let b be supposed $= \frac{c^2}{2}$, then we shall have

$$r^2 = \frac{c^4 - 4p^2 q^2}{4c^2} = \frac{(c^2 + 2pq)(c^2 - 2pq)}{4c^2},$$

from whence it is evident by inspection, that when $c^2 = p^2 + q^2$, the expression will at all times be a square number, and that r will

then be $= \frac{(p+q)(p-q)}{2c}$; consequently if s and n be respectively

put for the sum and difference of any two numbers p and q , the sum of

whose squares $p^2 + q^2 = c^2$, we shall have $r = \frac{sd}{2c}$, $s = \left(\frac{pq}{r}\right) =$

$\frac{c \times (s^2 - n^2)}{2sd}$, $x = (r+s) = \frac{s^2 d^2 + c^2 \times (s^2 - n^2)}{2csd}$, $y = (r-s) =$

$\frac{s^2 d^2 - c^2 \times (s^2 - n^2)}{2csd}$, $z = (p+q) = s$, and $u = (p-q) = n$,

and the three required numbers will be expressed by

$$\frac{\{s^2 d^2 + c^2 \times (s^2 - n^2)\}^2 \times s^2}{4c^2 s^2 d^2} + \frac{\{s^2 d^2 - c^2 \times (s^2 - n^2)\}^2 \times n^2}{4c^2 s^2 d^2},$$

$$\frac{\{s^4 d^4 - c^2 \times (s^2 - n^2)\} \times 2sd}{4c^2 s^2 d^2}, \text{ and}$$

$$\frac{\{s^2 d^2 + c^2 \times (s^2 - n^2)\}^2 \times d^2}{4c^2 s^2 d^2} + \frac{\{s^2 d^2 - c^2 \times (s^2 - n^2)\}^2 \times s^2}{4c^2 s^2 d^2};$$

or by neglecting the denominator which is a square number, and

writing for c^2 its equal $\frac{s^2 + n^2}{2}$, we shall have

$$s^2 \times (s^2 d^2 + \frac{s^4 - n^4}{2})^2 + n^2 \times (s^2 d^2 - \frac{s^4 - n^4}{2})^2,$$

$$2sd \times \{s^4 d^4 - (\frac{s^4 - n^4}{2})^2\}, \text{ and}$$

$D^2 \times (s^2 D^2 + \frac{s^4 - D^4}{2})^2 + s^2 \times (s^2 D^2 - \frac{s^4 - D^4}{2})^2$, for a general expression of three numbers, which answers the conditions of the question.

Example. If $p = 4$, $q = 3$, then will $s = 7$, and $D = 1$, and the three required numbers will in this case be 77764850, 20126386, and 66475250.

QUESTION 415. (Vol. II.)

Additional Solution.

In order to render the solution of the question complete, we shall in the first place, put down an easy method of finding and expressing the sum of n terms of the series

$$\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \&c. \dots \dots \frac{n}{r^n}.$$

It is obvious that $\frac{x}{1-x} = x + x^2 + x^3 + x^4 + \&c. \text{ ad infinitum}$,
and $\frac{x^{n+1}}{1-x} = x^{n+1} + x^{n+2} + x^{n+3} + x^{n+4} + \&c. \text{ ad infinitum}$.

Now, the latter series is manifestly the continuation of the first after the n leading terms, and consequently their difference

$$\frac{x - x^{n+1}}{1-x} = \frac{x}{x-1} (x^n - 1) = x + x^2 + x^3 + x^4 + \&c. \dots x^n.$$

Taking the fluxion of the last expressions, and dividing by $\dot{x}$, gives
 $\frac{nx^{n+1} - x^n(n+1) + 1}{(x-1)^2} = 1 + 2x + 3x^2 + 4x^3 + \&c. \dots nx^{n-1}$, or

$$\frac{nx^{n+2} - x^{n+1}(n+1) + x}{(x-1)^2} = x + 2x^2 + 3x^3 + 4x^4 + \&c. \dots nx^n;$$

writing $\frac{1}{r}$ in the place of x , the preceding expression becomes

$$\frac{\frac{n}{r^{n+2}} - \frac{1}{r^{n+1}}(1+n) + \frac{1}{r}}{\left(\frac{1}{r} - 1\right)^2} = \frac{\frac{n}{r^n} - \frac{1+n}{r^{n-1}} + r}{(r-1)^2} = \frac{1}{r^n} \times$$

$$\frac{r^{n+1} - (n+1)r + n}{(r-1)^2} = \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \&c. \dots \dots \frac{n}{r^n}.$$

Solution of the Question.

Put $p = 2000$, $r^4 = 1.05$ or $r = 1.012272 =$ the amount of 1*l.* at the end of the first quarter at the given rate of interest, and let n denote the time required in quarters.

Then after the first quarterly payment the debt will be denoted by $pr - 1$;

after the second, by $pr^2 - r - 2$;

after the third, by $pr^3 - r^2 - 2r - 3$;

after the fourth, by $pr^4 - r^3 - 2r^2 - 3r - 4$;

and therefore after the n th quarterly payment, the debt will be denoted by

$$\begin{aligned} & pr^n - r^{n-1} - 2r^{n-2} - 3r^{n-3} - 4r^{n-4} - \&c. \dots n. \\ & = r^n \times \left\{ p - \left(\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \&c. \dots + \frac{n}{r^n} \right) \right\}. \end{aligned}$$

Now, from the preceding problem,

$$\frac{1}{r^n} \times \frac{r^{n+1} - (n+1)r + n}{(r-1)^2} = \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \&c. \dots + \frac{n}{r^n};$$

$$\text{wherefore } r^n \times \left\{ p - \left(\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \&c. \dots + \frac{n}{r^n} \right) \right\} =$$

$$pr^n - \frac{r^{n+1} - (n+1)r + n}{(r-1)^2} = \frac{r^n \{ p(r-1)^2 - r \} + (n+1)r - n}{(r-1)^2}$$

$$= \frac{r^n a + (n+1)r - n}{(r-1)^2}, \text{ putting } a = p(r-1)^2 - r, \text{ which ex-}$$

presses the amount of the debt immediately after the n th quarterly payment.

But it is very evident that the debt is greater the moment before any quarterly payment than the moment after it. And the moment before the n th

quarterly payment the debt will be denoted by $\frac{r^n a + (n+1)r - n}{(r-1)^2} +$

$$n = \frac{r^n a + (n+1)r - n + n(r-1)^2}{(r-1)^2} = \frac{r^n a + n(r^2 - r) + r}{(r-1)^2},$$

which expression is to be a maximum by the question, or $r^n a + n(r^2 - r) + r$ is to be a maximum, because the denominator $(r-1)^2$ is constant. Therefore, putting the fluxion of the expression $= 0$,

supposing n variable, we shall have $nar^n \times \text{h. l. } r + n \times (r^2 - r) = 0$,

$$\text{or } r^n = \frac{r - r^2}{a \times \text{h. l. } r} = \frac{r(r-1)}{a \times \text{h. l. } r} = 1.433973 = g, \text{ and hence}$$

$$n = \frac{\log. g}{\log. r} = 29.551, \text{ consequently the debt will be the greatest}$$

possible just before the 30th quarterly payment.

The principles upon which the foregoing solution is founded are too clear and evident to admit of the least doubt.

Elements of the calculation.

$$\begin{aligned}
 r &= 1.012272, (r-1)^2 = .000150601984, r(r-1)^2 = \\
 &.301203968, a = r(r-1)^2 - r = -.7102325, \text{h. l. } r = \\
 &.0121975, -a \times \text{h. l. } r = .008663067, r(r-1) = .012422601984, \\
 \frac{r(r-1)}{-a \times \text{h. l. } r} &= 1.433973 = g; \text{ common log. of } 1.433973 = \\
 &.156541, \text{ and common log. of } r = .0052973 \text{ and } n = \frac{\log. g}{\log. r} = 29.551.
 \end{aligned}$$

c.

QUESTION 430. (Vol. II.)

First Additional Solution, by Mr. Cunliffe, R. M. College.

An equilateral triangle, is the least triangle that can possibly be circumscribed about a given circle, as is very well known.

Suppose an equilateral triangle, with its inscribed circle to be projected orthographically, upon any plane; the projection will be a triangle with its greatest inscribed ellipse, as is pretty evident; and the sides of the triangular projection will be bisected at the points of contact of the ellipse. For the sides of the equilateral triangle are bisected by the points of contact of the inscribed circle; and the sides of the triangular projection are obviously divided in the same ratio by the contact of the ellipse.

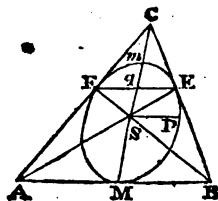
Suppose lines from the angles of the equilateral triangle to be drawn to the middle of the opposite sides; these lines will intersect in the centre of the inscribed circle; and consequently, in the projection, lines from the angles to the middle of the opposite sides will pass through the centre of the ellipse.

From the preceding conclusions we derive the following easy method of determining the greatest ellipse that can possibly be inscribed in a given triangle.

Let ACB be a given plane triangle, in which it is required to inscribe the greatest ellipse possible.

Draw AE and BF to the middle of the opposite sides; and through their intersection s draw CM, which will bisect AB, as is very well known; also, draw FE intersecting CM in q. Then FE is parallel to AB, and equal to $\frac{1}{2}AB$; and therefore $qe = \frac{1}{2}FE = \frac{1}{2}MB$, wherefore qe is an ordinate to the diameter mm.

Also $sq = \frac{1}{2}sm = \frac{1}{2}sm = \frac{1}{2}cm = qm$, and $qm = \frac{1}{2}cm$.



Moreover, by the principles of mensuration, $2cs \times 2sp \times \sin \angle CMB \times .7854 = \frac{1}{2} CM \times \frac{1}{2} MB \times \sqrt{3} \times \sin \angle CMB \times .7854 = \frac{1}{8} \times CM \times MB \times \sqrt{3} \times \sin \angle CMB \times .7854$, will express the area of the least ellipse that can possibly be circumscribed about the given plane triangle ACB.

Second Additional Solution.

The following solution to this problem, and another closely related to it, is taken from *Annales de Mathematiques*, vol. IV, page 284. It is by M. Berard, principal and professor of mathematics in the college of Briançon.

PROBLEM 1. To determine the ellipse of the greatest area that can be inscribed in a given triangle.

Solution. Let a and b be two sides of the given triangle, and γ the included angle; Let the vertex of this angle be taken as an origin, the side a for the axis of x , and the side b for the axis of y : If, then, x' and y' denote the unknown co-ordinates of the centre of the ellipse sought, its equation will have the form

$$A(x-x')^2 + B(y-y')^2 + 2C(x-x')(y-y') + 1 = 0;$$

A, B, C being coefficients to be determined, and which, with x' and y' are the unknown quantities of the problem.

In the first place, to express that this ellipse touches the sides of the triangle: let us make $y = 0$; then, as the resulting values of x must be equal, we find for the abscissa of the point of contact with the side a ;

$$x = x' + \frac{C}{A} y',$$

with the condition

$$A + y'^2 (AB - C^2) = 0 \dots \dots \dots (1)$$

If, in the same equation, we make $x = 0$, and consider that then the values of y are equal, we shall find for the ordinate of the point of contact with the side b ,

$$y = y' + \frac{C}{B} x',$$

with the condition

$$B + x'^2 (AB - C^2) = 0 \dots \dots \dots (2)$$

lastly, combining the same equation with the equation $bx + ay - ab = 0$ of the third side, put, for greater convenience, under this form, $b(x-x') + a(y-y') + (bx' + ay' - ab) = 0$, and expressing that the two systems of values which result from it, for x and y , are reducible to one only, we find, for the co-ordinates of the point of contact with the third side,

$$x = x' + \frac{(ac - bB)(bx' + ay' - ab)}{a^2 A + b^2 B - 2abc},$$

$$y = y' + \frac{(bc - aA)(bx' + ay' - ab)}{a^2 A + b^2 B - 2abc},$$

with the condition

$$a^2 A + b^2 B - 2abc + (bx' + ay' - ab)^2 (AB - c^2) = 0,$$

which, if we subtract from it the respective products of equations (1), (2) by a^2, b^2 , is reduced simply to

$$2c + (2bx' + 2ay' - 2x'y' - ab)(AB - c^2) = 0 \dots \dots \dots (3).$$

If, therefore, x' and y' were known; that is, if the centre of the ellipse were given, the only unknown quantities A, B, c of the problem would be given by the equations (1), (2), (3), from which we should find, neglecting the values 0, which cannot be admitted,

$$A = \frac{4y'^2}{(2bx' + 2ay' - 2x'y' - ab)^2 - 4x'^2 y'^2},$$

$$B = \frac{4x'^2}{(2bx' + 2ay' - 2x'y' - ab)^2 - 4x'^2 y'^2},$$

$$c = \frac{2(2bx' + 2ay' - 2x'y' - ab)}{(2bx' + 2ay' - 2x'y' - ab)^2 - 4x'^2 y'^2}.$$

We have, therefore, resolved the problem that requires an ellipse to be described with a given centre, which shall touch the three sides of a given triangle.

But let us now consider x' and y' as indeterminate, and require the ellipse to be the greatest possible. It is known that the area of an ellipse is merely the number π (3.1415927, &c.) multiplied by the product of the two principal semidiameters. Hence it follows that the required condition will be satisfied, if this product, or its square, be a *maximum*, but this square is

$$\frac{\sin^2 \gamma}{AB - c^2} *$$

* This is proved in another place by the author of this solution, as follows:

Let $Ax^2 + By^2 + 2Cxy = D$ (1)

be an equation of a line of the second order referred to its centre, and two axes containing an angle γ .

Let r denote the distance of any point of this curve from its centre, then we have

$$x^2 + y^2 + 2xy \cos \gamma = r^2. \dots \dots \dots (2)$$

and the property which characterizes the four vertices of the curve is, that for each, r is a maximum or a minimum.

Suppose that x and y are the co-ordinates of any one of these vertices, then r will be half of one of the principal diameters; Let us put $y = px$, the equations (1) and (2) then become

$$(Ap^2 + 2Cp + A)x^2 = D; (p^2 + 2p \cos \gamma + 1)x^2 = r^2,$$

hence, by eliminating x^2 , we find

$$(Ar^2 - D)p^2 + 2(Cr^2 - D \cos \gamma)p + (Ar^2 - D) = 0; \dots \dots \dots (3)$$

which gives for the condition of the maximum

$$d(AB - C^2) = 0, \dots\dots\dots (M)$$

or, by developing,

$$bda + adb - 2cdc = 0; \dots\dots\dots (P)$$

In like manner, differentiating the equations (1) (2) (3), and having respect to the equation (M), and making x' and y' to vary, there is had

$$dA + 2y'(AB - C^2) dy' = 0, \dots\dots\dots (Q)$$

$$dB + 2x'(AB - C^2) dx' = 0, \dots\dots\dots (R)$$

$$dc - (AB - C^2) \{ (x' - a) dy' + (y' - b) dx' \} = 0. \dots\dots (S)$$

If, among the four equations (P, Q, R, S) we eliminate dA , dB , dc , we shall find, all reductions being made,

$$\{ x'A + (y' - b) c \} dx + \{ y'B + (x' - a) c \} dy = 0;$$

and as the variables x' , y' are independent, we, hence, conclude

$$x'A + (y' - b) c = 0, (4); y'B + (x' - a) c = 0 \dots\dots\dots (5)$$

equations, which, joined to equations (1), (2), (3), resolve the problem.

By putting in these last for A and B , their values given by the equations (4) and (5), and dividing by c , we shall find for c three values of the first degree; and in equalling each of the two first to the third, the two resulting equations in x and y may be put under this form,

$$\begin{aligned} (2x' - a)(bx' + 2ay' - ab) &= 0, \\ (2y' - b)(ay' + 2bx' - ab) &= 0. \end{aligned}$$

As these may be satisfied, by putting either factor of the first member of each = 0, they would seem to give four systems of values for the unknown quantities x' , y' ; but of these, three ought to be rejected, because that they belong to the middle of the sides of the given triangle, which cannot be the centres of the inscribed

differentiating now this equation in respect of p only; since by hypothesis $dr=0$, it becomes,

$$(br^2 - D)p + (cr^2 - D \cos \gamma) = 0,$$

from which we find $p = - \frac{cr^2 - D \cos \gamma}{br^2 - D}$,

hence, and from equation (3), we find

$$r^4 - \frac{D(A + B - 2C \cos \gamma)}{AB - C^2} r^2 + \frac{D^2 \sin^2 \gamma}{AB - C^2} = 0.$$

The four roots of this equation (two and two of which are equal, with contrary signs), are the four principal semidiameters, hence the square of the product of the principal semidiameters is $\frac{D \sin^2 \gamma}{AB - C^2}$.

ellipses; the fourth system, which results from the equality of the last factors to 0, gives

$$x' = \frac{1}{3}a, y' = \frac{1}{3}b;$$

and therefore

$$c = -\frac{6}{ab}, A = -\frac{12}{a^2}, B = -\frac{12}{b^2};$$

$$AB - C^2 = \frac{108}{a^2 b^2};$$

and hence, $\frac{\pi \sin \gamma}{\sqrt{(AB - C^2)}} = \frac{\pi}{3\sqrt{3}} \cdot \frac{1}{ab} \sin \gamma;$

and the points of contact are the middle of the sides.

Thus, the centre of the greatest ellipse that can be inscribed in a triangle is at the centre of gravity of the area of the triangle, and it touches each of the sides in the middle, hence it follows that the triangle which has its angles at the points of contact has its sides parallel to those of the given triangle: The area of the ellipse is to that of the given triangle :: $\pi : 3\sqrt{3}$. Its equation is

$$12b^2(x - \frac{1}{3}a)^2 + 12a^2(y - \frac{1}{3}b)^2 + 12ab(x - \frac{1}{3}a)(y - \frac{1}{3}b) = a^2 b^2.$$

PROBLEM 2. To determine the ellipse of the least area that can circumscribe a given triangle.

Solution. Preserving the same conventions, and notation as in the former problem, the equation of the ellipse will, as before, be

$$A(x - x')^2 + B(y - y')^2 + 2C(x - x')(y - y') + 1 = 0.$$

Since this ellipse must pass through the origin, we have in the first place,

$$Ax'^2 + By'^2 + 2Cxy' + 1 = 0, \dots\dots\dots (c)$$

and the equation of its tangent, at that point, will be

$$(Ax' + cy')x + (By' + cx')y = 0. \dots\dots\dots (p)$$

This ellipse must also pass through the point of which the co-ordinates are a and 0; we, have, therefore,

$$A(x' - a)^2 + By'^2 + 2Cy'(x' - a) + 1 = 0;$$

an equation which, when equation (c) is subtracted from it, becomes

$$2y's + (2x - a)A = 0. \dots\dots\dots (1)$$

and the equation of the tangent at that point is

$$\{By' + c(x' - a)\}y + \{Cy' + A(x' - a)\}(x - a) = 0. \dots\dots (q)$$

The ellipse must also pass through the point of which the co-ordinates are 0 and b , hence also we must have

$$B(y' - b)^2 + Ax'^2 + 2Cx'(y' - b) + 1 = 0,$$

an equation from which, if equation (c) be subtracted, there will be had

$$2x'c + (2y' - b)B = 0, \dots\dots\dots (2)$$

and the equation of the tangent at that point is

$$\{ax' + c(y' - b)\}x + \{cx' + b(y' - b)\}(y - b) = 0 \dots (r)$$

If again the double of equation (c) be subtracted from the sum of the respective products of equations (1) and (2) by x' and y' , there is had

$$ax'A + by'B + 2 = 0 : \dots \dots \dots (3)$$

And by means of equations (1), (2), (3) those of the tangents to the three vertices become, respectively,

$$aAx + bBy = 0, \dots \dots \dots (p)$$

$$(bx - 2ac)y - aA(x - a) = 0, \dots \dots \dots (q)$$

$$(aA - 2bc)x - bB(y - b) = 0. \dots \dots \dots (r)$$

The same equations (1), (2), (3) give

$$A = -\frac{2y' - b}{x'(bx' + ay' - ab)}, \quad B = -\frac{2x' - a}{y'(bx' + ay' - ab)},$$

$$C = \frac{(2x' - a)(2y' - b)}{2x'y'(bx' + ay' - ab)};$$

and such would be the values of the unknown quantities, if the co-ordinates x', y' were given; that is, if it were proposed to describe an ellipse with a given centre, which should pass through the three angles of the triangle.

Let us now regard x' and y' as indeterminate, and restrict the ellipse to the condition of being the least possible; For this, it is yet necessary that the differential of $AB - C^2$ shall be nothing; but according to the values which have been assigned to A, B, C , we have

$$4(AB - C^2) = \frac{(2x' - a)(2y' - b)(2bx' + 2ay' - ab)}{x'y'(bx' + ay' - ab)}.$$

Taking now the differential of this fraction, and making the multipliers of dx and dy separately equal to 0, we find, all reductions being made,

$$y'(2y' - b)(2bx' + ay' - ab)\{2(x' - a)(bx' + ay') + a^2b\} = 0,$$

$$x'(2x' - a)(2ay' + bx' - ab)\{2(y' - b)(ay' + bx') + b^2a\} = 0.$$

The combination of these factors would seem to furnish sixteen solutions of the problem; but, in discussing them, it appears that the only one which can be admitted is that given by the two equations

$$2bx' + ay' - ab = 0, \quad 2ay' + bx' - ab = 0,$$

from which we deduce

$$x = \frac{1}{3}a, \quad y = \frac{1}{3}b.$$

And again,

$$A = -\frac{3}{a^2}, \quad B = -\frac{3}{b^2}, \quad C = -\frac{3}{2ab};$$

$$AB - c^2 = \frac{27}{4a^2 b^2}$$

$$\text{and hence } \frac{\pi \sin \gamma}{\sqrt{(AB - c^2)}} = \frac{4\pi}{3\sqrt{3}} \frac{1}{2} ab \sin \gamma,$$

from these values of x', y' , the equations (P), (Q), (R) of the tangents drawn to the ellipse through the angles of the triangle become simply

$$ay + bx = 0, \quad x - a = 0, \quad y - b = 0.$$

Thus, the least ellipse that can circumscribe a given triangle has its centre at the centre of gravity of the area of the triangle, and its tangents at the three angles are respectively parallel to the opposite sides. Hence it follows that the given triangle has its angles at the middle of the sides of that formed by the three tangents. The area of this ellipse is to that of the triangle as 4π to $3\sqrt{3}$. Its equation is $3b^2(x - \frac{1}{3}a)^2 + 3a^2(y - \frac{1}{3}b)^2 + 3ab(x - \frac{1}{3}a)(y - \frac{1}{3}b) = a^2 b^2$.

It also appears, that if two triangles are inscribed and circumscribed, the one in the other, in such a manner that their sides are parallel each to each, one and the same ellipse shall be, at once, the greatest that can be inscribed in the greater triangle and the smallest that can circumscribe the lesser.

The Editor of the Diary (Simpson, under the name of Patrick O'Cavanah) has observed, that the sides of the least triangle which can be described about any oval figure are bisected at the points of contact. This may be proved as follows: Imagine one side of the triangle to change its position while it incloses with the other two, which remain fixed, the minimum space, considered as given, this side will, as is well known, touch a given hyperbola which has the other two for its asymptotes. And as the hyperbola and the oval curve are to have a common tangent, it is easy to see that they cannot intersect each other; also, since the triangle is the smallest possible, the axis of the hyperbola must be the shortest possible; but this will manifestly be the case if the hyperbola and oval curve touch each other, and as their common tangent always lies between them it must pass through the point of contact; and, by the nature of the hyperbola, it will be bisected at that point.

Page 194, Vol. II.

In the Diary for 1760; the Editor, Mr. Simpson, says, "The question, by the very ingenious *Peter Walton* (Mr. Landen), re-

"quiring the sum of the series $x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \&c.$ by means

"of circular arcs and logarithms, when $x = \frac{1}{2}$; is of a limited, and
"very intricate nature, and what we could not venture to give in

“form among the others, as some of our correspondents, we are convinced, would not allow it to be a fair one. It is worthy, however, to be observed, that it is only in the case here proposed (where $x = \frac{1}{2}$) and in that where $x = 1$ (x being positive,) in which the series seems possible to be summed in the manner specified. With respect to the latter case, it is pretty well known, at this time, that the sum is expressed by two-thirds of the square of the semi-circumference of the circle whose diameter is unity. And if from the half of this sum, half the square of the hyp. log. of the number 2 be subtracted, the remainder will be the sum in the other case, where $x = \frac{1}{2}$. How this is derived, is left to be considered (agreeably to the design of the proposer) by those who have a taste for these subtle and abstracted speculations.”

A question similar to this was proposed in the *Mathematical Repository*, where it was required to shew under what circumstances

the sum of the infinite series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.$ could

be exhibited by the multiples or powers of the arcs of a circle.

To this a solution was given by Mr. Ivory as follows (see vol. II, page 130, part I.)

The investigation of the sums of the powers of the reciprocals of the natural numbers seems first to have engaged the attention of the two brothers, James and John Bernoulli. In a short treatise on the sums of infinite series written by James Bernoulli, (*Tractatus de seriebus infinitis, &c.* published with the *Ars Conjectandi* of the same author in one volume), we are informed (prop. 14.) that John Bernoulli first of all demonstrated that the sum of the infinite

series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ is greater than any finite number. In

the same treatise, the sum of the infinite series, $\frac{1}{1.3} + \frac{1}{2.4} +$

$\frac{1}{3.5} + \&c. = \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \&c.$ is shewn to be equal to $\frac{1}{2}$: and from hence it follows that the sum of the infinite

series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ is a finite quantity less than $\frac{1}{2}$. But the

summation of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ was a problem

for the solution of which the artifices then known were insufficient: and James Bernoulli speaking of this series, uses the following words, “difficilior est, quam quis expectaverit, summæ investigatio; quam tamen finitam esse, ex altera, qua manifesto minor est, colligimus; si quis inveniatur, nobisque communicet, quod industriam elusit hactenus; magnas de nobis gratias feret.” The further discoveries

of the Bernoullis, on this subject, are confined to the demonstration of the following properties; viz.

$$\left\{ 1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \&c. \right\} \times \frac{2^m - 1}{2^m} = 1 + \frac{1}{3^m} + \frac{1}{5^m} + \frac{1}{7^m} + \&c.$$

$$\left\{ 1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \&c. \right\} \times \frac{2^{m-1} - 1}{2^{m-1}} = 1 - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} + \&c.$$

The summation of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \&c.$ was accomplished by Euler; but the discovery was too late to gratify the curiosity of James Bernoulli, who was dead. The method of Euler extends to the general series $1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c.$ when

the index is any even number: and it is derived from the expression obtained by resolving the series for the sine of an arch into its binomial factors. Let z be any arch, and π the semi-circumference to the radius 1; then

$$\sin z = z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \&c. = z \left(1 - \frac{z^2}{\pi^2} \right) \left(1 - \frac{z^2}{2^2\pi^2} \right) \left(1 - \frac{z^2}{3^2\pi^2} \right) \&c.$$

and, by taking the fluxions of the logarithms of both sides of this equation and dividing by dz ,

$$\frac{\cos z}{\sin z} = \frac{1}{z} - \frac{2z}{\pi^2 - z^2} - \frac{2z}{2^2\pi^2 - z^2} - \frac{2z}{3^2\pi^2 - z^2} - \&c.$$

$$\text{let } \frac{\cos z}{\sin z} = \cot z = \frac{1}{z} - A^{(1)} \frac{1}{z^3} - A^{(3)} \frac{1}{z^5} - A^{(5)} \frac{1}{z^7} - \&c.$$

the general term of the developement being $-A^{(2n-1)} \frac{1}{z^{2n-1}}$; and

let the several fractions in the value of $\frac{\cos z}{\sin z}$ be developed into series:

then by equating the coefficients of the same powers of z , on different sides of the equation, we obtain,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = \frac{A^{(1)}}{2} \cdot \pi^2 = \frac{\pi^2}{6}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c. = \frac{A^{(3)}}{2} \cdot \pi^4 = \frac{\pi^4}{90}$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. = \frac{A^{(5)}}{2} \cdot \pi^6 = \frac{\pi^6}{945}$$

and in general,

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. = \frac{A^{(2n-1)}}{2} \times \pi^{2n}.$$

From the value of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$$

we immediately derive,

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \&c. = \frac{\pi^2}{12}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. = \frac{\pi^2}{8}.$$

These summations, first obtained by Euler, have been investigated by other mathematicians in different ways; and, particularly, by Mr. Landen in various parts of his writings.

If we put $x = 1$, in the series proposed in the question, it becomes

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \&c. \text{ the sum of which is } \frac{\pi^2}{12}, \text{ as we have seen;}$$

and thus one case is obtained in which the sum of that series is expressed by the square of an arc of a circle. But, if I mistake not, the ingenuity of mathematicians has hitherto discovered no other case, that of $x = 1$ excepted, in which the sum of the series $x -$

$$\frac{x^3}{2^2} + \frac{x^5}{3^2} - \frac{x^7}{4^2} + \&c. \text{ can be exhibited by means of arcs of a circle}$$

alone, without logarithms. Mr. Landen, indeed, has succeeded in determining the sum of the series $x - \frac{x^3}{2^2} + \frac{x^5}{3^2} - \frac{x^7}{4^2} + \&c.$ and the

sums of the kindred series is $x + \frac{x^3}{2^2} + \frac{x^5}{3^2} + \frac{x^7}{4^2} + \&c.$ and $x +$

$\frac{x^3}{3^2} + \frac{x^5}{5^2} + \frac{x^7}{7^2} + \&c.$ in some other particular instances by means of circular arcs and logarithms; and these cases I shall now proceed to notice.

For the sake of abridging language, I shall use the following notations: viz.

the symbol $\pi(x)$ for the series $x + \frac{x^3}{2^2} + \frac{x^5}{3^2} + \&c.$

the symbol $f(x)$ for the series $x - \frac{x^3}{2^2} + \frac{x^5}{3^2} - \&c.$

and the symbol $\phi(x)$ for the series $x + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \&c.$

Since $\int \frac{dx}{1 \mp x} = x \pm \frac{x^2}{2} + \frac{x^3}{3} \pm \frac{x^4}{4} + \&c.$ therefore

$\int \frac{dx}{x} \int \frac{dx}{1 \mp x} = x \pm \frac{x^2}{2} + \frac{x^3}{3} \pm \frac{x^4}{4} + \&c.$: these formulas be-

long to De Lorgna: by putting $\log$.

$\frac{1}{1-x}$ for $\int \frac{dx}{1-x}$ and $\log. (1+x)$ for $\int \frac{dx}{1+x}$, and attending to the meaning of the symbols just explained, we get

$$\int \frac{dx}{x} \log. \frac{1}{1-x} = r(x)$$

$$\int \frac{dx}{x} \log. (1+x) = f(x)$$

and, adding these expressions,

$$\int \frac{dx}{x} \log. \frac{1+x}{1-x} = 2\phi(x).$$

Now, let $x+y=1$; then $y=1-x$; and $-\int \frac{dx}{x} \log. y = -$

$\int \frac{dx}{x} \log. (1-x) = r(x)$: in like manner, $-\int \frac{dy}{y} \log. x =$

$r(y)$; therefore, $r(x) + r(y) = -\int \left\{ \frac{dx}{x} \log. y + \frac{dy}{y} \log. x \right\}$

$= c - \log. x \times \log. y$: to determine c , let $x=0$, then $r(x)=0$,

$\log. x \times \log. y = 0$, and $r(y) = r(1) = 1 + \frac{1}{2} + \frac{1}{3} + \&c. = \frac{\pi^2}{6}$

therefore

$$r(x) + r(1-x) = \frac{\pi^2}{6} - \log. x \times \log. (1-x) \dots \dots \dots (1).$$

Let $x = \frac{1}{2}$, or $x = \frac{1}{2}$ then $r\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$

$$\left(\frac{1}{2}\right)^4 + \frac{1}{4^2} \left(\frac{1}{2}\right)^4 + \&c. = \frac{\pi^2}{12} - \frac{1}{2} \log.^2 2.$$

Again, let $y = \frac{x}{1+x}$: then $\frac{dy}{y} = \frac{dx}{x} - \frac{dx}{1+x}$, and $1-y =$

$\frac{1}{1+x}$; therefore $\frac{dy}{y} \log. \frac{1}{1-y} = \frac{dx}{x} \log. (1+x) - \frac{dx}{1+x} \log.$

$(1+x)$; and by integrating $r(y) = f(x) - \frac{1}{2} \log.^2 (1+x)$, that is,

$$f(x) - r\left(\frac{x}{1+x}\right) = \frac{1}{2} \log.^2 (1+x) \dots \dots \dots (2).$$

By the formula (1), we get

$$F\left(\frac{1}{1+x}\right) + F\left(\frac{x}{1+x}\right) = \frac{\pi^2}{6} - \log. \frac{1}{1+x} \times \log. \frac{x}{1+x} :$$

and by combining this equation with the formula (2), we get

$$f(x) + F\left(\frac{1}{1+x}\right) = \frac{\pi^2}{6} - \frac{1}{2} \log.^2(1+x) + \log. (1+x) \times \log. x \quad (3).$$

Let $a = \frac{1}{1+x}$; then $a^2 + a = 1$, and $a = \frac{\sqrt{5}-1}{2}$: put a for x in the formula (3), then

$$f(a) + F(a) = \frac{\pi^2}{6} - \frac{3}{2} \log.^2 a.$$

Put a for x in formula (2), and because $a^2 = \frac{a}{1+a}$, therefore

$$f(a) - F(a^2) = \frac{1}{2} \log.^2 a.$$

But, from the nature of the functions $f(a)$ and $F(a)$, it is obvious that we have $F(a) - \frac{1}{2} F(a^2) = f(a)$; and thus three equations are obtained that are sufficient to determine the quantities $F(a)$, $f(a)$,

and $F(a^2)$; put $b = a^2 = \frac{3-\sqrt{5}}{2}$; and we thus get

$$F(a) = a + \frac{a^2}{2^2} + \frac{a^3}{3^2} + \&c. = \frac{\pi^2}{10} - \log.^2 a;$$

$$f(a) = a - \frac{a^2}{2^2} + \frac{a^3}{3^2} - \&c. = \frac{\pi^2}{15} - \frac{1}{2} \log.^2 a;$$

$$F(a^2), \text{ or } F(b) = b + \frac{b^2}{2^2} + \frac{b^3}{3^2} + \&c. = \frac{\pi^2}{15} - \frac{1}{2} \log.^2 b;$$

where $a = \frac{\sqrt{5}-1}{2}$ and $b = \frac{3-\sqrt{5}}{2}$.

Suppose $y = \frac{1-x}{1+x}$: then $-\int \frac{dx}{x} \log. y = \int \frac{dx}{x} \log. \frac{1+x}{1-x} = 2\phi(x)$: and because $x = \frac{1-y}{1+y}$, therefore, in like manner, $-\int \frac{dy}{y} \log. x = 2\phi(y)$: therefore

$$\phi(x) + \phi(y) = -\frac{1}{2} \int \left\{ \frac{dx}{x} \log. y + \frac{dy}{y} \log. x \right\} = c - \frac{1}{2} \log. x \times \log. y :$$

but when $x = 0$, then $\phi(x) = 0$, $\log. x \times \log. y = 0$, and

$$\phi(y) = \phi(1) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{\pi^2}{8} = c : \text{ therefore,}$$

$$\phi(x) + \phi\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{8} - \frac{1}{2} \log. x \times \log. \frac{1-x}{1+x} \dots\dots\dots (4)$$

let $e = \frac{1-e}{1+e}$; then $e^2 + 2e = 1$, and $e = \sqrt{2} - 1$: and by

putting $x = e$, the formula (4) gives

$$\phi(e) = e + \frac{e^3}{3} + \frac{e^5}{5} + \&c. = \frac{\pi^2}{16} - \frac{1}{4} \log.^2 e.$$

Again, a being $= \frac{\sqrt{5}-1}{2}$, we have obtained above,

$$f(a) + r(a) = \frac{\pi^2}{6} - \frac{1}{2} \log.^2 a: \text{ but } f(a) + r(a) = 2\phi(a):$$

therefore,

$$\phi(a) = a + \frac{a^3}{3} + \frac{a^5}{5} + \&c. = \frac{\pi^2}{12} - \frac{1}{4} \log.^2 a.$$

Lastly, let $d = \frac{1-a}{1+a} = \sqrt{5} - 2$: then, by substituting the value of $\phi(a)$ just obtained, in the formula (4), we obtain

$$\phi(d) = d + \frac{d^3}{3} + \frac{d^5}{5} + \&c. = \frac{\pi^2}{24} + \frac{1}{4} \log.^2 a - \frac{1}{2} \log. a \times \log. d.$$

The formulas (2) and (3) deserve notice, because the computation of $r(x)$, when x is less than 1, is, by them, reduced to the computation of $f(x)$, when x is less than 1. And, because $x = \frac{1+x}{1-\frac{1}{x}}$, therefore $\log. x = \log. (1+x) - \log. (1-\frac{1}{x})$: therefore

$$\frac{dx}{x} \log. (1-x) - \frac{dx}{x} \log. (1-\frac{1}{x}) = \frac{dx}{x} \log. x: \text{ therefore } f(x) + f(\frac{1}{x}) = c + \frac{1}{2} \log.^2 x: \text{ to determine } c \text{ put } x = 1, \text{ then } c = 2f(1) = \frac{\pi^2}{6}; \text{ therefore}$$

$$f(x) + f(\frac{1}{x}) = \frac{\pi^2}{6} + \frac{1}{2} \log.^2 x \dots\dots\dots (5).$$

By this formula, the computation of $f(x)$ when x is greater than 1, is reduced to the computation of $f(x)$ when x is less than 1. Thus the computation of $r(x)$ when x is less than 1, and of $f(x)$ in all cases whatsoever, is reduced to the computation of $f(x)$ when x is less than 1. Now this may be accomplished by a converging series as follows:

Let $z = \log. \sqrt{1+x}$ and let c be the number whose hyperbolic logarithm is 1: then $c^{2z} = 1+x$: therefore $dx = 2dz \times c^{2z}$, $x = c^{2z} - 1$, and $\frac{dx}{x} \log. (1+x) = \frac{4zc^{2z} \times dz}{c^{2z} - 1} = \frac{4zc^{2z} \times dz}{c^z - c^{-z}} =$

$$2adz + 2dz \times \frac{z(c^x + c^{-x})}{c^x - c^{-x}}. \text{ Let } \frac{z(c^x + c^{-x})}{c^x - c^{-x}} =$$

$$\frac{1 + \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6} + \&c.}{1 + \frac{z^2}{1.2.3} + \frac{z^4}{1.2.3.4.5} + \frac{z^6}{1.2.3.4.5.6.7} + \&c.}$$

be developed into

a series $1 + A^{(2)}z^2 - A^{(4)}z^4 + A^{(6)}z^6 - \&c.$ then by substituting, and taking the fluents, we get

$$f(x) = z^2 + 2 \times \left\{ z + A^{(2)} \cdot \frac{z^3}{3} - A^{(4)} \cdot \frac{z^5}{5} + A^{(6)} \cdot \frac{z^7}{7} - \&c. \right\}.$$



The reader who may wish for information respecting the summation of series of this form,

$$\pm x - \frac{x^3}{2^n} \pm \frac{x^5}{3^n} - \frac{x^7}{4^n} \pm \&c.$$

may consult *An Essay on the Theory of the various orders of Logarithmic Transcendents*, by William Spence, printed in 1809. This work shews that its ingenious author belonged to the higher class of analysts, and at the time it appeared, a just expectation was entertained that mathematical science would be advanced by his future labours: But bad health checked his progress, and he died at Glasgow, on the 20th of May, 1815, aged 35. We have heard that he had a work on Analysis prepared for the press at the time of his death.



QUESTION 478. (Vol. II.)

Additional Solution, by the Rev. Mr. L. Evans, Royal Military Academy.

Let $AE = CE = 80 = a$, $BE = z$, x and $y = \sin$ and cosine of $\angle ACB$ or the cosine and sine of $\angle ADB = \angle BAC$ radius 1. Then, by Prop. II. cor. 3, of Emerson's Trigonomet. $2xy$ will be the sine, and $y^2 - x^2$ will be the cosine of $\angle AEB$; and by trigonometry,

$$\sin \angle BAE : BE :: \sin \angle AEB : BA, \text{ i. e.}$$

$$y^2 - x^2 : z :: 2xy : \frac{2xyz}{y^2 - x^2} = AB;$$

$$\text{also } \sin \angle BAC : CB :: \sin \angle ACB : BA, \text{ i. e.}$$

$$y : z + a :: x : \frac{x}{y} \times (z + a) = AB,$$

and $\sin BAD : BD :: \sin ADB : AB$, i. e.

$$x : z + \frac{1}{4}a :: y : \frac{y}{x} \times (z + \frac{1}{4}a) = AB.$$

Therefore $\frac{x}{y} \times (z + a) = \frac{y}{x} \times (z + \frac{1}{4}a) = \frac{2xyz}{y^2 - x^2}$ the equi-pollent equation.

First, $\frac{x}{y} \times (z + a) = \frac{y}{x} \times (z + \frac{1}{4}a)$. Multiply both sides by xy , then we have $x^2 \times (z + a) = y^2 \times (z + \frac{1}{4}a)$, and $y^2 = \frac{z + a}{z + \frac{1}{4}a} \times x^2$. Let x^2 be subtracted from both sides, then $y^2 - x^2 = \frac{z + a}{z + \frac{1}{4}a} \times x^2 - x^2 = \frac{\frac{1}{4}ax^2}{z + \frac{1}{4}a}$. Substitute this for $y^2 - x^2$ in

$\frac{2xyz}{y^2 - x^2}$ the first value of AB , we have

$$\frac{2xyz}{\frac{\frac{1}{4}ax^2}{z + \frac{1}{4}a}} = \frac{2xyz \times (z + \frac{1}{4}a)}{\frac{1}{4}ax^2} = \frac{y}{x} \times \frac{8z}{3a} \times (z + \frac{1}{4}a). \text{ Now let}$$

the first and third values of AB , be equated, we then have $\frac{y}{x} \times \frac{8z}{3a} \times$

$(z + \frac{1}{4}a) = \frac{y}{x} \times (z + \frac{1}{4}a)$ or $\frac{8z}{3a} = 1$, hence $z = \frac{3}{8}a$, and $\sqrt{(AB^2 - x^2)} = AB$, the height of the steeple.

N. B. In the first solution to this question, line 6, for hypotenuse, read hypothesis; and for o in the diagram, read c.

QUESTION 641. (Vol II.)

Remarks on the two answers given to question 641; together with two New Solutions of the Problem.

By the Rev. Mr. John Hellins, B. D. F. R. S. and Vicar of Potter's Pury, in Northamptonshire.

Both of these answers are erroneous.

Mr. *Crakell's* error is the omission of the correction of the fluent of $\sqrt{\left(\frac{2xx-1}{xx-1}\right)}$ taken in descending series, as will quickly appear to any one who will finish his operation, and compare it with my papers on the *Rectification of the Hyperbola*, inserted in the *Philosophical Transactions* for the year 1802. It is to be regretted, that, in this instance, and in many others to be found in the periodical pamphlets of this

kind, part of the operation, and not unfrequently the most difficult part, is omitted: the effect of which is always a puzzling of the learners, (whose instruction should be the chief object of such publications) and sometimes a waste of the time of the learned in supplying omissions and correcting errors.

Mr. *Wildbore's* error is in the arithmetical calculation; Simpson's series which he used on that occasion being right, although its convergency is but slow.

The following general rectification of the Hyperbola will show where the errors of the before mentioned respectable mathematicians lie.

Let the transverse and conjugate semi-axes of an Hyperbola be denoted by 1 and b respectively; then will the eccentricity be $= \sqrt{1 + bb}$, which put $= e$. Let the abscissa, counted from the centre, be called x , and the length of the corresponding arch z , and we shall have the well-known equation $z = \dot{x} \sqrt{\left(\frac{eexx - 1}{xx - 1}\right)}$. Now

$$\dot{x} \sqrt{\left(\frac{eexx - 1}{xx - 1}\right)} \text{ is evidently } = \frac{ex\dot{x}}{\sqrt{(xx - 1)}} \times \sqrt{1 - \frac{1}{eexx}},$$

which is,

$$= \frac{ex\dot{x}}{\sqrt{(xx - 1)}} \times \left(1 - \frac{1}{2eexx} - \frac{1}{2.4e^2x^2} - \frac{3}{2.4.6e^3x^3} - \frac{3.5}{2.4.6.8e^4x^4}, \&c.\right)$$

$$= \left\{ \frac{ex\dot{x}}{\sqrt{(xx - 1)}} - \frac{\dot{x}}{\sqrt{(xx - 1)}} \times \left(\frac{1}{2ex} + \frac{1}{2.4e^3x^3} + \frac{3}{2.4.6e^5x^5} + \frac{3.5}{2.4.6.8e^7x^7}, \&c. \right) \right\}$$

Now, the fluent of $\frac{ex\dot{x}}{\sqrt{(xx - 1)}}$ is $e\sqrt{(xx - 1)}$; and if the fluents of $\frac{\dot{x}}{x\sqrt{(xx - 1)}}$, $\frac{\dot{x}}{x^3\sqrt{(xx - 1)}}$, $\frac{\dot{x}}{x^5\sqrt{(xx - 1)}}$, $\frac{\dot{x}}{x^7\sqrt{(xx - 1)}}$, &c. are denoted by A, B, C, D, &c. we shall have

A = circ. arch of which rad. is 1, and secant x ,

$$B = \frac{\sqrt{(xx - 1)}}{2xx} + \frac{1}{2}A,$$

$$C = \frac{\sqrt{(xx - 1)}}{4x^3} + \frac{3}{4}B,$$

$$D = \frac{\sqrt{(xx - 1)}}{6x^5} + \frac{5}{8}C,$$

&c. &c. &c. And, by multiplying these quantities by their proper factors, and placing them in due order, we shall have

$$(1). z = e\sqrt{(xx - 1)} - \frac{A}{2e} - \frac{B}{2.4e^3} - \frac{3C}{2.4.6e^5} - \frac{3.5D}{2.4.6.8e^7}, \&c.$$

which series, it is evident, will always converge; because, when x has any magnitude, x is greater than 1. It is obvious also, that, when x is = 1, then z is = 0, at which time A also is = 0, and $\sqrt{(xx-1)}$ is = 0, and consequently b , and c , and d , &c. are, each, = 0; so that this series needs no correction. It is further evident, that, when x becomes immensely great, the first term $e\sqrt{(xx-1)}$, will become barely ex , which denotes the length of the

asymptote corresponding to x ; and then $\frac{\sqrt{(xx-1)}}{2xx}$, and $\frac{\sqrt{(xx-1)}}{4x^3}$,

and $\frac{\sqrt{(xx-1)}}{6x^5}$, &c. becoming, each, = 0, we shall have $b = \frac{1}{2}A$,

and $c = \frac{3}{2.4}A$, and $d = \frac{3.5}{2.4.6}A$, &c. And these va-

lues being written for b , and c , and d , &c. in the series, we have, in this case,

$$(2). z = ex - A \times \left(\frac{1}{2e} + \frac{1}{2.2.4e^3} + \frac{3.3}{2.2.4.4.6e^5} + \frac{3.3.5.5}{2.2.4.4.6.6.8e^7} + \&c. \right)$$

an expression exactly agreeing with that in Simpson's Fluxions, A being = the quadrantal arch of a circle of which the radius is 1. And this series, it is evident, expresses the difference of the lengths of an infinite arch of an Hyperbola and its asymptote; and, when $e = \sqrt{2}$, its value is well known to be 0.5990701.*

The equation marked (1), as far as I know, is a new theorem. Let us now compute the value of z by it, writing $\sqrt{2}$ instead of e , and 1000 instead of x .

With these values of e and x we have $e\sqrt{(xx-1)} = 1414.2128553$. The value of A , continued to seven places of decimals, will differ from

q the quadrantal arch, by only $\frac{1}{1000}$, and may conveniently be de-

noted by $q - \frac{1}{1000}$. The value of $\frac{\sqrt{(xx-1)}}{2xx}$ is $\frac{1}{2000}$; from which

we have $b = \frac{1}{2000} + \frac{A}{2} = \frac{1}{2000} + \frac{q}{2} - \frac{1}{2000}$, barely = $\frac{1}{2}q$. And

since $\frac{\sqrt{(xx-1)}}{4x^3}$ has no value in nine places of decimals, and

$\frac{\sqrt{(xx-1)}}{6x^5}$, and the other algebraic expressions in which $\sqrt{(xx-1)}$

is found, are of less value, all of them may safely be omitted: so that

we shall have $c = \frac{3}{2.4}b = \frac{3}{2.4}q$, and $d = \frac{3.5}{2.4.6}c$, &c. And

* See the Philosophical Transactions for 1702, pp. 463 and 465.

these values of A, B, C, D , &c. being written for them in the equation numbered (1), the series there given will become $\frac{1}{2000e} - q \times \left(\frac{1}{2e} + \frac{1}{2 \cdot 2 \cdot 4e^3} + \frac{3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6e^5} + \frac{3 \cdot 3 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8e^7} \right.$ &c.); the value of which (e being $= \sqrt{2}$, and $q = 1.5707963$), is $0.0003535 - 0.5990701$. We, therefore, now have $z = 1414.2128553 + 0.0003535 - 0.5990701 = 1414.2132088 - 0.5990701 = 1413.6141387$, the length required; differing from the answer given by Mr. *Crankell* by the whole of the value of the series given in the equation numbered (2).

Mr. *Wildbore* erroneously made the sum of the infinite series last mentioned (e being $= \sqrt{2}$, and $q = 1.5707963$) to be $= 0.32111$, instead of 0.5990701 . And, when this error is corrected, the result will agree with that which I have now given.

But the computation of the length of any arch of an equilateral Hyperbola may be much more easily made by theorem vii, in p. 458 of the *Philos. Trans.* for the year 1802, than by the general theorem above given.

For the satisfaction of those who have not that vol. of the *Trans.* by them, I will here give the investigation of that theorem, and apply it to answer the 641st question in the *Ladies' Diary*.

If the transverse semi-axis of an equilateral Hyperbola be called 1, the ordinate y , and the corresponding arch z ; then (as is well known,)

will $z = y \sqrt{\left(\frac{1 + 2yy}{1 + yy} \right)}$. Put $uu = 1 + 2yy$, and we shall obtain

$$z = \frac{uuu}{\sqrt{(u^2 - 1)}} = u \left(1 + \frac{1}{2u^2} + \frac{3}{2 \cdot 4u^4} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6u^6} \right. \text{&c.} \text{); and,}$$

by taking the fluents, and correcting those on the second side of the equation, (because when y is $= 0$, then z is $= 0$, and u is $= 1$,) we

$$\text{shall have } z = \begin{cases} u - \frac{1}{2 \cdot 3u^3} - \frac{3}{2 \cdot 4 \cdot 7u^7} - \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 11u^{11}} \\ -d \end{cases} \quad (3)$$

where d , the correction of the fluent taken in descending series, is evidently $= 1 - \frac{1}{2 \cdot 3} - \frac{3}{2 \cdot 4 \cdot 7} - \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 11}$, &c. $= 0.5990701$.*

The application of this theorem to the case now before us, (as it stands in p. 472 of the *Philos. Trans.* for 1802,) is as follows:

" Since $y = \sqrt{(1000000 - 1)}$, we have $u = \sqrt{(2yy + 1)} = \sqrt{2} \times \sqrt{(1000000 - \frac{1}{2})} = \sqrt{2} \times (1000 - \frac{1}{4000})$ very nearly, =

* Methods of summing this slowly converging series are well known; and its value, when once found, will serve for innumerable arches of the Hyperbola.

problem, but a solution of it was given by Dr. Taylor in the Philosophical Transactions for 1717, which answered the question in its full extent.

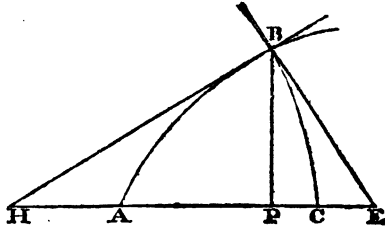
Other solutions were also given in the Leipsick Acts by John Bernoulli, the two Nicholas Bernoullis, and Herman : These, as well as Newton's and Taylor's are all collected in vol. second of Bernoulli's works published at Lausanne.

Since that time the problem of Trajectories has been solved, either partially, or generally, by most of the writers on Fluxions ; their solutions are chiefly founded on the same principle as the following

Solution.

Though the case of the right angle is easily deduced from the general one, yet perhaps it may be as well to consider that case separately.

Let AB be a curve of which $f(x', y', a) = 0$ is the equation, x' and y' being the co-ordinates AP and PB , and a a quantity which is constant for the same curve, but varies in passing from curve to curve, so that by giving it all possible values an indefinite number of curves will be formed, and it is required to find the curve BC that shall cut them all at right angles. Draw the tangents BH and BE .



Because HBE is a right angle, BE is a normal to the curve AB ; therefore PE is the subnormal to AB , and it is also the subtangent to BC , wherefore the subtangent of the required curve must be equal to the subnormal of the given one ; and as this condition must be satisfied at every point of intersection of the curves, or for every value of a , it is manifest that it must be expressed independent of that quantity.

The value of the subnormal is $= \frac{y' dy'}{dx'}$, and

of the subtangent $\frac{-y dx}{dy}$.

Therefore $\frac{y' dy'}{dx'} = \frac{-y dx}{dy}$;

or writing x for x' and y for y' , because at the point B the two curves have the same co-ordinates,

$$\frac{dy'}{dx'} = \frac{-dx}{dy} \dots\dots\dots (a)$$

Therefore if the value of $\frac{dy'}{dx'}$ be found from the given equation

$f(u', y', a) = 0$, by taking the fluxion and exterminating a , and be substituted in equation (a) it will give the fluxional equation of the curve sought.

Taking for example the parabola of which the equation is $ax^m = y^n$; then $\frac{dy'}{dx'} = \frac{amx^{m-1}}{ny^{n-1}}$, and putting for a its value $\frac{y^n}{x^m}$, deduced from the equation $ax^m = y^n$,

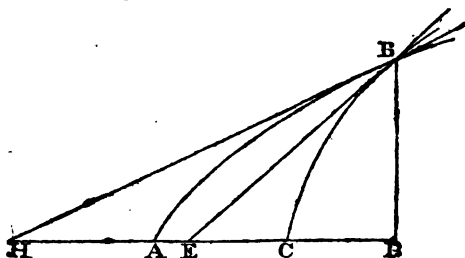
$$\frac{dy'}{dx'} = \frac{my'}{nx} = \frac{my}{nx}; \text{ therefore from equation (a)}$$

$$\frac{my}{nx} = \frac{-dx}{dy}$$

$$\text{or } \frac{my}{nx} + \frac{dx}{dy} = 0.$$

Therefore $mydy + nxdx = 0$, the fluent of which is $my^2 + nx^2 = c$; an equation to an ellipse whose centre is A, and axes in the proportion of $\sqrt{n}$ to $\sqrt{m}$.

Next, when the angle HBE is not a right angle. Because HBE is the difference of the angles HBP and EBP,



$$\tan HBE = \frac{\tan HBP - \tan EBP}{1 + \tan HBP \tan EBP},$$

or if t be put for the tangent of the given angle HBE,
 $t \times (1 + \tan HBP \tan EBP) = \tan HBP - \tan EBP.$

$$\text{But } \tan HBP = \frac{PH}{PB} = \frac{dx'}{dy'};$$

$$\text{and } \tan EBP = \frac{PE}{PB} = \frac{dx}{dy};$$

therefore, by substitution,

$$t \left(1 + \frac{dx'}{dy'} \frac{dx}{dy} \right) = \frac{dx'}{dy'} - \frac{dx}{dy};$$

$$\text{or } \frac{dy'}{dx'} = \frac{dy - tdx}{dx + tdy} \dots \dots \dots (b)$$

and substituting for $\frac{dy'}{dx'}$ its value, found as before, we shall have the fluxional equation of the curve.

When the given angle is a right one, t is infinite, consequently its factor

$$1 + \frac{dx'}{dy} \frac{dx}{dy} \text{ must be } = 0; \text{ therefore}$$

$$\frac{dy'}{dx'} + \frac{dx}{dy} = 0, \quad \frac{dy'}{dx'} = -\frac{dx}{dy}, \text{ the same as before.}$$

Or, if, by taking fluxions, we find $dy' = p dx'$, and substitute p for $\frac{dy'}{dx'}$ in formula (b), it gives

$$t(p dy + dx) + p dx - dy = 0, \dots\dots\dots (c)$$

and in the case of the right angle,

$$p dy + dx = 0.$$

In the preceding example, for the parabola, $\frac{dy'}{dx'}$ was found $= \frac{my}{nx}$, therefore from equation (b),

$$\frac{my}{nx} = \frac{dy - t dx}{dx + t dy'}$$

or $t(n x dx + m y dy) + m y dx - n x dy = 0$, which may be integrated by putting $y = xz$. When $m = n = 1$ the equation becomes

$$t(x dx + y dy) + y dx - x dy = 0,$$

or dividing the whole by $x^2 + y^2$;

$$t \times \left(\frac{x dx + y dy}{x^2 + y^2} \right) + \frac{y dx - x dy}{x^2 + y^2} = 0,$$

the fluent of which is

$$t . l (x^2 + y^2)^{\frac{1}{2}} + \text{arc} \left(\tan = \frac{x}{y} \right) = \text{constant},$$

$$\text{or } t . l \left(\frac{x^2 + y^2}{c} \right)^{\frac{1}{2}} = \text{arc} \left(\tan = \frac{x}{y} \right)$$

an equation to the logarithmic spiral.

It is indeed a well known property of that curve that it cuts all its radii vectores in a given angle, and the given curves are then a series of straight lines described from the equation $ax' = y'$.

If any difficulty arises in exterminating a we must try to obtain one of the variables x' or y' in terms of the other and the constant a .

Suppose, for instance, that we can find y' in terms of x' and a , the ordinate consecutive to y' will be found by making both x' and a to vary, that is

dy' will be $= p dx' + q da$ and
 $dy = p dx + q da$ changing x' and y' into x and y ; and
 substituting this value of dy in equation (c) we get

$$t(p^2 + 1) dx + t(py - q) da = 0 \dots \dots \dots (d)$$

and by integrating this equation we shall obtain the relation between the constant a and the abscissa x .

QUESTION 669. (Vol. II.)

*Remarks on the Answer given to Question 669; together with a
 New Solution.*

*By the Rev. Mr. John Hellins, B. D. F. R. S. and Vicar of Potter's
 Pury.*

This answer is grounded on a supposition that the annuity, after it commences, is to continue certain for near 28 years; whereas, according to the question, both its commencement and its continuance must depend upon a contingency; viz. the life of the person. The plain meaning of the question is this: 'A man aged 24 years gives 1000*l.* for an annuity of 100*l.* per annum, to commence at a future time and to continue till he attains the age of 60 years, (if he should live so long,) but to cease immediately after his death, if that should happen before he attains the age of 60: Now, allowing interest of money at 5 per cent., at what time ought he to receive the first 100*l.*?'

Solution:

The easiest way of solving questions of this kind, that I know of, is by trial and error, as follows:

By Dr. Price's Tables,* grounded upon the Northampton bills of mortality, the present value of an annuity of 1*l.* per annum, on the whole life of a person aged 24 years, allowing 5 per cent. interest of money, is 13.658*l.* If from this we subtract the present value of an annuity of 1*l.* on the life of the same person after he attains the age of 60 years, which, by the same Tables, is $(8.392 \times 0.172657 \times \frac{2038}{4835}) = 0.61074*l.*$ the remainder is 13.04726*l.* which is the value of an annuity of 1*l.* on the life of a person aged 24 years to continue till he attains the age of 60 years. This value, we see, is too great by a little more than three years purchase; the number of years

* I use these Tables because they are generally known.

purchase given for the annuity being only 10. Let us now compute the present values of an annuity of 1*l.* on the same life, to be deferred 3 years, and 4 years, and to end at the age of 60, as before; and the results will be 10·4071*l.* and 9·6353*l.* respectively. Then, by proportion, we have, as 0·7718 (the difference of these values) is to 1 year, so is 0·4071 to 0·5275 of a year, or 6 months and 10 days; which added to the 3 years give the time when the title to the annuity commences. But it is to be considered, that, when a person purchases an annuity (properly so called,) or an estate, the first year's rent does not become due till the end of one year; and this is the time which must be added to that above found, in order to have the time when Jack ought to receive the first 100*l.*; which, therefore, is at the end of 4 years, 6 months, and 10 days.

Now, for the reason above given, 1 ought to be added to the value of *t*, in the answer that was printed to this question; which would make it 9·127 years, or 9 years and 46 days; the error being no less than 4 years, 6 months, and 36 days!

—————
QUESTION 768. (Vol. III.)

Additional Solution. By Mr. Wallace, R. M. College.

This problem requires the greatest triangle possible to be inscribed in a given circle, so that two of its sides may have to each other a given ratio. As the Diary solution seems to be considerably intricate, we shall give another which appears to be more simple.

Let *A*, *B*, *C* denote the three angles of the triangle, and let the sides *a*, *b*, opposite to the angles *A*, *B*, be those which are to have a given ratio. Put *d* for the diameter of the circumscribing circle: Then $a = d \sin A$, $b = d \sin B$. Now, by a known theorem, the area is expressed by $\frac{1}{2}ab \sin C = \frac{1}{2}d^2 \sin A \sin B \sin C$; therefore,

$$\sin A \sin B \sin C = \text{a maximum.}$$

But, by trigonometry, $\sin A : \sin B :: a : b$, that is in a given ratio, therefore $\sin A \sin B$ will have a given ratio to $\sin^2 A$, and hence

$$\sin^2 A \sin C = \text{a max.}$$

and at the same time, $\sin B = \frac{b}{a} \sin A$.

From the first of these equations, by taking the fluxions, we have

$$\frac{\dot{\sin C}}{\sin C} = \frac{2 \cos A}{\sin A} \times \frac{\dot{\sin C}}{\cos C} = \frac{2 \tan C}{\tan A}.$$

But since $A + B + C = 2$ right angles,

$$\frac{\dot{\sin C}}{\sin C} = 1 + \frac{\dot{B}}{B}, \text{ therefore,}$$

$$1 + \frac{\dot{B}}{A} = \frac{2 \tan c}{\tan A}.$$

The other equation, $\sin B = \frac{b}{a} \sin A$, by taking the fluxions, and then exterminating $\frac{b}{a}$, gives

$$\frac{\dot{B}}{A} = \frac{\cos A}{\sin A} \times \frac{\sin B}{\cos B} = \frac{\tan B}{\tan A}.$$

Therefore, $1 + \frac{\tan B}{\tan A} = \frac{2 \tan c}{\tan A}$, and hence

$$\tan A + \tan B = 2 \tan c \dots \dots \dots (1)$$

By this elegant property of the triangle, and the given ratio of the sides opposite to A and B, the triangle might be found.

By the arithmetic of sines

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Now in any triangle, $\tan (A + B) = -\tan c$, therefore in the triangle in question,

$$\frac{1}{2} (\tan A + \tan B) = -\frac{\tan A + \tan B}{1 - \tan A \tan B}$$

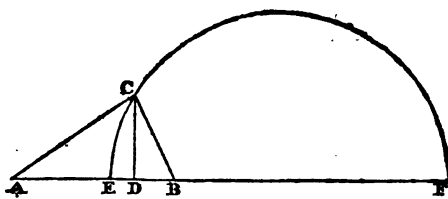
from which it follows that

$$\tan A \tan B = 3 \dots \dots \dots (2)$$

This is another very elegant property of the triangle required.

To apply this property to the construction of the triangle, let us suppose that ABC is a triangle similar to it, and that AB the base is given. In AB take E and F, so that AE may be to EB, and AF to FB in the

given ratio of the sides AC, CB. Then, by a well known theorem, the point C is in the circumference of a circle, having EF for its diameter,



which will be given, because the points E, F are given. Draw CD

perpendicular to EF: Then $\tan A = \frac{CD}{AD}$, and $\tan B = \frac{CD}{BD}$, hence,

$\tan A \tan B = \frac{CD^2}{AD \cdot BD}$. Now we have found that $\tan A \tan B = 3$, therefore

$$3AD \times BD = CD^2 = ED \times DF.$$

The problem is now evidently reduced to this; A, B, E, F are four given points in a straight line; it is required to determine a fifth point D, so that $AD \times DB$ may have to $ED \times DF$ the given ratio of 1 to 3, which is a case of the well known problem of *Determinate Section*; see Leslie's *Geometry, Geometrical Analysis*, Book 2nd. Also Wales and Lawson on *Determinate Section*. The point D being found as directed by these writers, which may be determined in various ways, the perpendicular DC will determine the point C, and the triangle ABC, which is similar to that required.

Having shewn how the problem may be constructed generally from the formulæ,

$$\tan A \tan B = 3, \quad \frac{\sin A}{\sin B} = \frac{a}{b},$$

we shall now investigate trigonometrical expressions for the angles of the triangle.

$$\text{Because } \sin A = \frac{\sqrt{(1 + \tan^2 A)}}{\tan A} \text{ and } \sin B = \frac{\sqrt{(1 + \tan^2 B)}}{\tan B},$$

therefore, from the second equation, we get

$$\frac{\tan^2 A}{1 + \tan^2 A} \times \frac{1 + \tan^2 B}{\tan^2 B} = \frac{a^2}{b^2},$$

and hence, by multiplying the numerators and putting for $\tan A \tan B$ its value, found from the first equation, we get

$$\frac{\tan^2 A + 9}{\tan^2 B + 9} = \frac{a^2}{b^2},$$

and hence again

$$b^2 \tan^2 A - a^2 \tan^2 B = 9(a^2 - b^2).$$

Let ϕ be such an angle that $a^2 \tan^2 B = 3ab \tan \phi$, then because $\tan A \tan B = 3$, we have $b^2 \tan^2 A = 3ab \cot \phi$, and $3ab(\cot \phi - \tan \phi) = 9(a^2 - b^2)$.

$$\text{But } \cot \phi - \tan \phi = 2 \cot 2\phi = \frac{2}{\tan 2\phi},$$

$$\text{therefore } \frac{2ab}{\tan 2\phi} = 3(a^2 - b^2).$$

From the preceding investigation we derive this rule.

Find an angle ϕ such, that

$$\tan 2\phi = \frac{2ab}{3(a-b)(a+b)}.$$

$$\text{Then, } \tan A = \sqrt{\frac{3a \times \cot \phi}{b}}, \quad \tan B = \sqrt{\frac{3b \times \tan \phi}{a}}.$$

QUESTION 788. (Vol. III.)

In order to exterminate the second term in the usual way, let $x = y + \frac{2}{3} = y + \frac{1}{3}$; this substituted for x in the given equation we have $y^2 - \frac{2}{3}y + \frac{4}{9} = a$, a quadratic form; hence $y = \frac{1}{3}\sqrt{3 \pm 2\sqrt{4a+1}}$ & $x = y + \frac{1}{3} = \frac{1}{3}\{1 + \sqrt{3 \pm 2\sqrt{4a+1}}\}$.

QUESTION 828. (Vol. III.)

Demonstration of the rule for finding the area of a trapezium inscribed in a circle.

LEMMA. If from the extremities of any chord in a circle, perpendiculars be drawn to a diameter, and from either extremity of that diameter a perpendicular be drawn to the chord, it will divide it into segments, which are mean proportionals between the segments of the diameter made by the perpendiculars.

Let AC be a chord and EF a diameter, CH and AG perpendiculars to EF, and FK perpendicular to AC; the segment KC is a mean proportional between FH and EG, and AK a mean proportional between FG and EH;

Join AE, AF, FC, and EC; the triangles KCF, AEF and AEG are similar, as are also the triangles FCH and FEC; therefore

$$FC : KC :: FE : AE$$

$$FH : FC :: FC : FE$$

therefore ex aeq. $FH : KC :: FC : AE$,

$$\text{or } FH : FC :: KC : AE;$$

$$\text{but } FC : KC :: AE : EG;$$

$$\text{therefore } FH : KC :: KC : EG.$$

In a similar way it is shewn that

$$FG : AK :: AK : EH.$$

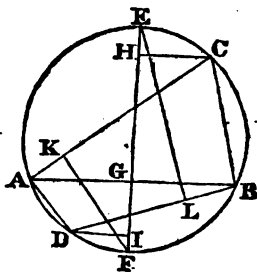
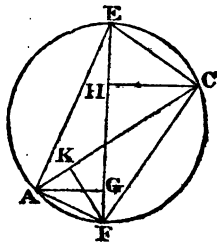
To apply this to the demonstration of the rule in the question, let ACBD be a trapezium inscribed in a circle, EF a diameter drawn perpendicular to one of the diagonals AB, and draw DI and CH perpendicular to EF, FK to AC, and EL to BD: By the property of the circle $EG : AG :: AG : GF$, therefore

$$EG \cdot HI : AG \cdot HI :: AG \cdot HI : GF \cdot HI.$$

But $AG \cdot HI$ = the trapezium ACBD, therefore

$$EG \cdot HI : \text{trapez.} :: \text{trapez.} : GF \cdot HI.$$

By the lemma $EG \cdot HF = KC^2$, $GF \cdot HE = AK^2$, $GF \cdot IE = DL^2$, $EG \cdot FI = LB^2$;



therefore $EG \cdot HI = KC^2 - LB^2$, and $GF \cdot HI = DL^2 - AK^2$,
therefore $KC^2 - LB^2 : \text{trap.} :: \text{trap.} : DL^2 - AK^2$.

Now it is evident that CK is half the sum of the sides AC, BC ; $DL =$ half the sum of the sides BD and AD ; $BC = CK - AK$, and $AD = DL - BL$; therefore $KC + DL$ is $=$ half the perimeter, and the excesses of half the perimeter above each of the sides AD, BD, CB , and AC are $KC + LB, KC - LB, DL + AK$ and $DL - AK$ respectively:

$$\text{But } (KC + LB)(KC - LB) = KC^2 - LB^2,$$

$$\text{and } (DL + AK)(DL - AK) = DL^2 - AK^2;$$

therefore the trapezium is a mean proportional between $(KC + LB)(KC - LB)$ and $(DL + AK)(DL - AK)$: or "between the rectangle under the excesses of half the perimeter above two of the sides and the rectangle under its excesses above the other two sides."

Cor. "Every trapezium inscribed in a circle is a mean proportional between the excess of the square of half the sum of two sides above the square of half the difference of the other two, and the excess of the square of half the sum of the latter above the square of half the difference of the former."

J. L.

QUESTION 918. (Vol. III.)

1. *Remarks by the author of the original Solution.*

The answer to this question is not conclusive. Professor Bugge, however, would have been right in estimating the effect, or depth to which a pile sinks at each stroke according to the square of the velocity, provided the resistance continued uniform: but it is known by experiment that the resistance increases with the depth, so that at last, the repeated strokes from the highest point to which the ram or weight can be raised, have no effect in driving the pile. Could the law of resistance be ascertained, it would not be difficult to compute how high the weight must be raised that would sink the pile to any proposed depth. And hence we may conclude that the higher the ram is raised, the greater will be the effect in a given time.

II. *Observations by Mr. P. Barlow, of the Royal Military Academy, Woolwich.*

This question is one which excited considerable interest amongst the Diarian correspondents at the time it was first proposed, in consequence of the obviously erroneous result which arose out of the mathematical investigation, viz. "that the effect would be the greatest when the ram was laid upon the top of the pile" estimating the momentum by the velocity into the mass. Or on a supposition

of its being measured by the mass into the square of the velocity, then "the effect would be the greatest the higher the ram was drawn up." It is obvious that these results are both erroneous, and it may therefore not be amiss to endeavour to discover what error in the solution has given rise to so singular a conclusion. I have already attempted this, in my "Dictionary," under the article PERCUSSION; but I intend in the present instance to take a rather different view of the subject, although the principle in both cases are the same.

It is easy to conceive that a pile may be driven so far into the ground, that no power which it is possible to apply in the usual way, can have an effect in producing *any* further motion in it. We are not, however, left to mere supposition and conjecture on this head, as this circumstance really occurred in driving the piles for the bridge of some mills, where the soil upon which the bridge was to be erected was pure sand of uniform density to the depth of 47 feet, yet no piles could be driven lower than 15 feet, although the rammers in the engines employed were enormously large. *Gregory's Mechanics*, vol. i, p. 292.

Hence it appears, that it is possible to drive a pile so far that all future operations of the ram become wholly useless; yet there can be no doubt that if a sufficient momentum was applied, motion might be produced. Hence it follows, that whatever may have been said with regard to the incongruity of percussion and pressure, there is in reality a momentum which is precisely equivalent to the pressure and resistance of the pile; for since, as in the case we have mentioned, a momentum may be too small to produce any motion, while with a greater momentum, motion will ensue, it follows that some where between the two a momentum may be found, that is just sufficient to produce an equilibrium between itself and the resistance and pressure of the pile.

This circumstance, it is obvious, must always have place in all operations of this kind; that is, after the pile has been driven to a little depth, the resistance will be such as to require a certain momentum to produce an equilibrium in the pile, and any momentum less than this will have no effect, however often repeated, in producing the required motion, and therefore if any greater momentum be applied, it is obviously only the difference between this and that which is expended in putting the pile in equilibrium that becomes effective, and which ought alone to be considered in computing the effect of the engine, and it is doubtless for want of duly attending to this circumstance that so many erroneous solutions have been given to this question. As to the momentum that may be necessary for producing the required equilibrium, it is obvious from the various circumstances that may arise in the application of the pile engine, that this can only be determined by experiment, on some principle similar to the following: Suppose that by experiment it is found that a momentum m drives the pile a inches; and a momentum n drives it b inches; and let be required, hence, to determine the momentum y , which is equal to the resistance of the pile.

The first effective momentum is $m - y$, and the second $m - y$; and now admitting each effect proportional to its cause, we have $m - y : m - y :: a : b$, whence

$$bm - by = am - ay, \text{ or } (a - b)y = am - bm,$$

or $y = \frac{am - bm}{a - b}$; which is the momentum equal to the resistance

of the pile. Find the height due to this momentum, and call it h , then the solution of the problem will stand as follows: (The remainder of the solution I shall transcribe from the article in my Dictionary above quoted, my only object having been to place the principles in a little different light, in consequence of some objections that have been made to them as given in that article.)

“ Let x be the required height to produce the maximum,

a the velocity per second in drawing up,

t the time in falling,

h the height necessary to produce the equilibrium,

$g = 16\frac{1}{12}$ feet,

w the weight of the ram.

Then supposing the momentum to vary as the velocity, we have

$$2gt = v; vw = m, \text{ the generated momentum.}$$

$$\frac{x}{a} = \text{time in drawing up.}$$

$$\frac{\sqrt{x}}{\sqrt{g}} = \text{time in falling} = t.$$

$$\frac{x\sqrt{g} + a\sqrt{x}}{a\sqrt{g}} = \text{the whole time.}$$

$$2\sqrt{gx} = v, \text{ or } 2w\sqrt{gx} = m, \text{ by substituting } \frac{\sqrt{x}}{\sqrt{g}} \text{ instead of } t.$$

And h being the height due to the momentum above determined, this will be expressed by $2w\sqrt{gh}$, and consequently $2w\sqrt{gx} - 2w\sqrt{gh} = 2w(\sqrt{gx} - \sqrt{gh})$ is the effective momentum, which divided by the time necessary for the production of it, must be a maximum, that is,

$$\frac{2w(\sqrt{gx} - \sqrt{gh})}{x\sqrt{g} + a\sqrt{x}} \times a\sqrt{g} = \text{a max., or}$$

$$\frac{\sqrt{x} - \sqrt{h}}{x\sqrt{g} + a\sqrt{x}} = \text{a max.}$$

This thrown into fluxions, gives

$$\frac{1}{2x} (x\sqrt{g} + a\sqrt{x}) - \sqrt{gx} + \sqrt{gh} - \frac{x}{2\sqrt{x}} (\sqrt{x} - \sqrt{h}) \Big\} = 0,$$

which reduced, gives

$$\frac{1}{2}\sqrt{gx} = \frac{1}{2}a\sqrt{\frac{h}{x}} + \sqrt{gh}, \text{ or}$$

$$x\sqrt{g} = a\sqrt{h} + 2\sqrt{gkx}, \text{ or}$$

$$x - 2\sqrt{hx} = a \frac{\sqrt{h}}{\sqrt{g}};$$

whence

$$\sqrt{x} = \sqrt{h} \pm \sqrt{\left(h + a \frac{\sqrt{h}}{\sqrt{g}}\right)}, \text{ or}$$

$$x = \left\{ \sqrt{h} \pm \sqrt{\left(h + a \frac{\sqrt{h}}{\sqrt{g}}\right)} \right\}^2;$$

which is the height required."

In the above solution I have only determined the height when any single momentum divided by the time necessary for the production of it, is a maximum; whereas the problem requires that the sum of all the momenta divided by the whole time should be a maximum. In order to determine this, we must assume a certain law of resistance, to which applying the above principle of solution, a general result will be obtained. But as this assumed law of resistance must be totally suppositions, and probably very different from that which arises in practice, the preceding determination (supposing the value of h to have been found by experiment towards the conclusion of the operation of the engine), is probably more accurate than that which would result from the above supposition of a variable law of resistance. For it must be observed that x , as found above, is always the same function of h , that h increases as the blows are repeated, and consequently x also increases each blow; but the distance between the top of the pile and the top of the engine also practically increases each stroke by the depth the pile has penetrated. It is therefore probable that the value of x found as above for any single stroke is very nearly, if not accurately, that which answers for the whole sum.

QUESTION 967. (Vol. III.)

Additional Solution.

Let AB be any position of the hypotenuse, and CP the perpendicular upon AB . Put $z = CP$,
 $\phi =$ the angle BCP , and $a =$

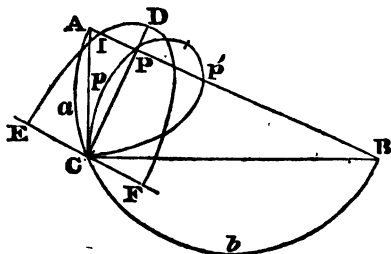
$$AB: \text{Then } AP = \frac{z \cos \phi}{\sin \phi} \text{ and}$$

$$BP = \frac{z \sin \phi}{\cos \phi};$$

therefore, since $AP + PB = AB$,

$$\frac{z \sin \phi}{\cos \phi} + \frac{z \cos \phi}{\sin \phi} = a,$$

$a = z \text{ or } \sin \phi \cos \phi$, the equation of the curve



To find the area, $z^2 = a^2 \sin^2 \phi \cos^2 \phi = a^2 \sin^2 \phi (1 - \sin^2 \phi)$,
whence, by completing the square,

$$\sin^2 \phi = \frac{1}{2} \pm \frac{1}{2a} \sqrt{(a^2 - 4z^2)},$$

and taking fluxions

$$2\dot{z} \sin \phi \cos \phi = \frac{\mp 2z\dot{z}}{a\sqrt{(a^2 - 4z^2)}},$$

$$\text{or } z\dot{\phi} = \frac{\mp z\dot{z}}{\sqrt{(a^2 - 4z^2)}};$$

$$\text{therefore } \frac{z^2\dot{\phi}}{2} = \frac{\mp z^2\dot{z}}{2\sqrt{(a^2 - 4z^2)}}, \text{ the fluxion of the area,}$$

the fluent of which is evidently equal to a fourth part of the semi-circular segment whose ordinate is z and diameter a . That is, if a semicircle ACB be described on the given line AB , the area cpr is a fourth part of the segment $APCA$, and the area rp/c a fourth part of the segment $BRCB$, therefore the area of the whole curve is a fourth part of the semicircle described on AB .

$$\text{Again, for the rectification, because } z\dot{\phi} = \frac{\mp z\dot{z}}{\sqrt{(a^2 - 4z^2)}}$$

$$z^2\dot{\phi} \text{ is } = \frac{z^2\dot{z}}{a^2 - 4z^2}, \text{ and } \frac{z^2\dot{\phi}^2}{z^2} = \frac{z^2}{a^2 - 4z^2};$$

$$\text{therefore } z\sqrt{(1 - \frac{z^2\dot{\phi}^2}{z^2})} = z\sqrt{(\frac{a^2 - 3z^2}{a^2 - 4z^2})}, \text{ the fluxion of the}$$

curve; but this expression is the fluxion of the elliptic arc intercepted between the extremities of the minor axis and an ordinate at the distance z from the centre, when the major axis is a , and the

minor $\frac{a}{2}$, therefore the length of the curve is equal to the length of

the said elliptic arc. On cp produced take $cd = \frac{1}{2}AB$, and on ecf , drawn perpendicular to cd , take ce and cf each equal to $\frac{1}{2}cd$ or $\frac{1}{2}AB$, and to the semiaxes cd and ce describe the semi-ellipse edf meeting BA in i ; the curve cpr is equal the elliptic arc zi , the curve rp/c equal the elliptic arc idf ; and the whole length of the curve cpr/c equal the periphery of the semi-ellipse edf .

J. L.

QUESTION 981. (Vol. III).

Remark on the answer of Amicus to this question.

By the Rev. Mr. John Hellins, B. D. F. R. S.

This solution is both clumsy and erroneous. Putting $1 + u = \frac{1 + 2x + xx}{1 - x + xx}$, he says, "If x be not greater than $2 + \sqrt{3}$, u will be greater than unity."—Now it is obvious that, when $x = 0$, u is also $= 0$. The fact is, that the fraction $\frac{1 + 2x + xx}{1 - x + xx}$ has a maximum, which is when $x = 1$; and when x has any value between $2 - \sqrt{3}$ and $2 + \sqrt{3}$, u will be greater than unity.



QUESTION 993. (Vol. III.)

A new Solution, by the Rev. Mr. John Hellins, B. D. F. R. S. and Vicar of Potter's Pury.

Let AB be the radius of a circle, and BC the tangent of 45° . Draw AC , cutting the arch of the circle in the point D . Take any point e in the tangent BC , and draw Ae cutting the arch DB in the point f . If now we put $AB (= BC) = 1$, $ce = v$, and the

arch $bf = z$, we shall have $z = \frac{v}{2 - 2v + vv} =$

$z \times \text{the series } \frac{1}{2} + \frac{v}{2} + \frac{vv}{4} - \frac{1}{4} \left(\frac{v^3}{2} + \frac{v^5}{2} + \right.$

$\left. \frac{v^9}{4} \right) + \frac{1}{4.4} \left(\frac{v^3}{2} + \frac{v^5}{2} + \frac{v^{10}}{4} \right) - \frac{1}{4.4.4} \left(\frac{v^{13}}{2} + \frac{v^{15}}{2} + \frac{v^{14}}{4} \right), \&c.$

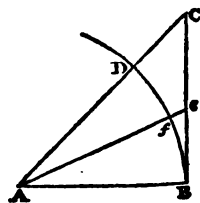
the law of continuation being obvious. And, by taking the fluents,

we have $z = \frac{v}{2} + \frac{vv}{2.2} + \frac{v^3}{4.3} - \frac{1}{4} \left(\frac{v^3}{2.5} + \frac{v^5}{2.6} + \frac{v^7}{4.7} \right) +$

$\frac{1}{4.4} \left(\frac{v^3}{2.9} + \frac{v^{10}}{2.10} + \frac{v^{11}}{4.11} \right) - \frac{1}{4.4.4} \left(\frac{v^{13}}{2.13} + \frac{v^{14}}{2.14} + \right.$

$\left. \frac{v^{15}}{4.15} \right), \&c.$ in which equation, z and v beginning together, no correction is necessary; and when v becomes $= 1$, z is $=$ the arch of

45° . We therefore have the arch of $90^\circ = 1 + \frac{1}{2} + \frac{1}{6} - \frac{1}{4} \left(\frac{1}{5} + \right.$



$$\frac{1}{6} + \frac{11}{14} + \frac{1}{4.4} \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{22} \right) - \frac{1}{4.4.4} \left(\frac{1}{13} + \frac{1}{14} + \frac{1}{30} \right),$$

Remark. This series, it is evident, admits of various transformations; and the ratio of the diameter of the circle to its circumference may easily be computed by it.

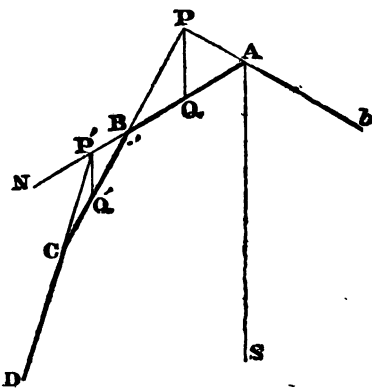
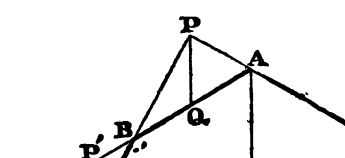
QUESTION 1087. (Vol. III.)

Additional Solution, by Mr. Wallace, R. M. College.

Because the spheres press on each other, and the supporting planes in the direction of straight lines drawn from their centres through the points of contact, the arch is exactly in the same state as if it were a polygon formed by straight lines joining the centres of every two contiguous spheres, and lines drawn from the centres of the lowermost perpendicular to the planes on which they rest: it being supposed that the lines are free to move about the angles, but that the whole is kept fast by a system of equal weights suspended from the angles, and in equilibrio.

Let BC be any side of the polygon, and BA , CD the adjoining sides ;

These will evidently have the same position as if the points A and D were absolutely fixed, and two equal weights were suspended from the extremities of the line BC , or both from Q' the middle of BC . The material line BC would then be kept in its position by three forces, viz. the two pressures on the fixed points A , D , which act in the directions AB , DC , and the force of gravity at Q' which acts in a vertical direction. But when three forces produce an equilibrium in this manner,



their directions must all pass through the same point : Hence we have this geometrical property of the figure : *A vertical line drawn from q' , the middle of bc any one of its sides, passes through e' the point in which the adjoining sides meet when produced.* From this property, the polygon may be determined thus : From any point A , draw two lines AB, Ab making angles of 60° on each side of the vertical line As . Take AB equal to the diameter of the spheres, and

A and B will be the centres of the uppermost and one of the adjoining spheres.

Bisect AB in q, draw qf parallel to As, meeting BA produced in r; join PB, and produce it so that BC = BA; then c will be the centre of the next sphere; and the centres of all the other spheres on each side may be determined by a like construction.

There is an elegant property of the angles by which the position of the lines may be found. Let α, β, γ be the angles which any three contiguous sides AB, BC, CD make with the vertical; then, in the triangles P'q'B, P'q'C, the angles NP'q' = α , P'q'B = β , CP'q' = γ , P'Bq' = $\alpha - \beta$, P'Cq' = $\beta - \gamma$; and because,

$$\frac{\sin P'Cq'}{\sin CP'q'} = \frac{P'q'}{Cq'} = \frac{P'q'}{Bq'} = \frac{\sin P'Bq'}{\sin BP'q'}$$

therefore $\frac{\sin(\beta - \gamma)}{\sin \gamma} = \frac{\sin(\alpha - \beta)}{\sin \alpha}$, and dividing by $\sin \beta$,

$$\frac{\sin(\beta - \gamma)}{\sin \beta \sin \gamma} = \frac{\sin(\alpha - \beta)}{\sin \alpha \sin \beta}.$$

But $\frac{\sin(\beta - \gamma)}{\sin \beta \sin \gamma} = \frac{\sin \beta \cos \gamma - \cos \beta \sin \gamma}{\sin \beta \sin \gamma} = \frac{\cos \gamma}{\sin \gamma} - \frac{\cos \beta}{\sin \beta}$,

and similarly, $\frac{\sin(\alpha - \beta)}{\sin \alpha \sin \beta} = \frac{\cos \beta}{\sin \beta} - \frac{\cos \alpha}{\sin \alpha}$;

therefore, $\frac{\cos \gamma}{\sin \gamma} - \frac{\cos \beta}{\sin \beta} = \frac{\cos \beta}{\sin \beta} - \frac{\cos \alpha}{\sin \alpha}$;

That is $\cotan \gamma - \cotan \beta = \cotan \beta - \cotan \alpha$; Thus it appears that the cotangents of the angles which the lines AB, BC, CD, &c. make with the vertical line form a series in arithmetical progression. Now, by hypothesis, the first AB makes with the vertical line As an angle of 60° ; and since Bq = Aq = qf, the angle BPA is a right angle, wherefore BPq the angle which the second makes with the vertical line, is 30° , and the common difference of the cotangents is $\cot 30^\circ - \cot 60^\circ = \tan 60^\circ - \tan 30^\circ = 2 \tan 30^\circ$. Hence, if we observe that $\cot 60^\circ = \tan 30^\circ = \sqrt{\frac{1}{3}}$, it follows that the cotangents of the angles which the lines AB, BC, CD, &c. make with the vertical are

$\sqrt{\frac{1}{3}}, 3\sqrt{\frac{1}{3}}, 5\sqrt{\frac{1}{3}}, 7\sqrt{\frac{1}{3}}, \&c.$ agreeing with the first solution; and from these, every thing else may be determined as in the other solutions.



PRIZE QUESTION 1801. (Vol. III.)

This problem is one of a class which derive considerable interest from their connection with the theory of terrestrial gravitation. It appears to have been resolved by the celebrated Italian mathematician Boscovich, but we cannot say at what time. However it has been

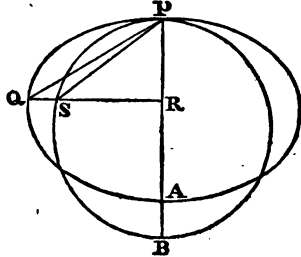
known to the English mathematicians ever since the year 1745, for in the *Miscellanea Curiosa Mathematica*, No. 1, question 20, we find it proposed by *Hurlothrumbo*, an appellation which Mr. Thomas Simpson appears frequently to have prefixed to questions which he proposed (see Dr. Hutton's Dictionary, art. Simpson). The question was this :

"What form must a mass of homogeneous matter take, that the attraction thereof on a corpuscle, somewhere in or without its surface, &c. may be the greatest possible? And what is the ratio of this attraction to the greatest whereby the corpuscle can be affected, when the same matter is in form of a sphere?"

In the following number an answer was given by *Philofluentine-canalgegeomastrolongo*, a whimsical name adopted by Mr. Emerson, as we learn from an account of his life prefixed to a vol. of Tracts. This ingenious mathematician's solution was as follows :

"Let PA be the axis of the solid, P the corpuscle, and let the attractive force be as the n^{th} power of the distance to any point q in the surface: draw PQ and QA perpendicular to the axis; then the force of a particle at A acting on P is as PA^n , and the force at q , in direction PQ , as PQ^n ; and $PQ : PR :: PQ^n : PR \times PQ^{n-1}$ the force of a particle at q acting on P in direction PA . Now this force of the particle q must be equal to that at A ; for if the force at q be greater or less than that at A , the matter must be removed from A to q , or from q to A , and then the whole body will attract more than before; so that at last when the body attracts the most possible, the force at A and at all points q must be equal; whence $PR \times PQ^{n-1} = PA^n$. Let $PR = x$, $QR = y$, $PA = a$. Then if $n = -2$, we shall have $a^2x = (x^2 + y^2)^{\frac{3}{2}}$, or $a^2x^3 = (x^2 + y^2)^3$ for the equation of the curve PQ .

Note. If $n = -1$, the body PQA will be a sphere; and if $n = 0$, 1, 2, 3, &c. the body will be a long slender cylinder, whose axis is PA ."



"To find the force of attraction.

"Let $\frac{2cfd}{m} = A$, $c = 3.1416$; then (by cor. 3, p. 277, of Mr.

Emerson's Doctrine of Fluxions) the force of the circle $QR = A \times$

$$\left\{ 1 - \frac{PR}{PQ} \right\} = A \times \left\{ 1 - \frac{x}{\sqrt{(x^2 + y^2)}} \right\} = A \times \left\{ 1 - \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right\}, \text{ and}$$

flux. force = $A \times \left\{ \dot{x} - \frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}} \right\}$, and the force of the solid = $A \times \left\{ x - \frac{3x^{\frac{3}{2}}}{5a^{\frac{3}{2}}} \right\}$, and the force of the whole solid = $A \times \frac{2}{3}a$.

"Let d = diameter of the sphere equal to the solid, the flux. force of the sphere = $A \times \left\{ \dot{x} - \frac{x\dot{x}}{\sqrt{dx}} \right\}$, and the fluent = $A \times \left\{ x - \frac{2x^{\frac{3}{2}}}{3\sqrt{d}} \right\}$, and the force of the whole sphere = $A \times \frac{1}{3}d$.

"For the content of the solid, its fluxion = $cy'\dot{x} = ca^{\frac{4}{3}}x^{\frac{2}{3}}\dot{x} - cx^3$, and the solid = $\frac{3}{2}ca^{\frac{4}{3}}x^{\frac{2}{3}} - \frac{cx^3}{3}$, and the whole solid = $\frac{4ca^3}{15}$, likewise the content of the sphere = $\frac{1}{2}cd^3$. Therefore $\frac{4ca^3}{15} = \frac{cd^3}{6}$, and $d = a \times \left(\frac{8}{5}\right)^{\frac{1}{3}}$. Therefore the force of the body to the force of the sphere as $A \times \frac{2}{3}a$ to $A \times \frac{1}{3}d$, or as $\frac{2}{3}a$ to $\frac{1}{3}d$; that is, as $\frac{2}{3}a$ to $\frac{1}{3}A \times \left(\frac{8}{5}\right)^{\frac{1}{3}}$, or as 3 to $(25)^{\frac{1}{3}}$."

At a later period, in 1764 we find the question again brought forward in the general magazine of Arts and Sciences for January, it runs thus,

"*A Physico-Mathematical PROBLEM, with its SOLUTION.*

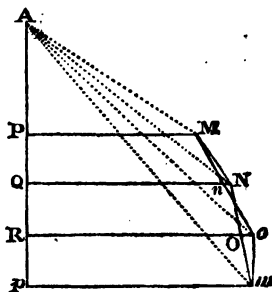
"**PROBLEM.** *Supposing the law of attraction to be in the inverse ratio of the square of the distance, to find the nature of the solid of the greatest attraction.*

"*Solution, by Mr. St. James.*

"It is plain in the first place, that the solid which attracts the given point A with the greatest possible force, of all solids under the same quantity of matter, must be a solid of rotation, since there can be no reason, why the matter should be distributed more on one side than on others. To find the nature of the curve which forms this solid by a rotation about its axis, let there be taken four equidistant and infinitely near ordinates, and let the lines MN , NO , OM be three in-

finitely small sides of the curve whose points M and m are supposed to be fixed, and the points N and o variable. Call the lines AP , AQ ; AR , Ap ; x , x' ; x'' , x''' ; the lines MP , NQ , &c. y , y' , &c. the lines AM , AN , &c. z , z' , &c.

"First by the nature of the problem, we have, $yyx + y'y'x + y''y''x = a$ a constant magnitude; whence, after having taken the differences, and observing that only y' and y'' are variable, we get $y'y' = y''y''$. Now, we shall find, by well known methods, that the attraction



of the solid formed by the rotation of $mnmpp$ about pp is $(1 - \frac{x}{z})x$

+ $(1 - \frac{x'}{z'})x + (1 - \frac{x''}{z''})x$, which should be a *maximum*; there-

fore, taking the difference, and observing that only x' and x'' are variable, and making it = 0, we get $\frac{x'z'}{x'(z'-z')} = \frac{x''z''}{x''(z''+z')}$; but

the right-angled triangles ANQ , AOR , as the points Q and R are fixed, give $y'y' = z'z'$ and $y''y'' = z''z''$: therefore $\frac{x'y'y'}{x'z'(z'-z')} = \frac{x''y''y''}{x''z''(z''+z')}$.

And since $y'y' = y''y''$, we shall have $\frac{x'}{x'z'(z'-z')} = \frac{x''}{x''z''(z''+z')} =$

a constant magnitude $\frac{1}{gg}$; therefore we shall have $z^3 = ggz$ for the

equation of the curve required.

"The problem might be solved another far more simple way, by taking the attraction of any one point of the surface of the solid; which attraction is $\frac{x}{z}$, and making it equal to a constant magnitude

$\frac{1}{gg}$, we shall have $z^3 = ggz$, for the equation of the required curve,

as before; For it is manifest, that if the attraction were less on any place of the surface than another, that point might be placed out of the solid, so as that it would attract more, and the solid would be no longer that of the greatest attraction, contrary to the hypothesis."

We know not who Mr. St. James was, but we observe that the solution is not in the style of English mathematics, probably it was copied from some foreign work.

The question has been also fully considered in a paper given in the Edin. Phil. Trans. vol. 6. *On the Solids of greatest Attraction*, by

Mr. J. Playfair, professor of natural philosophy in the University of Edinburgh. He there, as the first of a series of problems, proposes .

I. *To find the solid into which a mass of homogeneous matter must be formed, in order to attract a particle given in position, with the greatest force possible, in a given direction, and to this he gives the following solution :*

Let A be the particle given in position, AB the direction in which it is to be attracted ; and ACBH a section of the solid required, by a plane passing through AB.

Since the attraction of the solid is a maximum, by hypothesis, any small variation in the figure of the solid, provided the quantity of matter remain the same, will not change the attraction in the direction AB. If, therefore, a small portion of matter be taken from any point c, in the superficies of the solid, and placed at D, another point in the same superficies, there will be no variation produced in the force which the solid exerts on the particle A, in the direction AB.

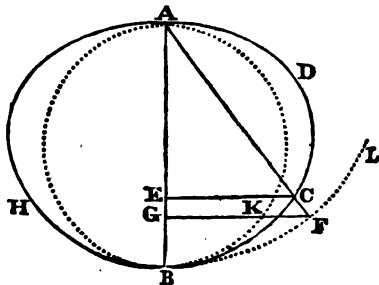
The curve ACB, therefore, is the locus of all the points in which a body being placed, will attract the particle A in the direction AB, with the same force.

This condition is sufficient to determine the nature of the curve ACB: From c, any point in that curve, draw CE perpendicular to AB ; then

if a mass of matter placed at c be called m^3 , $\frac{m^3}{AC^2}$ will be the attraction of that mass on A, in the direction AC, and $\frac{m^3 \times AE}{AC^3}$ will be its attraction in the direction AB. As this is constant, it will be equal to $\frac{m^3}{AB^2}$, and therefore $AB^2 \times AE = AC^3$.

All the sections of the required solid, therefore, by planes passing through AB, have this property, that $ac^3 = AB^2 \times AE$; and as this equation is sufficient to determine the nature of the curve to which it belongs, therefore all the sections of the solid, by planes that pass through AB, are similar and equal curves ; and the solid of consequence may be conceived to be generated by the revolution of ACB, any one of these curves, about AB as an axis.

The solid so generated may be called the *Solid of greatest Attraction* ; and the line ACB, the *Curve of equal Attraction*.



II. *To find the equation between the co-ordinates of ACB, the curve of equal attraction.*

From c draw CE perpendicular to AB; let AB = a, AE = x, EC = y. We have found $AB^3 \times AE = AC^3$, that is, $a^3 x = (x^2 + y^2)^{\frac{3}{2}}$, or $a^4 x^3 = (x^2 + y^2)^3$, which is an equation to a line of the 6th order.

To have y in terms of x, $x^2 + y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}}$, $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, and $y = x^{\frac{1}{3}} \sqrt{a^{\frac{4}{3}} - x^{\frac{4}{3}}}$.

Hence $y = 0$, both when $x = 0$, and when $x = a$. Also if x be supposed greater than a , y is impossible. No part of the curve, therefore, lies beyond B.

The parts of the curve on opposite sides of the line AB, are similar and equal, because the positive and negative values of y are equal. There is also another part of the curve on the side of A, opposite to B, similar and equal to ACB; for the values of y are the same whether x be positive or negative.

III. *The curve may easily be constructed without having recourse to the value of y just obtained.*

Let AB = a, AC = z, and the angle BAC = ϕ . Then AE = AC $\times$ $\cos \phi = z \cos \phi$, and so $a^3 z \cos \phi = z^3$, or $a^3 \cos \phi = z^2$; hence $z = a \sqrt{\cos \phi}$.

From this formula, a value of AC or z may be found, if ϕ or the angle BAC be given; and if it be required to find z in numbers, it may be conveniently calculated from this expression. A geometrical construction may also be easily derived from it. For if with the radius AB, a circle BFL be described from the centre A; if AC be produced to meet the circumference in F, and if FG be drawn at right

angles to AB, then $\frac{AG}{AB} = \cos \phi$, and so $z = a \times \sqrt{\frac{AG}{AB}} = \sqrt{(AB \times AG)} = AC$.

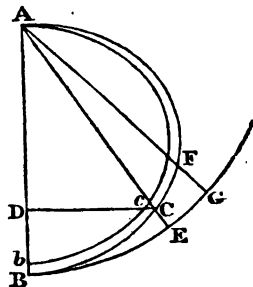
Therefore, if from the centre A, with the distance AB, a circle BFL be described, and if a circle be also described on the diameter AB, as AKB, then drawing any line AF from A, meeting the circle BFH in F, and from F letting fall FG perpendicular on AB, intersecting the semicircle AKB in K; if AK be joined, and AC made equal to AK, the point c is in the curve.

For $AK = \sqrt{(AB \times AG)}$, from the nature of the semicircle, and therefore $AC = \sqrt{(AB \times AG)}$, which has been shewn to be a property of the curve. In this way, any number of points of the curve may be determined; and the *Solid of greatest attraction* will be described, as already explained, by the revolution of this curve about the axis AB.

IV. *To find the area of the curve ACB.*

1. Let ACE, AFG be two radii, indefinitely near to one another,

meeting the curve ACB in c and r , and the circle, described with the radius AB , in E and G . Let $AC = z$ as before, the angle $BAC = \phi$, and $AB = a$. Then $GE = a\phi$, and the area $AGE = \frac{1}{2}a^2\phi$, and since $AE^2 : AC^2 :: \text{Sect. } AEG : \text{Sect. } ACF$, the sector $ACF = \frac{1}{2}z^2\phi$. But $z^2 = a^2 \cos \phi$, (§ III.), whence the sector ACF , or the fluxion of the area $ABC = \frac{1}{2}a^2\phi \cos \phi$, and consequently the area $ABC = \frac{1}{2}a^2 \sin \phi$, to which no constant quantity need be added, because it vanishes when $\phi = 0$, or when the area ABC vanishes.



The whole area of the curve, therefore, is $\frac{1}{2}a^2$, or $\frac{1}{2}AB^2$; for when ϕ is a right angle $\sin \phi = 1$. Hence the area of the curve on both sides of AB is equal to the square of AB .

2. The value of x , when y is a maximum, is easily found. For when y , and therefore y^2 is a maximum, $\frac{2}{3}a^{\frac{2}{3}}x^{-\frac{1}{3}} = 2x$, or $3x^{\frac{4}{3}} = a^{\frac{2}{3}}$, that is $x = \frac{a}{3^{\frac{3}{4}}} = \frac{a}{\sqrt[4]{27}}$.

Hence, calling b the value of y when a is a maximum, $b^2 = a^{\frac{2}{3}} \times \frac{a^{\frac{2}{3}}}{27^{\frac{1}{4}}} - \frac{a^2}{27^{\frac{1}{2}}} = a^2 \left(\frac{27^{\frac{1}{4}} - 1}{27^{\frac{1}{2}}} \right) = \frac{2a^2}{\sqrt[4]{27}}$, and so $b = a \frac{\sqrt[4]{2}}{\sqrt[4]{27}}$, and therefore $a : b :: \sqrt[4]{27} : \sqrt[4]{2}$, or as 11 : 7 nearly:

3. It is material to observe, that the radius of curvature at A is infinite. For since $y^2 = a^{\frac{2}{3}}x^{\frac{2}{3}} - x^2$, $\frac{y^2}{x} = \frac{a^{\frac{2}{3}}}{x^{\frac{1}{3}}} - x$. But when x is very small, or y indefinitely near to A , $\frac{y^2}{x}$ becomes the diameter of the circle having the same curvature with ACB at A , and when x vanishes, this value of $\frac{y^2}{x}$, or $\frac{a^{\frac{2}{3}}}{x^{\frac{1}{3}}} - x$, becomes infinite because of

the divisor $x^{\frac{1}{3}}$ being in that case $= 0$. The diameter, therefore, and the radius of curvature at A are infinite. In other words, no circle, having its centre in AB produced, and passing through A , can be described with so great a radius, but that, at the point A , it will be within the curve of equal attraction.

The solid of greatest attraction, then, at the extremity of its axis, where the attracted particle is placed, is exceedingly flat, approaching more nearly to a plane than the superficies of any sphere can do, however great its radius.

4. To find the radius of curvature at B , the other extremity of the axis.

Since $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$, if we divide by $a - x$, we have $\frac{y^2}{a - x} = \frac{a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2}{a - x}$. But at B , when $a - x$, or the abscissa reckoned from B vanishes, $\frac{y^2}{a - x}$ is the diameter of the circle having the same curvature with ACB in B . But when $a - x = 0$, or $a = x$, both the numerator and denominator of the fraction $\frac{a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2}{a - x}$ vanish, so that its ultimate value does not appear. To remove this difficulty, let $a - x = z$, or $x = a - z$, then we have $y^2 = a^{\frac{4}{3}} (a - z)^{\frac{2}{3}} - (a - z)^2$. But when z is extremely small, its powers, higher than the first, may be rejected; and therefore $(a - z)^{\frac{2}{3}} = a^{\frac{2}{3}} \left(1 - \frac{z}{a}\right)^{\frac{2}{3}} = a^{\frac{2}{3}} \left(1 - \frac{2z}{3a}, \&c.\right)$. Therefore the equation to the curve becomes in this case,

$$y^2 = a^{\frac{4}{3}} \times a^{\frac{2}{3}} \left(1 - \frac{2z}{3a}\right) - a^2 + 2az = a^2 - \frac{2}{3}az - a^2 + 2az = \frac{4}{3}az.$$

Hence $\frac{y^2}{2z}$ or the radius of curvature at $B = \frac{2}{3}a$. The curve,

therefore, at B falls wholly without the circle BKA , described on the diameter AB , as its radius of curvature is greater. This is also evident from the construction.

V. To find the force with which the solid above defined attracts the particle A in the direction AB .

Let b be a point indefinitely near to B , and let the curve acb be described similar to ACB . Through c draw ccd perpendicular to AB , and suppose the figure thus constructed to revolve about AB ; then

plane in which that curve is being parallel to AB , the line in which the attraction is estimated, and distant from it by the space b .

Instead of reckoning the abscissa from x , it may be made to begin at c . If AL or $CK = h$, then the value of h is determined from the equation $b^2 = a^{\frac{4}{3}} h^{\frac{2}{3}} - h^2$, and if $x = h + u$, u being put for CP , $v^2 = a^{\frac{4}{3}} (h+u)^{\frac{2}{3}} - (h+u)^2 - a^{\frac{4}{3}} h^{\frac{2}{3}} + h^2$, or $v^2 + (h+u)^2 + b^2 = a^{\frac{4}{3}} (h+u)^{\frac{2}{3}}$, or $(v^2 + (h+u)^2 + b^2)^3 = a^4 (h+u)^2$.

When b is equal to the maximum value of the ordinate EH , (§ IV. 2.) the curve CGD goes away into a point; and if b be supposed greater than this, the equation to the curve is impossible.

X. *The solid of greatest attraction may be found, and its properties investigated, in the way that has now been exemplified, whatever be the law of the attracting force. It will be sufficient, in any case, to find the equation of the generating curve, or the curve of equal attraction.*

Thus, if the attraction which the particle c (see the fig. on page 392) exerts on the given particle at A , be inversely as the m power of the distance, or as $\frac{1}{AC^m}$, then the attraction in the direction AE will be

$$\frac{AE}{AC^{m+1}}, \text{ and if we make this } = \frac{1}{AB^m}, \text{ we have } \frac{AE}{AC^{m+1}} = \frac{1}{AB^m},$$

$$\text{or making } AE = x, EC = y, \text{ and } AB = a, \text{ as before, } \frac{x}{(x^2 + y^2)^{\frac{m+1}{2}}} = \frac{1}{a^m},$$

$$= \frac{1}{a^m}, \text{ or } a^m x = (x^2 + y^2)^{\frac{m+1}{2}}, \text{ and } x^2 + y^2 = a^{\frac{2m}{m+1}} x^{\frac{2}{m+1}},$$

$$\text{or } y^2 = a^{\frac{2m}{m+1}} x^{\frac{2}{m+1}} - x^2.$$

If $m = 1$, or $m + 1 = 2$, this equation becomes $y^2 = ax - x^2$, being that of a circle of which the diameter is AB . If, therefore, the attracting force were inversely as the distance, the solid of greatest attraction would be a sphere.

If the force be inversely as the cube of the distance, or $m = 3$, and $m + 1 = 4$, the equation is $y^2 = a^{\frac{3}{2}} x^{\frac{1}{2}} - x^2$, which belongs to a line of the 4th order.

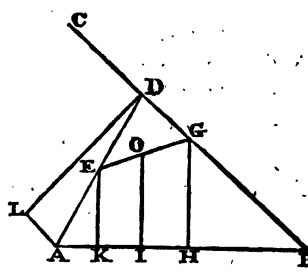
If $m = 4$, and $m + 1 = 5$, the equation is $y^2 = a^{\frac{4}{3}} x^{\frac{2}{3}} - x^2$; which belongs to a line of the 10th order.

In general, if m be an even number, the order of the curve is $(m + 1) \times 2$; but if m be an odd number, it is $m + 1$ simply.

QUESTION 1115. (Vol. IV.)

Two Additional Solutions, by Mr. Wallace, R. M. College.

I. Supposing the beams to be prisms, and leaving their thickness out of consideration, let e be the centre of gravity of AD , and g that of BC : Put w for the weight of the beam AD , w for that of BC , and v for the pressure exerted on BC by its leaning on AD at the point D .



The lever BC is held in equilibrium by two forces; one is its weight w , applied in a vertical direction GH , at g its centre of gravity; and the other is v , the pressure at D , exerted in the direction of DL , a perpendicular to BC . The lever AD is also held in equilibrium by two forces, viz. its weight w , in the vertical direction EK , and the reaction of the pressure at D in the direction DL .

Draw AL perpendicular to DL , then, by the nature of the lever,

$$w : v :: BD : BH,$$

$$v : w :: AK : AL;$$

$$\text{therefore } w : w :: BD \times AK : BH \times AL.$$

$$\text{Let } AB = a, AD = b, DB = x, AE = c, BG = d,$$

$$\text{then } AK = AE \times \cos A = \frac{c(b^2 + a^2 - x^2)}{2ab},$$

$$BH = BG \times \cos B = \frac{d(a^2 + x^2 - b^2)}{2ax},$$

$$AL = AD \times \cos D = \frac{b^2 + x^2 - a^2}{2x} :$$

Hence, by substitution,

$$w : w :: \frac{cx(b^2 + a^2 - x^2)}{2ab} : \frac{d(a^2 + x^2 - b^2)(b^2 + x^2 - a^2)}{4ax};$$

$$\text{therefore } bdw(a^2 + x^2 - b^2)(b^2 + x^2 - a^2) = 2cwx^2(b^2 + a^2 - x^2);$$

This equation which is of the fifth degree, exhibits a solution of the problem.

If we suppose $AD = AB$, that is $b = a$, the equation reduces to

$$adwx = 2cw(2a^2 - x^2)$$

which is of the second degree only.

II. Retaining the notation and figure of the last solution. The virtual velocity of the point e in a vertical direction is $w \times \frac{e}{EF}$, and

that of $\vec{c}, \vec{w} \times \vec{GH}$, therefore, by the principle of *virtual velocities* (Playfair's Outlines, § 127, or Gregory's Mechanics, vol. i, page 60).

$$\begin{aligned} \vec{w} \times \vec{EF} + \vec{w} \times \vec{GH} &= 0; \\ \text{but } EF &= c \times \sin A, \quad GH = d \times \sin B, \\ \vec{EF} &= c A \cos A, \quad \vec{GH} = d B \cos B: \end{aligned}$$

therefore, $c w A \cos A + d w B \cos B = 0$.

And since in the triangle ABD, the sides $AB = a$, and $AD = b$ are constant, we have

$$\dot{B} = -\frac{b\dot{A}}{x} \times \cos D \text{ (Leslie's Geom. 2nd. Ed. pa. 473).}$$

This equation, compared with the foregoing, gives

$$c w x \cos A = b d w \cos B \cos D.$$

$$\text{Now } \cos A = \frac{a^2 + b^2 - x^2}{2ab},$$

$$\cos B = \frac{a^2 + x^2 - b^2}{2ax},$$

$$\cos D = \frac{b^2 + x^2 - a^2}{2bx}.$$

These values being substituted, we obtain

$$b d w (a^2 + x^2 - b^2) (b^2 + x^2 - a^2) = 2 c w x^3 (a^2 + b^2 - x^2)$$

the same final equation as in the first solution.

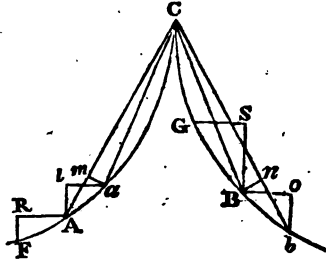
PRIZE QUESTION 1810. (1223.—Vol. IV.)

Additional Solution, by Mr. Lowry; from the Mathematical Repository, Vol. III. Part II. Art. 5.

The general problem, of which this question is a particular case, has often engaged the attention of mathematicians. It was first proposed, by John Bernoulli, in the *Comment. Petrop.* for 1727: at that time he stated only the result of his calculation; and it was not till 1735 that he published his solution in the *Act. Erud. Leips.* Herman, indeed, attempted a solution, but fell into some mistakes which rendered his conclusions erroneous. A complete and general solution was afterwards given by D'Alembert in his *Traité Dynamique*, published in 1744; and since then it has been resolved by several mathematicians of our own country: viz. by Mr. Atwood, in his *Treatise on Rectilineal Motion*, by Mr. Wildbore in the *Gentleman's Diary* for 1788, and either (partially or generally) by several of the Correspondents to the *Mathematical Repository* (see Art. 6, Vol. I, part II; Art. 1, 2, and 21, Vol. II, part II.)

The following solution is considerably different from any of them.

PROBLEM. Suppose a given weight B to descend by the force of gravity along a given curve COB ; and by means of a string passing over a pulley fixed at C , in the same vertical plane, to draw up another given weight A , along another curve FAA : required the velocities of the bodies at the points A and B , supposing the motion to commence when they are at the given points F and G .



SOLUTION. Let ab be the space descended by the weight B in the

indefinitely short interval of time $\dot{t}$, and Aa , the contemporary space ascended by the weight A . Draw the vertical lines FR , Al , bb , and BS , meeting the horizontal lines AB , al , BO and GS , in R , l , O and s ; and draw am and bn perpendicular to Ac and cb .

Put $x = FR$, the vertical distance ascended by A ;

$y = SB$, the vertical distance descended by B ;

$Ac = z$; $Al = \dot{x}$; $bo = \dot{y}$; $Am = bn = \dot{z}$;

$Aa = \dot{r}$; $bb = \dot{s}$; and $g = 32\frac{1}{2}$ feet, the measure of the accelerative force of gravity.

Then the velocity of the weight A in the direction of the curve

Aa is $\frac{\dot{r}}{\dot{t}}$ (the time being supposed to flow uniformly), and the va-

riation of the velocity or the measure of the accelerating force in that direction $= \frac{\ddot{r}}{\dot{t}^2}$. But the weight A cannot ascend in the

curve unless its gravity be counteracted and an additional force impressed upon it sufficient to produce the actual velocity $\frac{\dot{r}}{\dot{t}}$. Now

the force of gravity in the direction Aa is $= g \frac{\dot{x}}{\dot{r}}$, therefore the accelerating force necessary to cause A to ascend in the curve with the

velocity $\frac{\dot{r}}{\dot{t}}$ is $= g \frac{\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}$, and the whole motive force acting on

A , in the direction $Aa = A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2} \right)$ which is equivalent to a

motive force $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2} \right) \times \frac{\dot{r}}{\dot{x}}$, acting in the direction of the string.

Now the motive force by which A is urged towards c , is derived from the action of the weight B ; therefore B will be reacted upon in the direction cb by an equal motive force; that is, B will be

acted upon in the direction bc by a motive force $= A \times \left(\frac{g\ddot{x}}{r} + \frac{\ddot{r}}{t^2} \right) \times \frac{\dot{r}}{z}$;

which force may be resolved into two others, namely, $A \times \left(\frac{g\ddot{x}}{r} + \frac{\ddot{r}}{t^2} \right) \times \frac{\dot{r}}{s}$ acting in the direction of the curve, and retarding the descent of B ; and $A \times \left(\frac{g\ddot{x}}{r} + \frac{\ddot{r}}{t^2} \right) \times \frac{\dot{r}\dot{m}}{zs}$ (m being $= bn = \sqrt{(s^2 - z^2)}$) acting in a direction perpendicular to the curve, and diminishing or augmenting the pressure.

Therefore, the whole effect of the weight A on the weight B , is equivalent to a motive force $A \times \left(\frac{g\ddot{x}}{r} + \frac{\ddot{r}}{t^2} \right) \times \frac{\dot{r}}{s}$, directly opposing the descent of B in the curve.

Again, the velocity with which B actually descends in the curve is $\frac{\dot{s}}{t}$, and the variation of that velocity, or the measure of the ac-

celerating force, $= \frac{\ddot{s}}{t^2}$. But the force of gravity in the direction

of the curve is $= \frac{g\dot{y}}{s}$; therefore $\frac{g\dot{y}}{s} - \frac{\ddot{s}}{t^2}$ is the accelerating force

constantly destroyed by the connexion of B with the weight A , and

the whole motive force constantly lost $= B \times \left(\frac{g\dot{y}}{s} - \frac{\ddot{s}}{t^2} \right)$. But the

motive force constantly lost by B , must be equal to the motive force constantly retarding its descent;

$$\text{therefore } A \times \left(\frac{g\ddot{x}}{r} + \frac{\ddot{r}}{t^2} \right) \times \frac{\dot{r}}{s} = B \times \left(\frac{g\dot{y}}{s} - \frac{\ddot{s}}{t^2} \right).$$

$$\text{or } \frac{Ar}{t^2} \ddot{r} + \frac{Bs}{t^2} \ddot{s} = g \times (B\dot{y} - A\dot{x}).$$

And, multiplying by 2 and taking the fluents,

we have $\frac{Ar^2}{t^2} + \frac{Bs^2}{t^2} = 2g \times (By - Ax);$

or $Av^2 + Bu^2 = 2g \times (By - Ax)$, no correction being necessary, because the motion is supposed to begin when $x = 0$ and $y = 0$.

Now $v : u :: \dot{r} : \dot{s}$, therefore $u^2 = \frac{v^2 \dot{s}^2}{r^2}$, and

$Av^2 + \frac{Bv^2 \dot{s}^2}{r^2} = 2g \times (By - Ax);$ hence we have

$v = \dot{r} \sqrt{\frac{2g \times (By - Ax)}{Ar^2 + Bs^2}}$, and $u = \dot{s} \sqrt{\frac{2g \times (By - Ax)}{Ar^2 + Bs^2}}$, for

the velocities required.

COR. 1. Let f and f' be the accelerative forces by which the bodies A and B are respectively animated; that is,

let $f = \frac{\ddot{r}}{t^2}$ and $f' = \frac{\ddot{s}}{t^2}$, then $Ar\dot{f} + Bs\dot{f}' = g \times (B\dot{y} - A\dot{x})$.

But $f : f' :: \ddot{r} : \ddot{s}$, or $f' = \frac{\ddot{s}}{\ddot{r}} f$; therefore $Ar\dot{f} + \frac{Bs\dot{s}}{r} \ddot{r}$

$= g \times (B\dot{y} - A\dot{x})$; hence $f = g \times \frac{(B\dot{y} - A\dot{x}) \ddot{r}}{Ar \ddot{r} + Bs \ddot{s}}$ and f'

$= g \times \frac{(B\dot{y} - A\dot{x}) \ddot{s}}{Ar \ddot{r} + Bs \ddot{s}}$ as determined by Mr. Atwood.

COR. 2. The motive force $B \times \left(\frac{g\dot{y}}{\dot{s}} - \frac{\ddot{s}}{\dot{s}^2} \right)$, or $B \times \left(\frac{g\dot{y}}{\dot{s}} - f' \right)$, acting in the direction of the curve, may be resolved into two others,

namely, $B \times \left(\frac{g\dot{y}}{\dot{s}} - f' \right) \frac{\dot{s}}{x}$, acting in the direction of the string

CB, and which is equal to the motive force $A \times \left(\frac{g\dot{x}}{\dot{r}} + f \right) \frac{\dot{r}}{x}$; and

$B \times \left(\frac{g\dot{y}}{\dot{s}} - f' \right) \frac{\dot{m}}{z}$, acting perpendicularly to the curve, and which

is equal to the motive force $A \times \left(\frac{g\dot{x}}{\dot{r}} + f \right) \frac{\dot{m} \dot{r}}{z \dot{s}}$. Now the string

is stretched in the direction ca by the motive force $A \times \left(\frac{g\dot{x}}{r} + f \right) \frac{\dot{r}}{z}$,

and in the direction cb by the equal motive force $B \times \left(\frac{g\dot{y}}{s} - f \right) \frac{\dot{s}}{z}$;

therefore the whole pressure on the axis of the pulley is equal to the resultant of these two equal forces; that is, the whole pressure on the pulley is =

$$A \times \left(\frac{g\dot{x}}{r} + f \right) \frac{\dot{r}}{z} \times \frac{\sin \angle ACB}{\sin \frac{1}{2} \angle ACB} = B \times \left(\frac{g\dot{y}}{s} - f \right) \frac{\dot{s}}{z} \times \frac{\sin \angle ACB}{\sin \frac{1}{2} \angle ACB}.$$

COR. 3. The pressure of the weight B , against the curve, if it had no connexion with the weight A , would evidently be $= Bg \frac{\dot{a}}{s}$, (a being = so): but in consequence of that connexion, the pressure is diminished or augmented by the force $A \times \left(\frac{g\dot{x}}{r} + f \right) \frac{\dot{m} \dot{r}}{zs}$, or $B \times \left(\frac{g\dot{y}}{s} - f \right) \frac{\dot{m}}{z}$; therefore the whole pressure on the curve, arising

from gravity, is $= Bg \frac{\dot{a}}{s} \mp B \times \left(\frac{g\dot{y}}{s} - f \right) \frac{\dot{m}}{z}$; also the pressure arising from the centrifugal force of the body is $= B \frac{u^2}{R}$, (R being the radius of curvature at B); therefore the whole pressure against the curve at

$$\begin{aligned} B \text{ is } &= B \times \left(\frac{g\dot{a}}{s} + \frac{u^2}{R} \right) \mp B \times \left(\frac{g\dot{y}}{s} - f \right) \frac{\dot{m}}{z} \\ &= B \times \left(\frac{g\dot{a}}{s} + \frac{u^2}{R} \right) \mp A \times \left(\frac{g\dot{x}}{r} + f \right) \frac{\dot{m} \dot{r}}{zs}. \end{aligned}$$

And in the same manner, putting $\beta = AR$ and $\dot{n} = am = \sqrt{(r^2 - z^2)}$, the pressure against the curve at A is found

$$\begin{aligned} &= A \times \left(\frac{g\dot{\beta}}{r} + \frac{v^2}{R'} \right) \mp A \times \left(\frac{g\dot{x}}{r} + f \right) \frac{\dot{n}}{z} \\ &= A \times \left(\frac{g\dot{\beta}}{r} + \frac{v^2}{R'} \right) \mp B \times \left(\frac{g\dot{y}}{s} - f \right) \frac{\dot{n} \dot{s}}{zs}. \end{aligned}$$

Or, because $f = \frac{u}{s} = \frac{wu}{s}$, and $f = \frac{v}{r} = \frac{v\dot{v}}{r}$; the pressure at B is

$= B \times \left(\frac{g\dot{a}}{s} + \frac{v^2}{R} \right) \mp B \times \left(\frac{g\dot{y}}{s} - \frac{w^2}{s} \right) \frac{\dot{m}}{s}$, and the pressure at

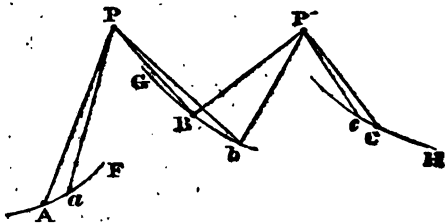
$A = A \times \left(\frac{g\dot{\beta}}{r} + \frac{v^2}{R'} \right) \mp A \times \left(\frac{g\dot{x}}{r} + \frac{v\dot{v}}{r} \right) \frac{\dot{n}}{s}$.

Hence the points where the bodies will quit the curves may be determined by making $\frac{g\dot{a}}{s} + \frac{v^2}{R} \mp \left(\frac{g\dot{y}}{s} - \frac{w^2}{s} \right) \frac{\dot{m}}{s} = 0$,

and $\frac{g\dot{\beta}}{r} + \frac{v^2}{R'} \mp \left(\frac{g\dot{x}}{r} + \frac{v\dot{v}}{r} \right) \frac{\dot{n}}{s} = 0$.

The problem may be resolved in a similar manner, when there are more weights than two.

For instance, let there be three weights A, B, C , and let a descend on the curve ca , and draw up the other weights A, c on the curves AP, CH , by means of a string passing over the pulleys P, P' .



The same construction and substitution being made as in the preceding case, let $cc = p$, be the little space described by c in the time t , and $h =$ the vertical height ascended by c . Then it has been shewn that, the effect of the weight A , in opposing the descent

of B , is equal to the motive force $A \times \left(\frac{g\dot{x}}{r} + \frac{\dot{r}}{t^2} \right) \times \frac{\dot{r}}{s}$; and, it

may be shewn in the same way that, the effect of the weight c , in opposing the descent of B , is equal to the motive force

$c \times \left(\frac{g\dot{h}}{p} + \frac{\dot{p}}{t^2} \right) \times \frac{\dot{p}}{s}$; therefore the whole effect of the

weights A and c , on the weight B , is equal to the motive force

$A \times \left(\frac{g\dot{x}}{r} + \frac{\dot{r}}{t^2} \right) \frac{\dot{r}}{s} + c \times \left(\frac{g\dot{h}}{p} + \frac{\dot{p}}{t^2} \right) \frac{\dot{p}}{s}$, directly opposing the descent of B in the curve. Hence, we have

$$A \times \left(\frac{g\dot{x}}{r} + \frac{\dot{r}}{t^2} \right) \frac{\dot{r}}{s} + c \times \left(\frac{g\dot{h}}{p} + \frac{\dot{p}}{t^2} \right) \frac{\dot{p}}{s} = B \times \left(\frac{g\dot{y}}{s} - \frac{\dot{s}}{t^2} \right),$$

$$\text{or } A \frac{\dot{r}}{t^2} + B \frac{\dot{s}}{t^2} + c \frac{\dot{p}}{t^2} = g \times (By - As - ch);$$

and, taking the fluents, we get

$$A \frac{\dot{r}^2}{r^2} + B \frac{\dot{s}^2}{s^2} + C \frac{\dot{p}^2}{p^2} = 2g \times (By - Ax - ch),$$

or (putting ω = the velocity of c)

$$Av^2 + Bu^2 + C\omega^2 = 2g \times (By - Ax - ch).$$

But $v = \frac{\dot{r}}{r} u = \frac{\dot{r}}{p} \omega$, therefore

$$Av^2 + Bv^2 \frac{s^2}{r^2} + Cv^2 \frac{p^2}{r^2} = 2g \times (By - Ax - ch),$$

$$\text{and } v = \dot{r} \sqrt{\frac{2g \times (By - Ax - ch)}{Ar^2 + Bs^2 + Cp^2}},$$

$$u = \dot{s} \sqrt{\frac{2g \times (By - Ax - ch)}{Ar^2 + Bs^2 + Cp^2}},$$

$$\omega = \dot{p} \sqrt{\frac{2g \times (By - Ax - ch)}{Ar^2 + Bs^2 + Cp^2}}.$$

By proceeding in a similar manner, the problem may be resolved for any number of weights, and the general result will be, that "The sum of the products of each body by the square of its velocity, is equal $2g$ multiplied into the products of each body by the space it has perpendicularly descended." The spaces being taken negatively when the bodies ascend, and positively when they descend.

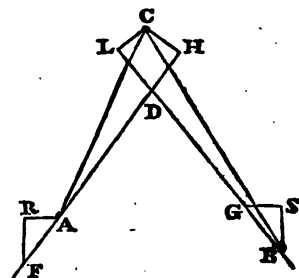
This is all that can be determined relating to the velocities of the bodies, unless the nature of the curves be known so as to enable us to find the relation of r, s, p , &c. We shall take a few particular examples, for the purpose of applying the preceding formula.

Ex. 1. To find the velocities of the bodies, when a descends on the inclined plane DCB , and draws up the weight A , on the inclined plane DAF , by means of a string passing over a pulley at c .

Suppose the motion to commence when the bodies are at F and G , and from c , let CH and CL be drawn perpendicular to the given planes.

Put $CH = a$, $CL = b$, $FH = c$, $GL = d$, the length of the string $= l$; and let, as in the general problem, $FA = r$, $GB = s$, $AC = z$, $FR = x$, and $BS = y$. Then $AH = \sqrt{(z^2 - a^2)}$, $r = c -$

$$\sqrt{(z^2 - a^2)}, \dot{r} = \frac{-z\dot{z}}{\sqrt{(z^2 - a^2)}}, BL = \sqrt{((l-z)^2 - b^2)}, s = \sqrt{((l-z)^2 - b^2)} - d, \text{ and } \dot{s} = \frac{-(l-z)\dot{z}}{\sqrt{((l-z)^2 - b^2)}}. \text{ These values of } \dot{r} \text{ and } \dot{s},$$



being substituted in the general formula for the values of v and u , we get, after proper reduction,

$$v = z\sqrt{2g((l-z)^2 - b^2)} \times \sqrt{\frac{By - Ax}{Ax^2((l-z)^2 - b^2) + B(l-z)^2(z^2 - a^2)}}$$

$$\text{and } u = (l-z)\sqrt{2g(z-a^2)} \times \sqrt{\frac{By - Ax}{Az^2((l-z)^2 - b^2) + B(l-z)^2(z^2 - a^2)}}.$$

But to exterminate x and y , let $\alpha =$ the angle FAN , and $\beta =$ the angle BGS ; then $x = r \sin \alpha = (c - \sqrt{(z^2 - a^2)}) \sin \alpha$, and $y = s \sin \beta = (\sqrt{(l - z)^2 - b^2} - d) \sin \beta$; and, by substitution, we have

$$v = z \sqrt{2g((l-z)^2 - b^2)} \times \frac{\sqrt{B(\sqrt{((l-z)^2 - b^2)} - d) \sin \phi - A(c - \sqrt{z^2 - a^2}) \sin \alpha}}{Az^2((l-z)^2 - b^2) + B(l-z)^2(2^2 - a^2)},$$

and

$$x = (l-z) \sqrt{2g(z^2 - a^2)} \times \sqrt{\frac{B(\sqrt{((l-z)^2 - b^2)} - d) \sin \beta - A(c - \sqrt{z^2 - a^2}) \sin \alpha}{A z^2 ((l-z)^2 - b^2) + B(l-z)^2 (z^2 - a^2)}}$$

or the velocities required in terms of x and known quantities.

When the pulley is at n , the intersection of the inclined planes, then $a = 0$, and $b = 0$; and we have, $v = u =$

$$\sqrt{2g} \times \sqrt{\frac{B(l-z-d) \sin \alpha - A(c-z) \sin \beta}{A+B}} =$$

$$\sqrt{\frac{2g}{A+B}} \times \sqrt{(Br \sin \alpha - As \sin \beta)}. \text{ And when the plane BD is}$$

vertical, $v = u = \sqrt{\frac{2g}{A+B}} \times \sqrt{(Br \sin \alpha - As)}$.

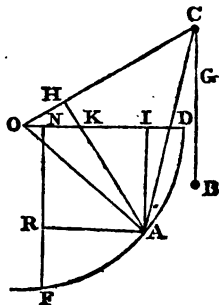
Again, when the plane xy is horizontal, and the weight w descends in a vertical direction; then $x = 0$, $a = 0$, $b = 0$, and $\sin \beta = 1$; and we have

$$v = z \times \sqrt{\frac{2g \times B(l - z - d)}{Az^3 + B(z^2 - a^2)}}, \text{ and } u = \sqrt{(z^2 - a^2)} \times \sqrt{\frac{2g \times B(l - z - d)}{Az^3 + B(z^2 - a^2)}}.$$

The method of calculation will be nearly the same when there are more bodies than two moving on inclined planes, but the expressions for the velocities are too complex to be put down here.

Ex. 2. To find the velocities of the bodies when **B** descends in a vertical direction, and draws up the body **A** in a circle **FAD**; or, which is the same thing, when the body **A** is fastened to an inflexible rod of a given length, without weight, moveable about the centre **O**, and drawn up by the weight **B**, by means of a string passing over a pulley at **c**.

Let o be the centre of the circle ; and draw ah , perpendicular to co , meeting the horizontal line od in k . Also draw ai and fn perpendicular to od . Put $oc = a$, radius of the circle $= c$, $oh = w$, $fa = r$, $cb = s$, and



$ac = z$; then, by the property of the circle, $r = \frac{cw}{\sqrt{(c^2 - w^2)}}$; also

$$z^2 = cn^2 - on^2 + oa^2 = a^2 - 2aw + c^2, \text{ and } z = -\frac{aw}{z} = -s.$$

These values of r and s being substituted in the general formula for v and u , we have, after proper reduction,

$$v = cz \times \sqrt{\frac{2g \times (By - Ax)}{Ac^2z^2 + Ba^2(c^2 - w^2)}}$$

$$\text{and } u = \frac{a}{\sqrt{(c^2 - w^2)}} \times \sqrt{\frac{2g \times (By - Ax)}{Ac^2z^2 + Ba^2(c^2 - w^2)}}$$

But to exterminate x and y ; let $a =$ the angle con , $d = rc$, $e = rn$; then $nk = w \tan a$, $na = \sqrt{(c^2 - w^2)}$, $ka = \sqrt{(c^2 - w^2)} - w \tan a$, $ai = \cos a (\sqrt{(c^2 - w^2)} - w \tan a)$ and $x = c - \cos a (\sqrt{(c^2 - w^2)} - w \tan a)$. Also, because $cr + oc = ac + co + on$, $d = z + y$, or $y = d - z = d - \sqrt{(a^2 - 2aw + c^2)}$; and we have

$$v = cz\sqrt{2g} \times \sqrt{\frac{B(d-z) - A(c - \cos a (\sqrt{(c^2 - w^2)} - w \tan a))}{Ac^2z^2 + Ba^2(c^2 - w^2)}}$$

$$\text{and } u = a\sqrt{\frac{2g}{c^2 - w^2}} \times \sqrt{\frac{B(d-z) - A(c - \cos a (\sqrt{(c^2 - w^2)} - w \tan a))}{Ac^2z^2 + Ba^2(c^2 - w^2)}}$$

$$\text{where } z = \sqrt{(a^2 - 2aw + c^2)}, \text{ or } w = \frac{a^2 + c^2 - z^2}{2a}$$

When the pulley is at n , and the arch rd is a quadrant, as in the question in the Diary, then $a = 0$, $w = or$, $a = c = e$, $d = c\sqrt{2}$, and we have

$$v = z\sqrt{2g} \times \sqrt{\frac{B(d-z) - A(c - \sqrt{(c^2 - w^2)})}{Az^2 + B(c^2 - w^2)}}$$

$$\text{and } u = a\sqrt{\frac{2g}{c^2 - w^2}} \times \sqrt{\frac{B(d-z) - A(c - \sqrt{(c^2 - w^2)})}{Az^2 + B(c^2 - w^2)}}$$

Or, substituting for z and w their proper values,

$$v = \sqrt{4gc} + \sqrt{\frac{B(c\sqrt{2} - \sqrt{(2c^2 - 2cw)}) - A(c - \sqrt{(c^2 - w^2)})}{2cA + B(c + w)}}$$

$$= 2c\sqrt{2g} \times \sqrt{\frac{B(c\sqrt{2} - z) - A(c - \frac{z}{2c}\sqrt{(4c^2 - z^2)})}{A \times 4c^2 + B(4c^2 - z^2)}}$$

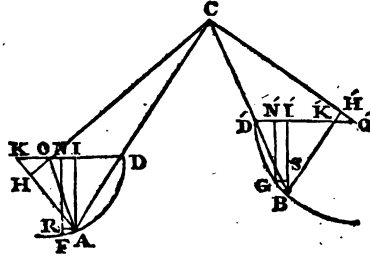
which agrees with the Diary solution.

When the weight A , descends from n , in the circle, and draws up the weight B , the problem is the same as quest. 545, Gents. Diary for 1785. In this case $x = ai = \sqrt{(c^2 - w^2)}$, y (the space ascended by B) $= ad = z$, $z = \sqrt{(2c^2 - 2cw)}$, and

$$v = z\sqrt{\frac{2g \times (Ax - By)}{Az^2 + B(c^2 - w^2)}} = \sqrt{2c} \times \sqrt{\frac{2g \times (A\sqrt{(c^2 - w^2)} - B\sqrt{(2c^2 - 2cw)})}{2Ac + B(c + w)}}$$

agreeing with the corrected expression for the velocity, given in the *Diary* for 1787.

Ex. 3. To find the velocities of the bodies, when x descends in a given circle $D'GB$, and draws up the weight A , which moves in another given circle FAD ; or, which is the same thing, when the bodies are fastened to the ends of two inflexible slender rods, without weight, moveable about the given points o, o' , and connected by a string passing over a pulley at c .



Let o, o' be the centres of the given circles, and draw AH , and BH' perpendicular to co and co' , meeting the horizontal lines $od, o'd'$ in κ and κ' ; and suppose the motion to commence when the bodies are at F and G .

Put $a = oc, b = o'c, c =$ the radius $od, c' =$ the radius $o'd', z = \kappa c, r = FA, s = GB, w = oH, \bar{w} = o'H', x = FK$ and $y = \kappa'S$.

$$\text{Then } \dot{r} = \frac{c\dot{w}}{\sqrt{(c^2 - w^2)}}, z^2 = a^2 - 2aw + c^2, \dot{z} = -\frac{a\dot{w}}{z}, \dot{s} = \frac{c'\dot{\bar{w}}}{\sqrt{(c'^2 - \bar{w}^2)}}$$

$$\kappa'^2 = (l-z)^2 = b^2 - 2b\bar{w} + c'^2, (l-z) \cdot = -\dot{z} = \frac{b\dot{\bar{w}}}{l-z}; \text{ hence } \frac{a\dot{w}}{z} =$$

$$\frac{b\dot{\bar{w}}}{l-z}, \text{ or } \dot{\bar{w}} = \frac{a(l-z)\dot{w}}{bz}, \text{ and therefore } \dot{s} = \frac{c'a(l-z)\dot{w}}{bz\sqrt{(c'^2 - \bar{w}^2)}}.$$

These values of $\dot{r}$ and $\dot{s}$, being substituted in the general formula for v and u , we shall have

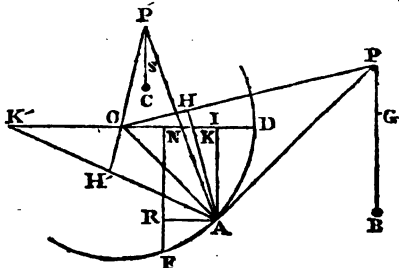
$$v = cbz \sqrt{\frac{2g \times (By - Ax) \times (c'^2 - \bar{w}^2)}{Ac^2b^2z^2(c'^2 - \bar{w}^2) + Bc'^2a^2(l-z)^2(c^2 - w^2)}}$$

$$u = c'a(l-z) \sqrt{\frac{2g \times (By - Ax) \times (c^2 - w^2)}{Ac^2b^2z^2(c'^2 - \bar{w}^2) + Bc'^2a^2(l-z)^2(c^2 - w^2)}}.$$

To exterminate x and y , let $\alpha =$ the angle $cod, \beta =$ the angle $co'd', e = FK, e' = \kappa'S$; then, as in the last example, $x = e - \cos \alpha (\sqrt{(c^2 - w^2)} - w \tan \alpha)$, and $y = \cos \beta (\sqrt{(c'^2 - \bar{w}^2)} - \bar{w} \tan \beta) - c'$, and these values being substituted in the above expressions, we shall have the velocities required in terms of z, w and $\bar{w}$; any two of which may be exterminated by means of the equations $z^2 = a^2 - 2aw + c^2$, and $(l-z)^2 = b^2 - 2b\bar{w} + c'^2$.

Ex. 4. A body A , being fastened to an inflexible rod of a given length, without weight, moveable about the point o , is drawn up by means of the weights B, C fastened to the ends of the strings $APB, A'C$, which pass over the fixed pullies P, P' . It is required to find the velocities of the bodies.

Draw AH and AN' perpendicular to OP and OP' , meeting the horizontal line OB in K and K' ; and suppose the bodies to be at r , g and s at the commencement of the motion. Put $OP = a$, $OP' = b$, radius $OA = c$, $OH = w$, $OH' = \bar{w}$, $AP = z$, $AP' = z'$, $AF = r$, the height BF , ascended by A , $= x$, the space GB , descended by B , $= y (= s)$, and the space GB descended by $C = h (= p)$.



$$\text{Then } \dot{r} = - \frac{c\dot{w}}{\sqrt{(c^2 - w^2)}}, \quad z^2 = a^2 - 2aw + c^2, \quad \dot{z} = - \frac{a\dot{w}}{z} = - \dot{s};$$

$$z'^2 = b^2 - 2b\bar{w} + c^2, \quad \dot{z}' = - \frac{b\dot{\bar{w}}}{z'} = - \dot{p}.$$

$$\text{But } - \frac{c\dot{w}}{\sqrt{(c^2 - w^2)}} = \dot{r} = \frac{c\dot{\bar{w}}}{\sqrt{(c^2 - \bar{w}^2)}};$$

$$\text{therefore } \dot{\bar{w}} = - \frac{\dot{w}\sqrt{(c^2 - \bar{w}^2)}}{\sqrt{(c^2 - w^2)}}, \text{ and } \frac{b\dot{\bar{w}}}{z'} = \frac{h\dot{w}}{z'} \times \sqrt{\frac{c^2 - \bar{w}^2}{c^2 - w^2}} = \dot{p};$$

$$\text{hence } \frac{\dot{s}}{r} = \frac{a\sqrt{(c^2 - w^2)}}{cz}, \text{ and } \frac{\dot{p}}{r} = \frac{b\sqrt{(c^2 - \bar{w}^2)}}{cz'}; \text{ and substituting}$$

these values of $\frac{\dot{s}}{r}$ and $\frac{\dot{p}}{r}$ in the general formula for the velocities,

we have

$$A\dot{v}^2 + B\dot{v}^2 \times \frac{a^2(c^2 - w^2)}{c^2 z^2} + C\dot{v}^2 \times \frac{b^2(c^2 - \bar{w}^2)}{c^2 z'^2} = 2g \times (By - Ax + Ch),$$

from whence we get

$$v = czx' \sqrt{\frac{2g \times (By - Ax + Ch)}{Ac^2 z^2 z'^2 + Ba^2(c^2 - w^2)z'^2 + Cb^2(c^2 - \bar{w}^2)z^2}}$$

$$\text{Then } u = ax' \sqrt{(c^2 - w^2)} \times \sqrt{\frac{2g \times (By - Ax + Ch)}{Ac^2 z^2 z'^2 + Ba^2(c^2 - w^2)z'^2 + Cb^2(c^2 - \bar{w}^2)z^2}}$$

$$\text{and } w' = bx' \sqrt{(c^2 - \bar{w}^2)} \times \sqrt{\frac{2g \times (By - Ax + Ch)}{Ac^2 z^2 z'^2 + Ba^2(c^2 - w^2)z'^2 + Cb^2(c^2 - \bar{w}^2)z^2}}$$

To exterminate x , y and h ; put $\alpha =$ the angle POB , $\beta =$ the angle $P'OD$, $e = FN$, $d = FP$, $d' = FP'$. Then, as in the last example, we find

$$AI = \cos \alpha (\sqrt{(c^2 - w^2)} - w \tan \alpha) = \cos \beta (\sqrt{(c^2 - \bar{w}^2)} + \bar{w} \tan \beta)$$

$x = e - \cos \alpha (\sqrt{(c^2 - w^2)} - w \tan \alpha)$, $y = d - z$, and $h = d' - z'$; these values of x , y and h being substituted in the above expressions, we shall have the velocities required, in terms of z , z' , w and $\bar{w}$; any

three of which may be exterminated by means of the equations $z^2 = a^2 - 2aw + c^2$, $z'^2 = b^2 - 2b\bar{w} + c^2$, and the equation expressing the relation between w and $\bar{w}$, viz. $\cos \alpha (\sqrt{c^2 - w^2} - w \tan \alpha) = \cos \beta (\sqrt{c^2 - \bar{w}^2} + \bar{w} \tan \beta)$.

QUESTION 1267. (Vol. iv.)

Additional Solution, by Mr. Edward Riddle, Newcastle-upon-Tyne.

Let FME be a section of the conical shadow, and HMKL another section perpendicular to the former. Draw GE, GF perpendicular to the tangents ME, MF, and GA perpendicular to the horizontal line LN. Then will N be the centre of the circle FIE, which circle is the base of the cone FME; and D'C' will be the transverse axis of the elliptic penumbral shadow, and CD that of the dark shadow.

Put $AG = a$; s and c = sine and cosine of BMC or BMD , the sun's semidiameter, and x = sine ABG the apparent altitude of the sun's centre. Then sine BDM , the apparent altitude of the sun's lower limb = sine $(ABG - BMD) = cx - s\sqrt{1 - x^2}$; and sine FEL the apparent altitude of the sun's upper limb = sine $(ABG + BMG) = cx + s\sqrt{1 - x^2}$. Now sine $ADG : GA :: \text{rad} : GB =$

$\frac{a}{x}$, and sine GMD : GX ::

$$\text{rad} : \text{GM} = \frac{a}{\delta}; \text{ hence}$$

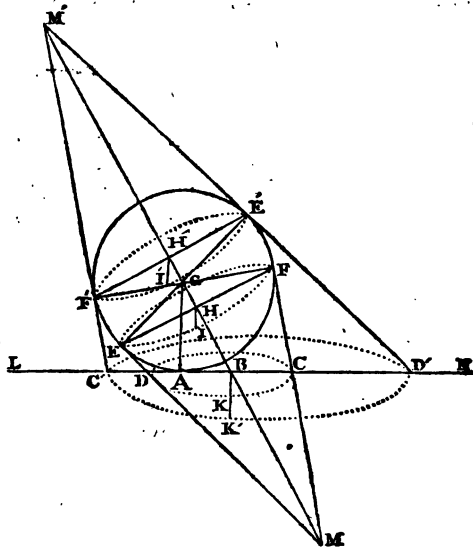
$$BM = GM \pm MB = \frac{a}{\rho} \pm$$

$$\frac{a}{x} = \frac{(x \pm s) \cdot a}{sx} \text{ . Again}$$

$$\sin \angle EDM : EM :: \sin \angle EMD : ED = \frac{(x \pm s) \cdot a}{cx' - sx\sqrt{1-x^2}}; \text{ and in the}$$

same way $BC = \frac{(x \pm s) \cdot a}{cx^2 + sx\sqrt{1-x^2}}$; hence $DC = DB + BC = \frac{2ac}{x \pm s}$.

But $\text{rad} : \text{GE} :: \cos \text{GME} : \text{EH}$ or $\text{HI} = ac$; and $\text{sine EMH} : \text{EH} :: \cos \text{EMH} : \text{MH} = \frac{ac^2}{s}$. Then by similar triangles $\text{HM} : \text{HI} :: \text{BM} :$



$EX = \frac{x \pm s}{cx}$, and by conics $\sqrt{(DB \cdot BC)} : EX :: DC : \text{conjugate} = 2a \cdot \sqrt{\frac{x \pm s}{x \mp s}}$. Put $m = .7853988$, &c. then $16a^2m = \text{superficies of the globe, and } m \cdot DC \cdot \text{conjugate} = 4a^2cm \sqrt{\frac{x \pm s}{(x \mp s)^3}} = \text{area of the elliptic shadow. Hence, per question, } 16a^2m = 4a^2cm \sqrt{\frac{x \pm s}{(x \mp s)^3}}$,

or by reduction, $c^2 \cdot (x \pm s) = 16 \cdot (x \mp s)^3$. In which equations the upper signs belong to the penumbral, and the lower to the dark shadow. By substituting for c and s their numerical values, we have $x = .2593044$ or $.240422 = \text{nat. sine } 15^\circ 1' 43''$ or $13^\circ 54' 41''$ apparent altitude of the sun's centre; with this corrected for parallax and refraction, the latitude and declination corrected to the reduced time, the time when the penumbral shadow would equal the superficies of the globe $10^h 22^m 2^s$, and for the dark shadow it is $10^h 5^m$.

PRIZE QUESTION 1811. (Vol. IV.)

*A Remark on a fluxional expression in the answer to this question;
By the Rev. Mr. John Hellins, B. D. F. R. S.*

If vv were substituted for x , in the expression $\frac{cc}{2dd\sqrt{g}} \times \frac{\dot{x}\sqrt{x}}{\sqrt{(l-cx-xx)}}$, and the numerical values of l and c were written for them, it would become $\frac{cc}{dd\sqrt{g}} \times \frac{vvv}{\sqrt{(272-28vv-v^2)}}$. Here the trinomial quantity under the vinculum is the product of the two binomials $35.633308 + vv$ and $7.633308 - vv$, which being denoted by $mm + vv$ and $nn - vv$, respectively, we have $\frac{vvv}{\sqrt{(272-28vv-v^2)}} = \frac{vvv}{\sqrt{(mm+vv)\sqrt{(nn-vv)}}$, which, by converting $\sqrt{(mm+vv)}$ into series, becomes $\frac{vvv}{m\sqrt{(nn-vv)}} \times (1 - \frac{vv}{2mm} + \frac{3v^2}{2 \cdot 4m^2} - \frac{3 \cdot 5v^3}{2 \cdot 4 \cdot 6m^3}$, &c.), an expression of which the fluents are easily found by means of a circular arch. And the series thus obtained will converge swiftly, since $\frac{vv}{mm}$, even when $vv = 6$, will be but about $\frac{1}{6}$.

INDEX

Of the Names of the Persons who have Proposed and Answered the Questions.

(N. B. The numbers affixed to the names shew which question each answered or proposed.)

A + B. Answered question 1055

Abbatt, Robert, Jun. *Ans.* 682, the prize question for 1783

Abram, John. *Proposed* question 1286

Ans. prize 1815

Adams, James. *Ans.* 798, 893, 913, 962, 969, 1035, 1037, 1059, 1062, 1149, 1154

Adams, T. *Ans.* 551

Adcock, Thomas. *Ans.* 627

Addison, John. *Prop.* 618

Ans. 570, 591, 600, 618

Adrastea. *Prop.* 89, prize 1722, prize 1724

Ans. prize 1722

Adway, Mrs. *Ans.* 30

Agriculturus. *Prop.* 668

Ainsworth, Jeremiah. *Prop.* 585

Allen, Mr. *Ans.* 79

Allen, Thomas. *Prop.* 502, 533, pr. 1766, 581, pr. 1768, 611, 626, 641

Ans. 447, 451, 474, 484, 485, 491, 493, 499, pr. 1762, 516, 517, 526, 527, 532, 545, 546, pr. 1766, 581, 593, pr. 1768, 608, 610, pr. 1769.

Allen, W. *Ans.* 409, 412

Allensis. *Ans.* 888

Amanda. *Prop.* 296

Amaryllis. *Ans.* 462

Ambler, J. R. *Ans.* 1291

Amicus. *Prop.* 264.

Ans. 251, 552

Amicus (the Rev. Mr. C. Wildbore).

Prop. 785, 795, pr. 1784, 842, 861, 876, 885, 905, pr. 1790, 937, pr. 1792, 967, 982, 996, 1012, 1023, pr. 1798, 1087, pr. 1802, pr. 1803

Ans. 766, 777, 778, 780, 781, 782, 786, 803, 808, 809, 811, 814, 815, pr. 1783, 818, 821, 823, 826, 827, 828, 830, 831, pr. 1784, 833, 840, 842, 844, 845, 846, 850, 852, 856, 857, 861, pr. 1786, 866, 868, 871, pr. 1787, 880, 886, 887, 889, 890, 891, pr. 1788, 904, 905, pr. 1789, 909, 914, 921, 922, 928, 930, 931, 936, 937, 942, 943, 947, 952, 958, 965, 966, 967, 974, 973, 977, 978, 980, 981, 982, 986, 988, 993, 995, 996, pr. 1795, 1006, 1010, 1011, 1012, pr. 1796, 1016, 1019, 1023, 1025, 1026, 1027, pr. 1797, pr. 1798, 1057, pr. 1799, 1062, 1087, pr. 1801, pr. 1802, pr. 1803

Amicus. *Ans.* 1237

- Anagramensis. *Prop.* 355
 Anderson, D. *Ans.* 1194, 1212, 1221
 Andrew, John. *Prop.* 92
 Ans. 38, 91, 92
 Annely, Bernard. *Prop.* 115
 Archimedes. *Prop.* 918
 Ans. 918
 Armstrong, Henry. *Prop.* 1033
 Armstrong, William. *Ans.* 66
 Armstrong, William (of Carlisle). *Prop.* 971
 Ans. 971, pr. 1794, 994, 1010, 1012
 Ash, J. *Prop.* 269, 286, 291, 310, 397, 411
 Ans. 266, 273, 276, 280, 282, 283, 284, 286, 290, 291, 339; pr. 1753, 381
 Ashby, Sam. *Prop.* 163, 170, pr. 1735
 Ans. 166
 Ashmore. *Ans.* 535
 Ashton, James. *Prop.* 893, 959
 Ans. 881, 901, 939, 959, 1046
 Ashton, John. *Prop.* 79
 Aspland, John. *Prop.* 718, 809, 867, 1140
 Ans. 642, 643, 669, 700, 708, 801, 866, pr. 1804
 Astronomicus. *Prop.* 653
 Ans. 546
 Atkinson, Henry. *Ans.* 1207, 1213, 1220, 1229, pr. 1811, 1240, 1251, pr. 1812,
 1270, 1271, 1275, 1281, 1282, 1296, pr. 1815
 Atkinson, Miss Maria. *Prop.* 377, 391
 Atkinson, Thomas. *Prop.* 490, 566
 Ans. 252
 Atticus. *Prop.* 826
 Ans. 826
 Auderson, George. *Prop.* 153
 Ans. 144, 145, 150, 155, 157, 159
 Bacon, William. *Ans.* 421
 Bagshaw, Wm. *Ans.* 1294
 Baines, John, Junr. *Prop.* 1210, 1230, 1243, 1263
 Ans. 1210, 1287, 1291
 Baker, Rev. Mr. Ant. *Prop.* 282, 312, 314, 328, 333, 357, 366, 368
 Ans. 303, 314, 318, 321, 328, 333, 357
 Baker, Thomas. *Prop.* 434, 489, 520
 Bamfield, Sam. *Prop.* prize 1747
 Ans. 384, 402, 411
 Bamfield, W. *Ans.* 450
 Bark, J. *Ans.* 428
 Barker, Thomas. *Prop.* 451, 468, 477, 498, 508, 520, 524, 536, 538, 575, 702,
 709, 723, 731
 Ans. 406, 428, 434, 444, 449, 452, 468, 476, 485, 521, 585, 709
 Barlow, Peter. *Prop.* 1123, 1143, 1175, 1189, 1205
 Ans. 1098, 1110, 1122, 1136, 1140, 1145, 1151, 1153, 1169, 1183,
 1189, 1190, 1207
 Barnes, George. *Ans.* 926
 Baron, George. *Ans.* 954, 972, 979
 Barrett, George. *Ans.* 1079, 1082, 1084, 1086, 1087
 Barron, John. *Ans.* 1068, 1075, 1083, 1098
 Bathonius. *Prop.* 381
 Batten, E. *Ans.* 466
 Battersbye, Thomas. *Prop.* 143
 Baxtonden, Thomas. *Prop.* 424
 Ans. 424, 452
 Bayley, Edward. *Ans.* 550
 Baylis, William. *Ans.* 1034
 Beacham, Mr. *Ans.* 180
 Beales, Rob. *Prop.* 54, 64
 Ans. 51

- Bean, William. *Ans.* 1215
 Beck, George. *Prop.* 688, 708, 827, 845
 Ans. 688, 708, 722, 811
 Beckerley, James. *Ans.* 1005, 1261
 Beighton, Celia. *Ans.* 97
 Beighton, Henry. *Prop.* 28
 Ans. prize 1712
 Beighton, Robert. *Prop.* 189
 Bell, Fr. *Prop.* 444
 Bell, Teasdale. *Ans.* 1260, 1271, 1274, 1277, 1291, 1295
 Bennett, J. *Prop.* 593
 Bennett, John. *Prop.* 1128
 Ans. 1119, 1128, 1132, 1166, 1188
 Bennett, R. *Prop.* 1047
 Benson, B. *Prop.* 957
 Ans. 893
 Bengal officer. *Prop.* 992
 Ans. 925, 992
 Beresford, Jas. *Prop.* 425
 Ans. 425
 Beriffe, Mr. *Prop.* 56
 Ans. 36, 51
 Beswick, George. *Prop.* 878
 Ans. 876
 Betts, J. *Prop.* 247
 Ans. 235
 Bevil, William. *Prop.* 339, 341, 372, 402, 417, 431, 439, 472
 Ans. 327, 341, 374, 387, 388, 396, 415, 431
 Bewby, George. *Ans.* 804
 Bewby, William. *Prop.* 1090
 Ans. 1074, 1078
 Birch, E. *Ans.* 1091
 Birch, J. *Prop.* 896
 Ans. 848
 Birchoverensis. *Prop.* 423
 Ans. 402, 409, 425, 443, 476, 480
 Bickford, John. *Prop.* 911
 Bird, Thomas. *Prop.* 199
 Ans. 201
 Birkin, William. *Ans.* 1263
 Birks, John. *Ans.* 370
 Birrinos. *Ans.* 237, 240, 257, 258
 Blackwell, John. *Prop.* 1081, 1098
 Ans. 1031, 1059
 Blakley, Luke. *Ans.* 1131
 Bleathman, Moses. *Ans.* 57, 58
 Blennerhassett, Wyvell. *Prop.* 504
 Boden, John. *Ans.* 884
 Bonycastle, John. *Prop.* 828, 843, pr. 1786, 904, 921
 Ans. 843, 904, 921
 Boote, Daniel. *Prop.* 222
 Boston, J. *Ans.* 392
 Boswell, John. *Ans.* 12, 31
 Bosworth, Newton. *Prop.* 1001
 Ans. 989, 1064
 Bosworth, Thomas. *Ans.* 539, pr. 1767, 641, 674, 758
 Botham, Abr. *Prop.* 515
 Ans. 397, 407
 Boucher, Edward. *Prop.* 716
 Boulby, George. *Prop.* 1045
 Bownas, Robert. *Prop.* 844, 922
 Ans. 783, 809, 820, 821, 823, 831, 844

- Boydall, Jos. *Ans.* 24
 Boydell, Mrs. *Prop.* 43
 Brady, Charles. *Ans.* prize 1792, 954, 1166
 Brampton, J. *Prop.* 483
 Branch, Billy. *Ans.* 335
 Bransby, John. *Ans.* 1036, 1040
 Bretherick, R. *Ans.* 864
 Brewer, Joseph. *Ans.* 1096, 1165, 1176, 1179, 1181, 1261
 Brickland, John. *Prop.* 436
 Ans. 436
 Brierly, J. *Ans.* 1248
 Brinkley, John. *Prop.* 791, 807, 820
 Ans. 780, 790, 791, 799, 807, 813
 Britannicus. *Ans.* 351
 Bromhall, T. *Ans.* 463, 479, 493, 495, 504
 Bronwin, Brice. *Prop.* 1200
 Ans. 1173
 Brook, J. *Ans.* 1256
 Brown, George. *Prop.* 110, 171, 177, 465
 Ans. 133, pr. 1728, 177
 Brown, George (Newcastle). *Ans.* 883, 972
 Brown, Hugh. *Prop.* 400, 415, 459, 485
 Ans. 379, 386, 400, 415, 428, 459, 477, 579
 Brown, Sarah. *Prop.* 29
 Brown, William, Bamburgh Castle. *Ans.* prize 1806
 Brownbridge, G. *Prop.* 393
 Ans. 398
 Browne, Nat. *Ans.* 14, 71, 110
 Browne, William. *Prop.* 141, 249
 Brownell, N. *Ans.* 573
 Bruce, E. *Ans.* 1006
 Bruce, John. *Ans.* 1005, 1228
 Bryhon, Mrs. Eliz. *Ans.* 1657
 Buchanan, A. Junr. *Prop.* 908, 943, 979
 Ans. 901, 964
 Buffham, George. *Ans.* 1076
 Bulman, John. *Prop.* 136, 179, 190, 290, 293, 299
 Ans. 135, pr. 1728, 160, 178, 190
 Bulmer, Thomas. *Ans.* 980
 Bumpkin, C. (Mr. J. Landen). *Prop.* 752
 Ans. 311, 322, pr. 1774, 740, pr. 1778
 Burdon, William. *Prop.* 1017, 1035, 1053, 1064, 1084, 1112, 1127, 1173, 1181,
 1212, 1213, 1234, 1235, 1261, 1262, 1277, 1293, 1294
 Ans. 944, 990, 1003, 1008, 1020, 1024, 1034, 1046, 1061, 1067,
 1070, 1072, 1081, 1084, 1085, 1090, 1097, 1100, 1197,
 1205, 1209, 1230, 1234, 1242, 1245, 1248, 1250, 1260,
 1261, 1272, 1278, 1289, 1290, 1291, 1293, 1294, 1295
 Burnet, John. *Prop.* 26
 Burrow, John. *Ans.* 865
 Burrow, Reuben. *Ans.* 645, 649, 656, 659, 660, 663, 666
 Burton, Ralph. *Prop.* 976
 Buscy, T. *Ans.* 31
 Butler, Robert. *Prop.* 510, 532, 540, 549, 562, 565, 570
 Ans. 398, 428, 442, 447, 482, 497, 499, 510, 525, 528, 529, 531,
 540, 549, 565
 Buttermann, W. *Ans.* 1091, 1092
 Buxton, German. *Prop.* 1170
 Byron, Hugh. *Ans.* 668
 Campbell Colin. *Prop.* 1010
 Ans. 960, 969, 981, 985, 997, 1011, pr. 1796, 1038
 Campbell, James. *Ans.* 1105, 1121

- Campbell, John. *Ans.* 1095
 Carr, John. *Prop.* 1101
 Ans. 1101
 Carss, William. *Prop.* 929
 Cary, Thomas. *Prop.* 60
 Gastieau, Miss M. *Ans.* 1059
 Cavill, John. *Ans.* 908, 958, 979, 1001, 1024, 1077, 1091
 Cetii, G. (Mr W. Wales). *Prop.* 629, 691
 Ans. 630, 636
 Chantry, John. *Prop.* 1125
 Ans. 1100, 1109, 1125
 Chapman, George. *Prop.* 1048
 Chapman, Thomas. *Ans.* 1291
 Chapman, William. *Prop.* 466, 478
 Ans. 484
 Chapple, William. *Prop.* prize 1746
 Charlton, Forster. *Ans.* 1289, 1292
 Charlton, John. *Ans.* 1273, 1275, 1289
 Charlton, Lionel. *Prop.* 428, 441
 Ans. 404, 411, 417, 426, 442, 446, 469
 Chartreux, W. (The Rev. Mr. W. Crakelt). *Prop.* 651
 Chipchase, John. *Prop.* 559, 568, 592, 608, 615, 637, 650
 Ans. 571, 588, 600, 637, 646
 Chrononmononpublicos. *Prop.* 345, prize 1752
 Clarke, Henry. *Prop.* 689, 706, 733, 754, 782, 815
 Ans. 627, 680, 691, 706, 713, 733, 762, 763, 764, 765, 768, 782, 810
 Clark, John. *Ans.* 131
 Clarke, James (Salford). *Ans.* 805, 806
 Clarke, James. *Ans.* pr. 1728
 Clarke, S. *Prop.* 634
 Clarke, Mr. *Prop.* 374
 Claxton, Miss Betty. *Ans.* 909
 Clayton, Josiah. *Prop.* 34
 Ans. 36
 Clement. *Ans.* 865
 Clericus. *Ans.* 606
 Clericus. *Ans.* prize 1795
 Clifton, T. *Ans.* 1197, 1212
 Coates, T. *Ans.* 371
 Cock, Thomas. *Ans.* 820
 Cockerill, C. *Prop.* 823
 Cockson, Cuthbert. *Prop.* 261, 284, 287
 Ans. 244
 Codling, Henry. *Ans.* 1106, 1130, prize 1805, 1198
 Colbourn, Elias. *Ans.* 126
 Cole, James. *Ans.* 23, 35
 Cole, William. *Prop.* 674, 687, 778, 1091, 1113, 1215, 1233.
 Ans. 553, 570, 659, 603, 666, 685, 745, 833, 1075, 1113, 1195
 Collinridge, Mr. *Prop.* 317
 Ans. 298
 Collins, John. *Prop.* 1082, 1155, 1259
 Ans. 1051, 1079, 1143, 1168, 1181, 1192, 1207, 1227, 1239, 1245,
 1264, 1265, 1277, 1288, 1290, 1292, 1293
 Collins, Thomas. *Ans.* 1292, 1294
 Conner, J. T. *Prop.* 986, 1008
 Cook, Anthony. *Ans.* 1273, 1276, 1287, 1291
 Cook, Thomas. *Ans.* 820, 1093
 Coquetarius. *Prop.* 670
 Corbett, J. *Prop.* 225, 323
 Ans. prize 1744
 Cornthwaite, John. *Ans.* 1111
 Corrie, A. *Ans.* 1077

- Cosier, Gideon. *Prop.* 21
 Cottam, William. *Ans.* 355, 356, 386
 Cottom, J. *Ans.* 655, 659
 Cotton, Johnson. *Ans.* 1168
 Coughron, George. *Prop.* 681
 Ans. 603, 614, 617, 626, 628, 631, 634, 642, 643, 647, 656, 659, 662, 681
 Coultherd, John. *Ans.* 1033
 Coultherd, Thomas. *Prop.* 1039, 1054, 1067
 Ans. 1024, 1030, 1037, 1040, 1048
 Cowen, Miss Sarah. *Prop.* 1030, 1059
 Cowley, Jos. *Ans.* 642
 Cowper, Thomas. *Prop.* 202, 235, 257, 325, 337, pr. 1751, 347, 364, 367
 Ans. 242, 243, 259, 294, 520, 525, 337, 342, 343, pr. 1751, 349, 351, 355
 Cox, Christopher. *Ans.* 932, 946, 950
 Crabb, William. *Prop.* 38, 74, 76, 80
 Ans. prize 1714, 77, 79
 Craggs, John. *Ans.* 878, 931, 999, 1018, 1024, 1036, pr. 1796, 1052, 1062, 1072,
 1237, 1270, 1228, 1284
 Crakeit, Rev. Mr. Wm. *Prop.* 607, 622, 635, 664, 739, 749, 779
 Ans. 596, pr. 1768, 602, 606, 607, pr. 1769, 615, 620, 622,
 638, 639, 641, 646, 647, 650, 651, 659, 662, 663, 664
 Cram, Thomas. *Ans.* 962
 Crane, William. *Ans.* 1164, 1198
 Craven, Jere. *Ans.* 1135, 1230
 Crocus Metallorum. *Prop.* 277
 Crosby, Thomas. *Prop.* 913
 Ans. 1000
 Croudace, Thomas. *Ans.* 1079, 1080
 Cullyer, John. *Prop.* 888, 901, 935, 944
 Ans. 852, 884, 1106, 1111
 Cunliffe, James. *Prop.* 974
 Cunliffe, Jas. Junr. *Prop.* 1246
 Ans. 1197, 1203, 1206, 1212, 1219, 1236, 1280
 Cunningham, David. *Prop.* 692, 705
 Ans. 670
 Curiosus. *Prop.* 534, 542, 558
 Ans. 523
 Dagger, Col. *Ans.* 198
 Dalby, Isaac. *Prop.* 648, 662, 669, 825, 831, 860, 871, 906
 Ans. 600, 629, 632, 644, 648, 659, 665, 666, 825, 831, 834, 836,
 839, 845, 856, 871, 906
 Dale, James. *Ans.* 937
 Dalton, George. *Ans.* 1289, 1290
 Dalton, John (Kendal). *Prop.* 884, 978
 Ans. 850, 864, 866, 875, 882, 885, 886, 897, 903, pr. 1789,
 908, 916, 921, 951, 963, 965, 978
 Dalton, John (South Cave). *Ans.* 925
 Daniel, William. *Prop.* 241, 248
 Ans. 224
 Darby, John. *Ans.* 1290, 1295
 Dawes, John. *Prop.* 1258, 1274
 Ans. 1136, 1150, 1264
 Dawson, M. *Ans.* 1265
 Davies, William. *Ans.* 428, 437, 455, 465, 467
 Davies, William (London). *Ans.* 985, 999
 Davis, Mr. *Prop.* 340
 Ans. 340
 Davis, William (schoolmaster at Crowan). *Ans.* 1029
 Davy, John. *Prop.* 1273, 1291
 Ans. 1179, 1181, 1211, 1213, 1218, 1228, 1254, 1257, 1258, 1266, 1275,
 1277, 1279, 1281, 1289, 1290, 1291, 1292, 1293, 1294, 1296

- Deare, W. *Prop.* 77, 90
 Dees, Ralph. *Ans.* 699
 Denning, Richard. *Prop.* 851
 Ans. 772
 Dewhurst, John. *Prop.* 1242
 Ans. 1237
 Dicker, Samuel. *Prop.* 84
 Dickinson, T. *Ans.* 1153
 Dicky, Master. *Prop.* 335
 Dictator Roffensis. *Prop.* 334
 Dillwyn, L. W. *Ans.* 1018
 Diophantoides. *Prop.* 853
 Diophantus. *Prop.* 254
 Dixon, Nichol. (Mr John Hunter). *Prop.* 363
 Ans. 347, 349
 Dixon, Lieut. Wm. *Prop.* 946
 Dobson, Peter. *Prop.* 1176, 1180
 Ans. 1171
 Dod, Eliz. *Ans.* 79, 95
 Dodd, Thomas. *Prop.* 41, 53, 70, 75, 112, 126, 131
 Ans. 37, 48, 76, 77
 Doidge, William. *Prop.* 59, 66, 82, 103, 109
 Doedle, Timothy. (Mr Simpson). *Prop.* 386, 388
 Ans. 362, 372
 Dotchen, J. A. *Prop.* 1248
 Ans. 1125, 1135, 1152, 1167, 1182, 1183, 1194, 1195, 1196, 1199,
 1200, pr. 1809, 1213, 1219, 1226, 1228, 1243, 1245, 1257,
 1263, 1268, 1272, 1279, pr. 1814, 1293
 Doubt, J. *Ans.* prize 1734
 Dougharty, Jos. *Prop.* 86
 Dover, Richard. *Prop.* 1032
 Ans. 1001
 Dowden, Robert. *Ans.* 757, 800, 822, 854
 Draconarius. *Ans.* 661
 Dubliniensis. *Prop.* 925
 Duckitt, George. *Prop.* 1265, 1276, 1296
 Ans. 1240, 1255, 1276, 1294
 Dunn, William. *Ans.* 1226
 Dunthorne, Richard. *Prop.* prize 1738, 210
 Ans. 175, 191, 193, 201, 203
 Dutton, Ralph. *Prop.* 715
 Dymond, Joseph. *Prop.* 588, 604, 765
 Eadon, George. *Prop.* 701
 Eadon, John, Junr. *Ans.* 743, 1029
 Eaton, William. *Prop.* 1108
 Ans. 1057, 1100, 1108, 1114
 Edens, John. *Prop.* 47
 Ans. 32, pr. 1713, 40, pr. 1714
 Edwards, J. *Prop.* 602
 Edwards, Joseph. *Prop.* 676
 Elley, T. *Ans.* 1125
 Elliott, Richard. *Ans.* 992, 1004, 1019
 Elphick, Ed. *Prop.* 46
 Elstob, Mark. *Prop.* 714, 774
 Ans. 643, 703, 714, 718
 Embleton, Richard. *Ans.* 991, 1003, 1009
 Embleton, William. *Prop.* 514, 522
 Ans. 477, 479, 490, 493, 495, 497, 506, 514, pr. 1763
 Emerson, William. *Ans.* 287
 Enefer, W. *Ans.* 378

- England, Edward. *Ans.* 1136, 1164, 1167, 1267
 Enotts, Edward. *Ans.* prize 1735
 Epps, John. *Prop.* 1272
 Evan, Lewis. *Prop.* 23, 63
 Ans. 38, 88, 98
 Evans, Rev. Mr. Lewis. *Prop.* 835, 914, 977, 1009, 1024
 Ans. 803, 804, 819, 835, 858, 870, 917, 961, 970, 982,
 990, 1009, 1015, 1070
 Evans, Tho. Simpson. *Ans.* 973, 1050, 1052, 1061
 Ewbank, Rev. Mr. J. *Prop.* 947
 Ans. 971, 977, 1002, 1049, 1069, 1090, 1120
 Exley, Thomas. *Ans.* 1259
 F. R. S. *Prop.* 240, 272
 Ans. 240, 242
 Facer, Joseph. *Ans.* 939, 1008
 Failes, Mr. *Ans.* 78
 Fall, R. *Ans.* 219
 Farey, John. *Prop.* 887, 902, 931
 Ans. 859, 876, 887, pr. 1788, pr. 1790, 1121, 1126
 Farnell, Thomas. *Prop.* 1068
 Farrah, J. *Ans.* 1055
 Farrer, Nich. *Prop.* 214, 224, 231, 260, pr. 1745, 266, 273, 283, 321
 Ans. 205, 207, 216, 221, 224, 228, 230, 244, 246, 247, 249, 250,
 251, 253, 259, 261, 262, 270, 271, 297
 Fatherly, John. *Prop.* 719
 Ans. 668, 704, 716, 747
 Fearn, Thomas. *Prop.* 183
 Fearnside, Robert. *Prop.* 145, 158, 159, pr. 1732, pr. 1733, 175, 185
 Ans. 135, 136, 139, pr. 1729, 148, 152, 158, 159, 162, pr. 1731,
 164, 166, 168, 170, 173, 185
 Featherstonhaugh, W. *Prop.* 1006, 1061
 Fellows, J. *Prop.* 440
 Fennell, John. *Ans.* 995
 Fenwick, Lieut. T. H. *Ans.* 946
 Ferdinando. *Prop.* 964
 Fermat, Mons. *Prop.* 606
 Fidelio. *Prop.* 942
 Fildes, John. *Prop.* 909
 Ans. 1021
 Filius-Diophanti. *Prop.* 275
 Finch, John. *Ans.* 70, 77, 78, 79, pr. 1728
 Fish, Amos. *Prop.* 22, 24
 Fish, John. *Prop.* 362
 Ans. 352
 Fisher, Joseph. *Prop.* 494, 561
 Ans. 561
 Fisher, Lydia. *Prop.* 18
 Fleck, Matthew. *Ans.* 848, 865, 879, 895
 Fletcher, John. *Ans.* 761, 777, 807
 Fletcher, Thomas. *Prop.* 51
 Ans. 73
 Flitcon, Mr. *Ans.* 281, 343, 400, 421
 Ford, John. *Ans.* 15, pr. 1711
 Ford, Thomas. *Prop.* 1228
 Ans. 1209, 1232
 Forster, Richard. *Ans.* 1225
 Fortunatus. *Prop.* 268
 Fox, Alfred. *Ans.* 1254, 1259, 1266, 1272, 1280, 1284, 1289, 1290, 1293, 1295
 Francis, William. *Ans.* 728, 744
 Francis, Wm. Junr. *Prop.* 1056, 1110
 Ans. 1056, 1066, 1076, 1083, 1098, 1110, 1124
 Franciscus. *Ans.* 25

- French, Joseph. *Ans.* 806
 Furnass, Rev. Mr. J. *Prop.* 1055, 1071, 1074, 1107, 1145, 1158, 1172, pr. 1809, 1241
Ans. 1000, 1042, 1045, 1098, 1127, 1135, 1172, 1174, 1192, 1216, 1236, 1237, pr. 1814, 1289, 1290
 G. J. *Prop.* 1102, 1116, 1129, 1144, 1159
Ans. 1102, 1116, 1129, 1144, 1159
 Gale, James. *Prop.* 1062
Ans. 1056
 Gallyatt, Edward. *Ans.* 736
 Garnett, Joseph. *Prop.* 1011
Ans. 924, 944, 948, 967
 Garth, George. *Ans.* 1167
 Gawith, William. *Prop.* 603, 614, 628
Ans. 586
 Gawthorpe, W. *Ans.* 377
 Gee, J. *Ans.* 1030
 Gemini. (Mr. W. Wales). *Ans.* 673
 Geometricus. (Mr. W. Wales). *Prop.* 506, 526, 675
 George, Philip. *Ans.* 895, 399
 Gibbons, Richard. *Prop.* 218, 398, 450, 484, 519, 555, 569
Ans. 247, 296, 338, 339, 363, 384, 393, 479, 483, 492, 506, 535, 551, 555, 578, 582
 Gill, W. *Ans.* 119
 Gittins, Joseph. 1002, 1092, 1119, 1137
 Glendinning, Adam. *Prop.* 1280
Ans. 1287, 1288, 1289, 1290, 1293, 1295
 Glenie, Lieut. *Prop.* 734, 793, 811, 840, 854, pr. 1794, 994, 1007, 1023, 1036
Ans. 793, 840, pr. 1794, 1007, 1023
 Glover, Charles. *Ans.* 84, 90
 Golding, Edward. *Ans.* 178, 184, 186
 Golding, Master John. *Ans.* 1089
 Gompertz, Benjamin. *Ans.* 994
 Gooch, William. *Prop.* 850, 872
Ans. 849
 Gorton, William. *Ans.* 851
 Gosling, Thomas. *Ans.* 11
 Gough, John. *Ans.* 827, 1041
 Gould, John. *Prop.* 874
Ans. 690, 692, 694
 Grant, Thomas. *Prop.* 113, 121, 134, 138, 146, 165
Ans. 121, 131, 154, 137, 146
 Green, Charles. *Prop.* 499
Ans. 486, 499
 Green, William. *Ans.* 1001, 1031, 1066, 1074
 Gregory, O. G. *Prop.* 972, 989, 1022, 1086, 1097, 1117, 1142, pr. 1805
Ans. 1046, 1050, 1055, 1068, 1074, 1083, 1094, 1097, 1100, 1105, 1107, 1112, 1115, 1117, 1119, 1123, 1128, 1132, pr. 1805
 Gregory, William. *Ans.* 1250
 Griffith, John. *Ans.* 899
 Griffiths, Edward. *Ans.* 503
 Grimmett, William. *Prop.* 156, 169
Ans. 147, 152, 153, 156, 162, 163, 164, 166
 Guady, John. *Prop.* 176, 180
 Gurney, Mr. *Ans.* 77
 H. G. *Ans.* 995
 Hackman, Mr. *Ans.* 1248
 Haldane, Major Henry. *Ans.* 987
 Habershon, Jos. Jones. *Prop.* 1244
Ans. 1250
 Hacking, Mr. *Ans.* 1248
 Hale, Christopher. *Prop.* 157, 172
Ans. 157, 172

- Hale, Joseph. *Ans.* 1165, 1187, 1258, 1265
 Hall, R. *Ans.* 36, 54
 Hall, Thomas. *Ans.* 712
 Hamilton, M. *Ans.* 1153
 Hampshire, John. *Ans.* 779, 813
 Hampson, John. *Prop.* 288, 304, 316, 476, 491
 Ans. prize 1728, 275, 288, 422, 483, 491, 505
 Hann, Henry. *Ans.* 1090, 1109
 Hannah, John. *Ans.* 898
 Hardy, William. *Prop.* 900
 Hare, E. *Ans.* 493
 Hare, G. *Prop.* 78
 Harker, Rev. Geo. *Prop.* 1550
 Ans. 1232, 1250
 Harlstone, T. *Prop.* 1289, 1290
 Ans. 1259, 1269, 1271
 Harmonicus. *Prop.* 319
 Harris, Christopher. *Prop.* 88
 Harris, J. *Ans.* 61
 Harris, J. B. *Ans.* 1135
 Harris, Thomas. *Prop.* 464, 486, 500, 557
 Ans. 467, 485, 486, 557
 Harrison, John. *Ans.* 910
 Harrison, William. *Ans.* 394
 Hartley, James. *Prop.* pr. 1753
 Ans. 349, 350, 354, 355, 356, 357, 358
 Hamley, John. *Prop.* 131
 Hartley, J. *Prop.* 955
 Ans. 902, 939, 1000, 1020, 1032, 1053
 Hartley, Robert. *Prop.* 799, 812
 Ans. 749, 761, 782, 791
 Harvey, Geo. Junr. *Ans.* 1203, 1215
 Hastings, David. *Ans.* 467, 478, 480
 Hatton, Daniel. *Ans.* 29
 Hauxley, Edward. *Prop.* 186
 Hawkes, John. *Prop.* 1029, 1066, 1137
 Ans. 1044
 Hawney, William. *Prop.* 20, 49, 67
 Ans. 78
 Haycock, John. *Prop.* 916
 Ans. 901, 911, 931, 1004
 Haydock, William. *Ans.* 1044
 Haywood, T. *Prop.* 44
 Hazard, Jack. *Prop.* 541
 Ans. 541
 Mearding, J. H. *Prop.* 1024
 Ans. 1089, 1105, 1110, 1112, 1132, 1186, 1161, 1169, 1179, 1186,
 1191, 1192, 1196, 1200, pr. 1809, 1214, 1219
 Heath, Robert. *Prop.* 196, 209, pr. 1739, 213, pr. 1740, pr. 1742, 239, 265, 277,
 278, pr. 1746, 289, 292, 295, 306, pr. 1748, 330
 Ans. 196, 197, pr. 1737, 199, 200, 202, 204, 205, 207, 208, 210, 211, 217,
 218, pr. 1740, 221, 245, 248, 260, 263, 264, pr. 1745, 265, 268,
 277, pr. 1746, 281, 289, 293, 296, 302, 306, pr. 1748, 317, 326
 Hedley, William. *Ans.* 676
 Heford, J. *Prop.* 1076
 Heford, S. *Prop.* 1092
 Hellins, Rev. Mr. John. *Prop.* 707, 810, 821, 841, 859, 889, 903, 919, 981, 991
 Ans. 707, 773, 788, 793, 792, 799, 902, 812, 816, 831,
 841, 903, 919, 981
 Hemmingway, Mr. *Ans.* 230
 Henry, David. *Prop.* 1063, 1115
 Ans. 1063, 1085, 1115

- Hepworth, Mr.** *Prop.* 910
Herschell, William. *Ans.* prize 1779
Hesychia. *Ans.* 49, 67
Heyshot, Lady Mer. *Prop.* 58
Hewitt, Thomas. *Prop.* 1100
Hilditch, Joseph. *Prop.* 370
Hill, John. *Prop.* 250
Ans. 192, 207
Hill, S. (of Welworth) *Prop.* 927
Hill, T. *Ans.* 74
Hind, T. *Prop.* 1014, 1051
Hine, Joseph. *Prop.* 1236, 1239
Ans. 1225, 1239
Hingeston, Peter. *Prop.* prize 1722
Hitchins, Malachy. *Prop.* 511
Ans. 467, 489, 518, 520
Hodge, John. *Prop.* 37
Hodges, Stephen. *Prop.* 342, 554, 645, 660, 672
Ans. 334
Holden, F. *Ans.* 355, prize 1752
Holden, Richard. *Ans.* 485
Holme, Christopher. *Ans.* 1014
Holt, Charles. *Ans.* 1233, 1292
Holt, John. *Prop.* 939
Ans. 924, 949
Honey, John. *Ans.* 350, 359, 400, 410, 471
Honor, William. *Prop.* 350
Honour, a person of. *Prop.* 361
Honorius. *Prop.* 339
Hopkinson, T. *Ans.* 439
Hopper, Thomas. *Prop.* 1120, 1151
Ans. 1048, 1104, 1120, 1122, 1134, 1140
Horn, Jonathan. *Prop.* 1016
Hornby, Jonathan. *Prop.* 880, 940
Hornby, Thos. *Prop.* 1005, 1077
Ans. 1041, 1069, 1109, 1112
Morner, W. G. *Prop.* 1282
Ans. 1214, 1248, 1269, 1276, 1282, pr. 1814, 1284, 1234, 1235
Monicultura. *Ans.* 776
Hoskins, N. *Prop.* 895
Houlgate, John. *Ans.* 1071
Howard, John. *Prop.* prize 1799
Ans. 869, 945, pr. 1797
Hubert, H. Junr. *Ans.* 1256
Hudson, John. *Prop.* 453, 505
Ans. 439, 490, 496
Hull, R. *Ans.* 54
Hunt, J. *Prop.* 848, 863, 924
Hunter, Henry. *Prop.* 1078
Ans. 1009, 1028, 1044, 1078
Muntly, Thomas. *Ans.* 358
Hurlothrumbo. (Mr. Simpson). *Prop.* 229, 236
Ans. 226, 228, 229, 230, 232
Hurlothundro. *Prop.* 278, 309
Ans. 309
Hutchinson, Hugh. *Ans.* 1017, 1135
Huthersall, John. *Prop.* 1002
Hutton, Charles. *Prop.* 580, 596, 610, 652
Ans. 505, 507, pr. 1763, 522, 523, 524, 544, 545, 549, 562, 565,
pr. 1766, 580, 581, 595, 624, 661, 634, pr. 1771, 652
I. B. S. *Prop.* 203
I. R. S. *Ans.* 19
Jackson, John. *Ans.* 225

- Jackson, John. *Prop.* 804, 857, 985
Ans. 715, 835, 957
 Jackson, John (of Sherburne.) *Prop.* 1179
 Jackson, Miss Susan. *Ans.* 1241, 1254, 1269, 1273
 James, Joseph. *Prop.* 669, 730
Ans. 702, 729, 759
 Jeacock, Joseph. *Ans.* 42
 Jennings, J. *Prop.* 81
 Jepson, W. *Prop.* 327
Ans. 285, 312, 316, 327
 Index. *Ans.* 1071
 Ingleborough, John. *Prop.* 142
 Johnson, Charles. *Ans.* 1045
 Johnson, Edward. *Prop.* 458
Ans. 437
 Johnson, John. *Ans.* 1253
 Jones, Samuel. *Prop.* 1150
Ans. 1110, 1292, 1294
 Jope, John. *Ans.* 75, prize 1720, 86
 Juvenis. *Prop.* 421
Ans. 557
 Juvenis. *Prop.* 864
 Kay, Peter. *Prop.* 221, 223, 227, 238, 246
Ans. 213, 218, 223, 227, 238
 Kay, William. *Prop.* 724
 Kaye, Joseph. *Ans.* 1089
 Kelly, Mr. *Ans.* 743
 Kemp, S. *Prop.* 479
 Kew, Nath. *Ans.* 37
 Kenabell, Jonathan. *Ans.* 406
 Kempton, Edward. *Prop.* 492
 King, Thomas. *Ans.* 1113
 King, William. *Prop.* 685, 717, 758
 Kingston, William. *Prop.* 394
Ans. 252, 379, 408, 423, 435, 467, 471, 487
 Kinnebrook, David. *Prop.* prize 1765
Ans. 493, 501, 508, 512, 515, pr. 1763, 526, 528, 534, pr. 1776
 Kinnebrook, David, Junr. *Prop.* 945, pr. 1793
Ans. 880, 885, 913, 916, 924, 927, pr. 1791, 947, 949, 1049, 1053
 Kirton, Thomas. *Prop.* 1046, 1152, 1209, 1229
Ans. 1142
 Knight, J. *Ans.* 972, 984
 Knipe, John. *Ans.* 106
 Knowles, John. *Ans.* 813
 Koit, Samuel. *Ans.* 377
 Kubernetes. (Mr. Simpson). *Ans.* 366, 369, pr. 1755, 420, pr. 1759
 Lady, A. *Ans.* 6
 Lady, A. Grubean. *Ans.* 170
 Lamplugh, Mr. *Ans.* 1295
 Landen, John. *Prop.* 245, 259, 276, 279, 285, 301, 307, 311, 313, 322, pr. 1749
Ans. 243, 255, 256, 272, 274, pr. 1746, 285, pr. 1747, 300, 301, 304, 305, 307
 Langdon, Robert. *Prop.* 1044
 Laughton, W. *Ans.* 60
 Laurent, P. M. *Ans.* 1054
 Lawson, Rev. Mr. John. *Prop.* 665, 680, 722, 736, 746, 764
Ans. 607, 680, 722, 736, 746
 Lawton, William. *Ans.* 930, 935
 Lax, William. *Ans.* 250
 Leader, J. *Ans.* 574
 Lean, Joel. *Prop.* 775, 886
Ans. 751, 775
 Loddell, John. *Prop.* 63

- Leming, P. *Ans.* 293
Leigh, Thomas. *Prop.* 833
Leighton, William. *Prop.* 338
Lewis, J. *Ans.* 34
Leybourn, Thomas. *Ans.* 900, 908, prize 1792
Liddell, John. *Prop.* 928, 960, 980, 1019
Ans. 960, 1019
Lightbourn, Henry. *Prop.* 1119
Ans. 1104, 1168
Line. *Ans.* 710, prize 1777
Lingen, Mr. *Ans.* 29, 37
Litson, W. *Ans.* 376, 391
Llerien. *Prop.* prize 1812, pr. 1813, pr. 1814, pr. 1815
Ans. prize 1812, pr. 1813, pr. 1815
Lockwood, J. *Prop.* 1121
Ans. 1046, 1111
Lodge, George. *Prop.* 582
Lord, Mr. *Ans.* 194
Lovatt, Richard. *Prop.* 149, 174, 200, 211
Ans. 141, 142, 143, 165, 167, 174, 200
Lover, Joshua. *Prop.* 32
Ans. 29
Lowe, John. *Prop.* 140
Ans. 140
Lucas, W. *Ans.* 423
Lunarian, A. *Ans.* 917
Lycett, Richard. *Prop.* 195
Ans. 195
Lycidas. *Prop.* 546
Ans. 528
Lydal, B. *Ans.* 391, 435
Lynn, John. *Ans.* 705, 706, 718, 719
M. J. *Ans.* 361
M. P. of Durham. *Prop.* 469, 481, 496
Ans. 455, 477, 478, 480, 481, 485, 488, 496, 497, 501, 503, 510, 512, 513
M. W. *Ans.* 442
Mabbott, Jonathan. *Prop.* 760
McCarthy, Felix. *Ans.* 556
Macdonald, J. *Prop.* 1098
McDoneld, J. T. *Ans.* 1202, 1207, 1244, 1251, prize 1812
McRoy, Mor. *Prop.* 456
Maddocks, William. *Prop.* 1105
Maffett, R. *Ans.* 1292, 1295
Makinson, William. *Ans.* 1270, 1289, 1291
Mallock, Richard. *Prop.* 482
Ans. 424, 441, 478, 491, 496
Mancuniensis. (Mr. John Holt). *Prop.* 915
Ans. 896, 912
Manlove, Miss. *Prop.* 298
Mark, John. *Prop.* 1245
Matrat, William. *Prop.* 1069, 1206, 1221
Ans. 1127, 1185, 1188, 1201, 1206, 1271
Marriot, Sam. *Prop.* 122
Ans. prize 1725
Marriott, William. *Prop.* 991, 1015
Ans. 991, 999, 1001
Marshall, Robert. *Ans.* 672
Maseres, John. *Prop.* 950
Mason, Charles. *Prop.* 91, 93, 99, 114, 118, 128, 151, 160, 166, 168, 178,
prize 1734, 182, 192, 198, 201, 204, 212, 271, 294, 326,
331, 368, 412, 452
Ans. 41, 42, 43, 44, 45, 46, prize 1717, 92, 97, 111, 117, 120,
128, 160, 162, 166, 181, 182, 184, prize 1735, 198, 201, 326

- Mason, James. *Ans.* 1050
 Mason, Miss Nancy. *Prop.* 984, 1003, 1018
 Massey, William. *Prop.* 48, 144
 Mathematicus. *Ans.* 557, 740
 Mathematicus, Juvenis. *Ans.* 178
 May, J. *Prop.* 207, 215, 219, 228, 242, pr. 1744, 281
 Ans. 201, pr. 1738, 218, 222, 228, 242, pr. 1744
 Mayo, R. *Prop.* 630
 Mellor, William. *Prop.* 1160
 Ans. 1123
 Merones. (Mr. Emerson). *Prop.* 193, 206, 220
 Ans. prize 1736, 193, 194, 195, 197, pr. 1737, 205, 206, 207, 209, 210,
 215, 217, 221, 223, pr. 1741, 226, 229, pr. 1742, 238, 240
 Metro. *Prop.* 1183
 Middleton, William. *Prop.* 1065
 Ans. 1061, 1064
 Middleton, Miss Maria. *Ans.* 1002, 1014, 1017, 1041
 Milbourn, J. *Ans.* 365, 378, 395, 440
 Milburn, Henry. *Ans.* 1264, prize 1813, 1270, 1278, 1289, 1292, 1294
 Mills, James. *Ans.* 550, 666, 588
 Milner, Thomas. *Prop.* 1042
 Milward, W. *Ans.* 132
 Mole, John. *Ans.* 772
 Molineux, Thomas. *Prop.* 1036
 Mooney, Michael. *Prop.* 958, 1139
 Ans. 925, 1139
 Moore, Edward. *Ans.* 52
 Moore, William. *Prop.* 1164, prize 1808, 1207, 1222
 Ans. 1169, 1183, 1191, prize 1808
 Morgan, Mrs. *Ans.* prize 1714
 Morland, John. *Prop.* 379
 Ans. 390
 Morley, T. *Ans.* prize 1808
 Morley, William. *Prop.* 1195
 Morun, John. *Prop.* 1134
 Moss, Thomas. *Prop.* 382, 384, 396, 403, 416, 427, 437, 455, 501, 503, 513,
 pr. 1763, 545, 548, 560, 577, 579, pr. 1767, 595, 604,
 623, 633, 636, 647, 658, 661, 678, 722, 737, 745, 765, 783,
 Ans. 382, 387, 405, 427, 453, 493, 501, 512, 513, pr. 1763, 548,
 560, 577, 579, 604, 623, 636, 647, 658, 737, 745
 Moyle, M. *Prop.* 85
 Ans. 49
 Mouldsdaie, Joseph. *Ans.* 994, 1007, 1025, 1031
 Mountjoy, R. *Prop.* 973
 Mudge, Col. William. *Prop.* 890, pr. 1789, 920, pr. 1791, 952, 997, pr. 1792,
 pr. 1800, pr. 1807, pr. 1811
 Ans. 887, 889, 891, 920, pr. 1791, 982, pr. 1797, pr. 1807
 Mundell, Thomas. *Ans.* 1274
 Murck, M. H. *Prop.* 1111
 Myers, Thomas. *Prop.* 1141, 1156, 1174, 1191, 1201
 Ans. 1126, 1141, 1151, 1156, pr. 1806, 1270, 1175, 1182, 1191
 Nairne, Da. *Ans.* 129
 Naughtley, Alex. *Prop.* 71, 95, 107
 Ans. 48, 73, 108, 114
 Nauticus. (Mr. W. Wales). *Prop.* 725, 738, 753, pr. 1780, 780, 796, 816
 Ans. 731, 738, 769, pr. 1780, 796
 Neild, Thomas. *Prop.* 879, 954
 Nelson, Mary. *Prop.* 72
 Ans. 27, 28, 33, 38, pr. 1714, pr. 1715
 Neshiz, A. *Prop.* 1198, 1225, 1254, 1271
 Ans. 1184, 1211, 1236, 1239
 Newbegin, Mr. *Ans.* 1236, 1239

- Newbold, John. *Prop.* 40
Ans. 24, 33, 36, 38, 39, pr. 1714
- Newby, William. *Prop.* 1031, 1070
Ans. 1051
- Newton, Sir Isaac. *Prop.* prize 1743
- Newtoniensis. *Ans.* 332, prize 1750
- Nichols, Ann. (Mr. W. Wales). *Prop.* 462, 475, 487, 516, 527, 543, 551, 556, 567, 588
Ans. 552
- Nichols, J. *Ans.* 392
- Nicholson, James. *Prop.* 712
Ans. 668, 758, 950, 966, 1011
- Nicholson, Richard. *Prop.* 1169
Ans. 1017, 1025, 1137, 1140, 1142, 1154, 1160
- Nondum, Esq. *Ans.* prize 1794
- North, James. *Prop.* prize 1806, 1279
Ans. 1158, 1162
- Nosnihctuh. *Prop.* 493
Ans. 482
- Nott, Edward. *Ans.* 464, 580
- Nujo Dargnas. (The Rev. Mr. W. Crakelt). *Prop.* 590
Ans. 590
- Nunn, Robert. *Prop.* 1197
Ans. 1124, 1180, 1258
- Nuttall, John. *Ans.* prize 1794
- Oats, N. *Prop.* 237
Ans. 179
- O'Cavanah, Patrick. (Mr. Simpson). *Prop.* 387, 405, 413, 420, pr. 1757, 435, 446,
pr. 1759, 470, pr. 1760
Ans. pr. 1753, 405, 410, pr. 1756, 425, 490, 432, 433, 438,
443, 445, 454, 460, pr. 1760
- O'Connor, Cullen. *Prop.* 694, 710
Ans. 694, 770, 971
- Ogle, Stephen. *Prop.* 497, 619, 649, 663, 853
Ans. 621, 663
- Oliver, R. *Ans.* 1030
- Omicron. *Prop.* 837, 1297
Ans. 1297
- Ommanney, John. *Ans.* 154, 175
- Orchard, Joseph. *Ans.* 342, 343, 347, 348, 351
- O'Riordan, Dr. M. S. *Prop.* 1185
- Osborne, Jer. *Ans.* 569, 589
- Page, Edward. *Ans.* 1293
- Painter, John. *Ans.* 103, prize 1723, 105
- Pamphagus. (The Rev. Mr. W. Crakelt). *Prop.* 589
Ans. 569, 589, 573, 575
- Pamphilus, Eumenes. *Prop.* 184, 188
Ans. 176, 181, 188
- Park, Jacob. *Ans.* 915, 928, 945
- Parker, Richard. *Ans.* 17
- Parglos, P. *Prop.* 1069
- Parlour, S. *Ans.* 1215, 1227
- Parnel, E. *Ans.* 600
- Parnell, Nathan. *Prop.* 744, 762, 779, 798, 813, 824
Ans. 700, 718, 734, 736, 739, 744, 751, 761, 766, 778, 779, 785,
792, 798, 799, 824
- Paty, J. *Prop.* 584, 598
Ans. 569
- Peace, Joseph. *Ans.* 883
- Peacock, Jos. *Ans.* 42
- Pearson, William. *Prop.* 1000
Ans. 900, 942, 976

- Peart, Thomas. *Prop.* 418
Ans. 399, 403, 407, 410, 418
- Pcirce, J. *Ans.* 53, 56, 57
- Penberthy, John. *Prop.* 742, 772
Ans. 772
- Penn, Wm. Junr. *Prop.* 866
- Penovius. *Ans.* 417, 430, 435, 438
- Perrott, George. *Ans.* 683
- Phelan, Peter. *Prop.* 1138
Ans. 1138, 1164
- Philaethes. *Prop.* 773
- Philaethes Cleasbyensis. (Mr. T. Todd). *Prop.* 894
- Philaethes Londinensis. (Mr. Geo. Witchel). *Prop.* prize 1764
Ans. prize 1764
- Philarithmus. *Ans.* 453, 457, 734
- Phillips, James. *Ans.* 713
- Phillips, Robert. *Prop.* 767, 792
Ans. 748, 763, 767, 784
- Philomathes. *Prop.* 50
- Philomathes, Anna. *Ans.* 59, 62, 63, 64
- Philo-Pesos. *Prop.* 407
- Philo-Nauticus. *Prop.* 1252
Ans. 1252
- Philosophicus. *Prop.* 124
- Philotechnus. *Prop.* 530
- Philotheoros. *Prop.* 354, 374
- Philo Tipperus. *Ans.* 43
- Phipps, W. *Ans.* 393
- Pickering, George. *Ans.* 1124, 1141, 1150, 1157, 1169, 1180
- Piercy, Richard. *Prop.* 233
- Pisanus, Leonardus. *Prop.* 1099
Ans. 1099, prize 1803
- Pfomley, J. *Prop.* 83
- Plus Minus. (The Rev. Mr. Jones). *Prop.* 445, 529, pr. 1769, 625, 639, pr. 1775,
755, 769, 786, pr. 1782, 814, 829, pr. 1785, pr. 1787
Ans. 529, 533, 625, 626, pr. 1744, pr. 1775, 752, 755, 769, 786,
796, pr. 1782, 814, 829, pr. 1785
- Pointin, Thomas. *Prop.* 52, pr. 1730, 161, pr. 1731
Ans. pr. 1730, 161
- Potter, John. *Ans.* 500
- Powle, J. *Prop.* 243, 251, 253, 258, 262, 267, 329, 332, 624
Ans. pr. 1740, pr. 1742, 237, 253, 258, 312, 315
- Pretty, Charles. *Prop.* 1257
Ans. 985, 1170
- Froson, P. *Ans.* 43
- Proser, Mr. *Ans.* 5
- Purver, William. *Prop.* 727, 818
- Putsey, W. *Prop.* 1166, 1202, 1260
Ans. 1159, 1190, 1209, 1218, 1231, 1255, 1290
- Puzzlem, Peter. (Mr. J. Landen). *Prop.* pr. 1772, pr. 1775, pr. 1776, pr. 1777,
pr. 1779, pr. 1781
Ans. pr. 1772, pr. 1773, pr. 1776, pr. 1777, pr. 1779, pr. 1781
- R. B. *Prop.* pr. 1810
Ans. pr. 1810
- Raby, Alexander. *Ans.* 1164
- Ragg, Abel. *Ans.* 58
- Ramsay, John. *Ans.* 1029, 1039, 1047, 1054
- Ramsay, Thomas. *Prop.* 226
- Randles, John. *Prop.* 343, 352
Ans. 380
- Rasberry, T. *Prop.* 98

- Rawle, Richard. *Prop.* 597, 483
 Reed, Edward. *Prop.* 663
 Ans. 686
 Rees, J. *Ans.* 991
 Reffshir, J. *Prop.* 1052
 Reynolds, William. *Prop.* 698, 723
 Ans. 703, 742
 Rhinoceros. *Prop.* 270
 Richard of the Vale. *Prop.* 448
 Richards, John. *Prop.* 59, 100, 106
 Richards, William. *Ans.* 1081
 Richardson, William. *Ans.* 774
 Rickaby, John. *Ans.* 356
 Riddle, Edward. *Ans.* 1199, 1214, 1222, 1225, 1235, 1242, 1251, 1266, pr. 1813,
 pr. 1814, 1289, 1292, 1293, 1294, pr. 1815
 Ridout, Thomas. *Ans.* 941
 Robbins, Reuben. *Prop.* 822
 Ans. pr. 1783
 Roberts, David. *Ans.* 1033
 Roberts, Stephen. *Prop.* 703, 757
 Ans. 752
 Robinson, James. *Ans.* 364, 368, 370
 Robinson, Thomas. *Prop.* 700, 790, 836
 Ans. 675, 750, 813, 824, 830, 836
 Robins, William. *Prop.* 975
 Ans. 988
 Rocktree, Henry. *Prop.* 1104
 Rodham, John. *Prop.* 965, pr. 1796, 1021
 Ans. 988, 1021
 Rcdomontado. *Prop.* 300
 Rogers, Robert. *Prop.* 1237
 Ans. 1210
 Rollinson, Edward. *Prop.* 389, 401, pr. 1755, 414, 432, 460, 471, pr. 1761
 Ans. 388, 389, 399, 401, 414, 416, pr. 1757, 444, 461, 470
 Ronkesley, Mr. *Ans.* prize 1716
 Roope, George. *Ans.* 863
 Roper, John. *Ans.* 568
 Rosamond. *Prop.* 305
 Rosser, John. *Ans.* pr. 1728
 Rotheram, J. M. D. *Ans.* 851
 Rouille, A. *Ans.* 986, 1014
 Rouse, Samuel. *Prop.* 129
 Ans. 112, 122
 Rowbottom, Isaac. *Prop.* 1060, 1080, pr. 1804, 1146, 1171, 1186, 1199
 Ans. 1060, 1080, pr. 1804, 1146
 Rowe, Alexander. *Prop.* 673, 702, 729, 797, 808, 834, 849, 875, 893, 930, 1004,
 1020, 1075, 1096
 Ans. 657, 675, 689, 707, 744, 808, 825, 881, 889, 899, 933,
 940, 994, 1008, 1075, 1079, 1089, 1960
 Rubins, J. *Ans.* 248
 Rundell, Master, F. T. C. *Ans.* 1059
 Russellus. *Ans.* 183
 Rusticus. *Prop.* 376, 406
 Ans. 503
 Rutherford, John. *Prop.* 1040, 1249
 Ans. 1038, 1237, 1241
 Ryley, John. *Prop.* 934, 962, 1041, 1132, 1232, 1251
 Ans. 860, 910, 929, 934, 952, 975, 987, 996, 1005, 1022, 1025,
 1026, 1039, 1065, 1069, pr. 1801, 1203, 1218, 1234
 Ryley, John, Junr. *Prop.* 1216
 Ans. 1199, 1203, 1205
 Ryley, Samuel. *Ans.* 1160, prize 1808

- S. T. Rector of Ormside. *Ans.* 969, 977, 984
 S. T. *Ans.* 1289, 1290, 1291
 S. Miss E. *Ans.* 391
 S——e. Miss T. *Prop.* 434
 Sadler, Thomas. *Prop.* 503, 518, 535, 550, 597, 612, 627
Ans. 475, 489
 Saint, William. *Prop.* 1187, 1203, 1217; 1218
Ans. 1177, 1207, 1218
 Sampson, Rev. Mr. John. *Prop.* 932, 963
Ans. 860, 963
 Sandalls, T. *Prop.* 252
 Sandiforth, Richard. *Ans.* prize 1714, 41
 Saul, Isaac. *Prop.* 865, 898
Ans. 837, 868, 888, 902, 995
 Saunderson, George. *Prop.* 747, 766, 781, 800, pr. 1783, 830, 846, 858, 873,
 pr. 1788, 922, 936, 951, 1027
Ans. 695, 739, 747, 783, 785, 794, 795, 800, pr. 1782,
 815, 816, pr. 1783, 822, 824, 830, 846, 873, 876,
 pr. 1787, 922, 936, 937
 Scarth, Capt. Wm. *Ans.* 149
 Scholefield, J. *Ans.* 863
 Scott, J. *Ans.* 727, 804
 Scott, Thomas. *Ans.* 1045
 Scurr, Rev. Mr. Thomas. *Prop.* 1063, 1114, 1147, 1177, 1188
Ans. 1065, 1114, 1120, 1138, 1147, 1152, 1161, pr. 1804,
 1177, 1216, 1237
 Scyth, Arch. *Ans.* 248, 252
 Selkirk, John. *Ans.* 1089
 Senex. *Ans.* 1204, pr. 1811
 Sergeant, Thomas. *Ans.* 772
 Sewell, William. *Prop.* 576, 693, 793, 750, 917
Ans. 663, 664, 666, 671, 677, 679, 681, pr. 1778, 750, 754,
 pr. 1782, pr. 1789
 Shadgett, John. *Prop.* 642, 655, 684, 713
Ans. 684
 Shallow, Ant. Esq. (Mr. Simpson). *Prop.* 383, 390, pr. 1754
Ans. 362, 363, 373, pr. 1754
 Sharp, Paul. *Prop.* 449, 525, 539, 547, 563, 572, 591, 600, 613, 632, 644, 789,
 805
Ans. 482, 520, 547, 563, 566, 583
 Shepherd, Thomas. *Prop.* 45
Ans. 26
 Sheridan, D. T. *Prop.* 1214, 1224, 1240, 1264, 1275
Ans. 1214, 1219, 1224, 1240, 1247, 1250, 1262, 1289, 1290,
 1295
 Shipman, J. *Ans.* 995
 Shireff, J. *Ans.* 1015
 Shutcliffe, Robert. *Ans.* 437
 Sidebottom, John. *Ans.* prize 1728
 Sidway, Barbara. *Prop.* 36
Ans. 17, 20, 21, 22, pr. 1712, pr. 171
 Sikes, B. *Ans.* 552, 560
 Silvia. *Ans.* 41, 44, 45, 46
 Simmons, John. *Prop.* 57, 102, 116, 120, 137
Ans. 55
 Simpson, Thomas. *Prop.* 187, pr. 1736
Ans. pr. 1736
 Simpson, Henry. *Ans.* 1169
 Simpson, Timothy. *Prop.* 881
 Simpson, William, Junr. *Ans.* 858, 880
 Skewes, J. *Prop.* 1162
Ans. 1162, pr. 1808

- Stettin, William. *Ans.* 396
 Smart, William. *Ans.* 642, 643, 671
 Smith, Ch. *Prop.* 587
 Ans. 313, 337, 350
 Smith, Jo. *Prop.* 69, 96
 Smith, John. *Prop.* 1266, 1292
 Ans. 1045, 1104, 1114, 1150, 1152, 1172, 1173, 1179, 1180, 1196,
 1225, 1253, 1247, 1257, 1284, 1293
 Smith, Thomas. *Prop.* 1126, 1184, 1226, 1270, 1285
 Ans. 1104, 1172, 1218, 1231, 1255, 1264, 1266, 1294
 Smith, William. *Ans.* 334, 407
 Smithson, Richard. *Prop.* 1109
 Snooke, W. D. *Ans.* 1265
 Sowerby, John. *Prop.* 1057, 1079
 Sparrow, James. *Prop.* 1034
 Sparrow, Thomas. *Prop.* 148
 Spencer, J. *Ans.* 598, 616
 Spencer, William. *Prop.* 517, 531
 Ans. 488, 502, 509, pr. 1765
 Spicer, William. *Prop.* 426, 467, 480, 495, 507, 523, 536, 553, 578, 595, 599,
 616, 631, 643, 656, 671, 683, 704, 748
 Ans. 396, 429, 467, 505, 536, 567, 584, 597, 599, 618
 Spry, William. *Prop.* 422
 Squire, Thomas. *Prop.* 1049, 1106
 Stamp, Charles. *Ans.* 1216
 Stevens, Nich. *Ans.* 34
 Stevens, Philip. *Prop.* 315
 Stevenson, George. *Ans.* 878, 893, 952, 957, pr. 1796
 Stiff, Sir Stately. (Mr. J. Landen). *Ans.* prize 1749
 Stoker, W. *Ans.* 392
 Surtees, John. *Prop.* 1072, 1085, 1130
 Ans. 986, 996, 1007, 1016, 1027, pr. 1797, 1032, pr. 1798, 1052,
 pr. 1799, 1130, pr. 1806, 1173, 1174, 1217, 1220, 1237
 Surtees, Robert. *Ans.* 1079, 1080
 Sutton, W. *Ans.* 313, 345
 Swanwick, T. *Prop.* 1094
 Ans. 1067
 Swift, William. *Prop.* 788
 Sylva, Jacobus de Vitak. *Prop.* 870
 Sylvia. *Ans.* 93, 107
 Sylvius. *Prop.* 392
 Ans. 378, 385
 T. S. *Prop.* 80, 465
 T—t, W. *Prop.* 380, 385
 Ans. 367
 Tapper, Richard. *Prop.* 97
 Ans. 92, 94, 99, 101, 102, 106, 109, pr. 1724, pr. 1725
 Taptinos. *Prop.* 351, 353
 Tarrat, Isaac. *Prop.* 552
 Ans. 518, 538, 554
 Tate, Charles. *Prop.* 378
 Ans. 394
 Tattam, Richard, Junr. *Prop.* 819
 Taylor, John. *Ans.* 1137
 Taylor, Michael. *Prop.* 721, 763, 799
 Ans. 672, 710, 725, 738, 753, 763, 794
 Taylor, Richard. *Prop.* 1281
 Ans. 1232, 1281
 Taylor, William. *Prop.* 42
 Telones, Palai. *Prop.* 133
 Terey, James. *Prop.* 318, 319, 348, 369, 371
 Ans. 212, 255, pr. 1744, 280, 299, 308, 313, 347, 346, 334, 440

- Terricola. (Dr. Nevil Maskelyne, the late Astronomer Royal). *Prop.* 784, 801,
 prize 1795, prize 1801
 Terril, William. *Ans.* 757, 789
 Terry, Mathew. *Prop.* 882, 897, 912, 1124
 Ans. 867, 989
 Terry, Richard. *Ans.* 440
 Thatker, Anthony. *Prop.* 197, 217, 232
 Ans. 214, 217, 220, pr. 1741, 241, pr. 1743
 Thompson, Charles. *Ans.* 1246, 1295
 Thompson, J. *Ans.* 427
 Thompson, Ralph. *Prop.* 732
 Ans. 717
 Thompson, Thomas. *Ans.* prize 1794, 1006
 Thoubren, James. *Ans.* 1048
 Tilney, Henry. *Ans.* 879
 Titlow, Samuel. *Prop.* 1211, 1227
 Ans. 1167, 1187, 1201, 1217
 Todd, Thomas. *Prop.* 666, 868, 886
 Ans. 378, 559, 576, 868, 886
 Toft, William. *Prop.* 454, 528, 537, 544, 564, 571
 Ans. 528, 537, 564
 Topham, John. *Ans.* 100
 Towan, Thomas. *Prop.* 1161
 Ans. 1094, 1145, 1161
 Travis, Henry. *Prop.* 208, 216
 Ans. 199, pr. 1738, 205, 206, 208, 216, pr. 1740
 Treeby, S. *Ans.* 1270, 1292, 1295
 Trelease, Charles. *Ans.* 788
 Trott, Walter. *Prop.* 395, 410, 429
 Ans. 201, 403, 409
 Truman, William. *Prop.* 1287
 Ans. 1232, 1294
 Truswell, Thomas. *Prop.* 883
 Ans. 742
 Turner, John. *Prop.* 119, 125, 135, 150, 154, 155, 164, 167, 173, 177, 181, 191,
 194, pr. 1737, 205, pr. 1741, 250, 244, 256, 297, 303
 Ans. 113, 115, 122, 123, 124, 128, pr. 1727, 130, 131, 154, 156,
 165, 167, 171, 173, 177, 184, pr. 1735, 187, 189, 194,
 pr. 1737, 205, 211, pr. 1739, 213, 217, 235, 241, 269, 271,
 275, 279, 282, 284, 292, 295, 300, 321
 Turner, J. *Prop.* prize 1750, 350, 605, 679, 776, 839, 856
 Ans. 317, 319, 643, 661
 Turner, John. *Ans.* 1143
 Vertigo. *Prop.* 966
 Verus. *Ans.* prize 1800
 Vessey, Thomas. *Ans.* 551
 Viator. *Ans.* prize 1796
 Vince, Sam. *Prop.* 574, 594, 617
 Ans. 558
 Virgo, W. *Ans.* 950, 954, 984, 1014
 Vorley, William. *Ans.* 35
 Upnorenensis. (Mr. Robert Heath). *Prop.* 302, 308, 330, 344, 358
 Ans. 344, 345
 Utting, James. *Ans.* prize 1814
 W A. *Prop.* 30
 Waine, J. *Ans.* 266, 267
 Wales, William. *Prop.* 609, 640
 Ans. 540, 542, 543, 572, 580, pr. 1767, 609, 625, 637, pr. 1771,
 653
 Walker, Edward. *Ans.* prize 1711
 Walker, Fr. *Ans.* 50
 Walker, Henry. *Prop.* 64

OF CONTRIBUTORS.

465

- Walker, Joseph. *Prop.* 512, 521, 578
Ans. 545, 578
- Walker, Joseph. *Prop.* 1231
- Walker, Thomas. *Ans.* 519
- Wall, Mr. *Ans.* 79
- Walter, Peter. *Prop.* 33
Ans. 35
- Walton, John. *Prop.* 999
- Walton, Peter. (Mr. J. Landen). *Prop.* 438, 447, 461
Ans. 438, 443, 461
- Walton, William. *Ans.* 735, 749
- Waltoniensis. (Mr. J. Landen). *Ans.* 330, prize 1750
- Ward, Peter. *Prop.* 61
- Warren, Edward. *Prop.* 1037
- Watkins, William. *Prop.* 1122, 1149, 1182, 1219
Ans. 1081, 1142
- Watson, Major Henry. *Prop.* 399, 404, 409, pr. 1756, 430, 443
Ans. 352, 353, 383, 385, 399, 413, 419, 431, 438, 456, 473
- Watts, John. *Ans.* 225, 227, 233, 234
- Waugh, Richard. *Ans.* 857, 869
- Wear, Edward. *Ans.* 955
- Weatherall, William. *Ans.* 902
- Weedon, Alexander. *Prop.* 25
Ans. 21
- Wells, William. *Ans.* 642
- Western, Jack. *Prop.* 949
- Weston, Miss Mary. *Ans.* 1195, 1210, 1231, 1247, 1258, 1286, 1291, 1294
- Weston, R. *Prop.* 419, 433
Ans. 419
- White, John. *Ans.* 5
- White, Mr. *Ans.* 77, 78
- White, Thomas. *Ans.* 721, 732, 746, 760, 781, 921
- White, William. *Prop.* 1095
- Whitehead, Richard. *Prop.* 94, 108, 111, 117, 123, 130
Ans. 114, 116, 118, 121, 123, 127
- Whitehouse, Abel. *Ans.* 855, 874, 884, 894
- Whiting, Thomas. *Prop.* 926
- Whitley, John. *Prop.* 1190, 1267, 1278, 1295
Ans. 1155, 1169, 1241, 1244, 1248, 1249, 1256, 1260, 1267, 1275,
1278, pr. 1814, 1293, 1294, 1295
- Whitton, John. *Ans.* 790, 793, 797, 811
- Whitworth, William. *Prop.* 62
Ans. 57, pr. 1718, pr. 1719
- Widd, Harland. *Ans.* 385
- Widger, Roger. *Ans.* 326, 329
- Wigglesworth, John. *Ans.* 359, 364, 371
- Wildbore, Rev. Mr. Charles. *Prop.* 473, 621, pr. 1770, 638, pr. 1771, 646, 654,
667, pr. 1774, 695, 740, pr. 1778, 751, 770,
Ans. 444, 592, 593, 594, 595, pr. 1768, 608, 611,
pr. 1769, 617, 619, 620, 623, 624, 626, pr. 1770,
632, 635, 638, 640, 641, pr. 1771, 653, 654,
665, 667, 678, 679, 680, 681, 695, pr. 1775,
718, 721, 724, 725, pr. 1777, 740, pr. 1778,
751, 753, pr. 1719, 770
- Wilding, James. *Prop.* 969, 1050
- Wilkin, Thomas. *Ans.* 406
- Wilkin, William. *Prop.* 657, 659, 677, 690
Ans. 612, 643, 691
- Wilkinson, Joseph. *Ans.* 57
- Wilkinson, Robert. *Prop.* 941, 956, 986, 1153
Ans. 927, 941, 956, 1032, 1143, 1161
- Willan, Thomas. *Ans.* 851

- Williams, Mr. *Ans.* 78
 Williams, Capt. Edw. *Prop.* 620, 891, 933, 1026
 Ans. 601, 1026
 Williams, John. *Prop.* 280, 356, 365, 373
 Ans. 357
 Williams, James. *Prop.* 863, 869
 Ans. 801, 805, 820, 833
 Williams, Joseph (Bristol). *Prop.* 1255, 1256, 1288
 Ans. 1269, 1268
 Williams, Joseph (Canterbury). *Prop.* 1269
 Williams, Phil. *Ans.* 376
 Williams, Thomas. *Prop.* 87, 104, 105, 127, 132, 139, 147, 152
 Ans. 128
 Williams, Stephen. *Prop.* 682
 Ans. 693, 744
 Willes, John. *Prop.* 735, 761
 Ans. 729
 Wilmett, Noah. *Ans.* 1201
 Willingham, John. *Prop.* 101
 Ans. 101
 Wills, John. *Ans.* 1286, 1290
 Wilson, James. *Prop.* 442
 Ans. 428
 Wilson, John. *Prop.* 27, 35
 Wilson, J. *Ans.* 854
 Wilson, Robert. *Prop.* 31
 Wilson, Thomas. *Ans.* 720
 Winward, Jesse. *Prop.* 1247
 Wirksworthiensis. *Ans.* 903
 Wiseman, William. *Prop.* 1154, 1165, 1192, 1196, 1284
 Ans. 1154, 1165, 1175, 1184, 1246, 1262, 1269, 1293
 Witchel, George. *Prop.* pr. 1758, 457, 474, 488, pr. 1762
 Ans. pr. 1758, 448, 458, 472, 474
 Wittam, Obadiah. *Prop.* 349
 Wood, James. *Prop.* 1194
 Ans. 1217, 1267
 Woodby, S. *Ans.* 537
 Woolcott, Simon. *Prop.* 748, 948
 Ans. 897, 948
 Woolfin, James. *Prop.* 1134, 1168
 Woolston, Thomas. *Prop.* 961, 970, 988
 Ans. 849, 863, 940, 961
 Wore, J. *Ans.* 579
 Wright, Anna. *Prop.* prize 1713
 Ans. prize 1711, 23
 Wright, Mary. *Ans.* 16, or prize 1710
 Wright, John. *Ans.* 976
 Wright, T. *Ans.* prize 1716
 Wright, Rev. Mr. W. *Prop.* 1135, 1136, 1167
 Wright, William. *Prop.* 1220
 Ans. prize 1813
 Wylde, Mr. *Ans.* 36
 X + Y. *Prop.* 838, 855
 Xantippe. *Prop.* 55
 Yeret, James. *Ans.* prize 1769
 York, Thomas. *Ans.* 1246
 Young, James. *Ans.* 576, 656, 687, 692, 698, 840, 883, 872
 Young, M. *Ans.* 1153
 Young, R. *Prop.* 408
 Ans. 381, 399
 Z. W. *Prop.* 1157
 — . *Prop.* 990, 995

INDEX

Referring the Questions to the different Branches of Mathematics.

ALGEBRA.

Problems purely Algebraic; 2, 5, 6, 7, 11, 13, 15, 18, 19, 21, 23, 24, 27, 29, 32, 41, 43, 49, 53, 54, 58, 65, 72, 81, 84, 89, 90, 97, 98, 105, 107, 109, 113, 118, 131, 136, 141, 144, 148, 151, 162, 166, 180, 183, 184, 204, 239, 247, 249, 255, 261, 273, 291, 292, 306, 308, 309, 334, 338, 352, 368, 376, 377, 378, 391, 393, 406, 412, 413, 421, 428, 434, 436, 442, 448, 463, 464, 475, 477, 489, 503, 518, 519, 520, 521, 535, 537, 550, 552, 554, 566, 582, 597, 612, 614, 627, 628, 642, 655, 668, 670, 682, 686, 699, 712, 715, 727, 728, 742, 757, 772, 788, 803, 818, 848, 863, 878, 895, 902, 908, 910, 924, 931, 939, 954, 977, 984, 990, prize 1796, 1029, 1060, 1119, 1168, 1266, 1272.

Algebraical Theorems; 821, 844, 889, 904, 921, 993, 996, 1205.

ARITHMETIC.

Questions; 1, 2, 3, 4, 5, 7, 23, 27, 35, 55, 83, 144, 234, 334, 335, 462, 584, 894.

ARITHMETIC OF SINES.

588.

ASTRONOMY, GEOGRAPHY, &c.

16, 30, prize 1713, prize 1716, 79, 85, 93, 110, 115, 120, 149, prize 1730, 157, 159, 179, 186, prize 1736, 201, 214, 220, 222, 231, 241, 242, prize 1743, 257, 271, 297, 289, prize 1748, 315, 320, 325, 337, 340, prize 1751, 349, 355, 364, 367, 389, 390, 404, prize 1758, 457, 458, prize 1759, 474, 486, 487, prize 1760, 492, 494, 499, 500, 505, 511, 516, 527, 528, 532, 539, 543, 546, 556, 557, 561, 564, 568, 572, 586, 609, 625, 630, prize 1771, 650, 651, 653, 654, 687, 709, 710, 731, 738, 753, 759, 769, 796, prize 1782, 808, 816, 823, 834, 835, 851, 856, 869, 873, 880, prize 1789, 913, 926, 927, 930, 936, 940, 941, 942, 951, 958, 960, 961, 988, 1017, 1019, 1035, 1040, 1046, 1048, 1049, 1054, 1068, 1101, 1106, 1117, 1136, 1139, 1145, 1152, 1184, prize 1808, 1267, 1288.

CHANCES, COMBINATIONS, PERMUTATIONS, &c.

20, 37, 163, 218, 221, 268, 298, 416, prize 1760, 509, 541, 579, 881, 899, 1120, 1131, 1189.

CURVES.

(This class includes problems concerning loci, and the rectification, quadrature, &c. of curves.)

Prize 1738, 262, 263, 265, 287, 301, 302, 312, 332, 373, prize 1755, 431, 460, 488, 525, 526, 529, 531, 533, 574, 575, 581, 593, prize 1769, 616, 617, 626, 641, 666, prize 1774, 690, 692, 693, 694, prize 1775, 705, 708, 765, 778, 792,

prize 1784, 838, pr. 1785, 876, prize 1787, 887, prize 1788, prize 1790, 935, 943, 948, 967, 979, 1068, prize 1802, 1122, 1128, prize 1804, 1143, 1248, 1249, 1296, 1297.

DYNAMICS.

10, 33, prize 1715, 50, prize 1722, 99, 112, 114, 121, 124, 129, 138, 152, 154, 155, prize 1736, 198, 201, 203, 206, 223, 227, 229, prize 1742, 236, 288, 243, 246, 253, 259, 272, 276, 284, 285, prize 1747, 300, 307, 310, 314, 321, 322, prize 1749, 326, 362, 392, 403, 418, 433, 447, 450, 458, 460, 468, 484, 498, 515, 534, prize 1765, 553, 559, prize 1766, 576, 583, prize 1768, 631, prize 1772, prize 1773, 681, 694, prize 1776, prize 1777, 735, 740, prize 1778, 748, prize 1779, 761, 767, 784, 786, prize 1781, 797, 801, 804, 823, prize 1784, 838, 841, 855, prize 1786, 870, 882, 888, 890, 897, 918, 920, 993, prize 1794, 962, 966, prize 1793, 995, 997, prize 1795, 1011, 1042, 1045, 1051, 1057, 1069, 1081, prize 1801, 1089, 1094, 1098, 1102, 1110, 1116, 1129, 1130, 1134, 1141, 1144, 1147, prize 1805, 1154, 1158, 1159, 1162, 1172, 1177, prize 1807, 1198, 1200, 1202, 1216, 1222, prize 1810, 1247, 1251, prize 1812, 1256, 1257, 1259, prize 1813, 1276, 1279, 1281, 1293, 1297, prize 1815.

FLUXIONS.

Problems of Maxima and Minima; 209, 233, 279, 426, 473, 573, 600, 683, 750, 758, 915, 1000.

Fluxional Equations; 405, 419, 432, 502, 517, 667, 760, 782, 842, 874, 919, 949, 963, 981, 982, 1024.

GEODESIC PROBLEMS AND THEOREMS.

794, 906, 1026, prize 1800, 1220.

GEOMETRY.

Geometrical Problems resolved by pure Geometry; 2, 9, 22, prize 1711, 28, prize 1715, 86, 94, 100, prize 1724, 123, prize 1727, 139, 147, 156, 160, 163, 167, 173, 182, 199, 202, 215, 219, 224, 225, 235, 248, 252, 260, 270, prize 1746, 280, 290, 347, 351, 353, 383, 384, 385, 386, 394, 395, 407, 424, 425, 437, 438, 440, 454, 455, 469, 470, 478, 481, 490, 501, 506, 507, 508, 512, 523, 524, 540, 567, 570, prize 1767, 588, 590, 602, 604, 606, 621, 622, 633, 634, 635, 636, 646, 647, 648, 661, 662, 663, 664, 665, 674, 675, 678, 679, 680, 688, 691, 700, 718, 720, 722, 736, 739, 744, 745, 747, 749, 762, 764, 770, 777, 779, 780, 781, 783, 791, 793, 795, 798, 807, 809, 813, prize 1783, 819, 820, 822, 824, 831, 837, 840, 843, 850, 852, 854, 858, 860, 865, 909, 917, 957, 971, prize 1794, 994, 1003, 1007, 1015, 1021, 1023, 1034, 1038, 1044, 1050, 1053, 1082, 1090, 1094, 1097, 1121, 1126, 1140, 1160, 1167, 1182, 1195, 1224, 1232, 1233, 1239, 1241, 1245, 1246, 1255, 1264, 1274, 1280, prize 1814, 1289, 1290, 1295.

Geometrical Problems resolved by Algebra.

(*This class comprehends questions in Mensuration.*)

2, 17, 22, 38, 40, prize 1714, 52, 60, prize 1715, 71, prize 1719, 77, 82, prize 1720, 86, 88, 94, 96, 100, 102, 103, 104, prize 1723, 106, 108, prize 1724, 111, 116, prize 1725, 117, 122, prize 1726, 123, prize 1727, 132, 135, prize 1728, 139, 140, 143, 145, 158, 160, 163, 165, 167, 169, 173, prize 1753, 176, 178, prize 1754, 181, 188, 191, 193, 194, 195, 196, 199, 202, 205, 208, 210, 212, 216, prize 1740, 224, 225, 228, 235, 240, prize 1743, 248, 252, prize 1744, 256, prize 1745, 266, 274, prize 1746, 280, 283, 286, 297, 313, 323, 333, 342, 343, 347, 351, 353, 358, 359, 363, 369, 370, 385, 386, 394, 397, 398, 407, 408, 422, 423, 424, 425, 436, 437, 438, 440, 444, 449, 466, 478, 482, 485, 490, 497, 504, 513, 538, 545, 551, 560, 565.

570, 578, 580, 589, 591, 596, 598, 603, 610, 618, 619, 643, 644, 657, 661, 671, 672, 675, 677, 684, 698, 700, 703, 706, 718, 714, 719, 728, 744, 752, 755, 773, 775, 790, 799, 807, 809, 812, 813, prize 1783, 820, 822, 850, 852, 854, 858, 864, 865, 879, 883, 898, 900, 902, 909, 911, 964, 971, 973, 978, prize 1794, 989, 994, 1001, 1002, 1004, 1015, 1007, 1016, 1020, 1021, 1032, 1033, 1036, 1039, 1047, 1050, 1070, 1072, 1077, 1091, 1097, 1109, 1112, 1121, 1123, 1135, 1149, 1157, 1159, prize 1806, 1164, 1165, 1166, 1167, 1173, 1179, 1181, 1191, prize 1808, 1195, 1199, 1210, 1213, 1228, 1229, 1235, 1236, 1237, 1255, 1265, 1278, 1277, 1284, 1292.

Geometrical Problems resolved by Arithmetic.

(This class comprehends questions in Mensuration.)

9, 14, 17, prize 1711, 25, 26, 31, 34, 42, 45, prize 1717, 59, 63, 68, 69, 70, 76, 80, 86, 91, 95, 99, 101, 104, 137, 171, 182, 196, 219, 294, 303, 343, 370, 384, 395, 467, 483, 715, 774, 864, 896, 900, 940, 941, 956, 959, 961, 969, 975, 986, 999, 1001, 1002, 1009, 1014, 1016, 1018, 1022, 1030, 1031, 1032, 1034, 1040, 1053, 1055, 1059, 1064, 1065, 1066, 1092, 1112, 1123, 1125, 1135, 1137, 1152, 1165, 1166, 1167, 1180, 1181, 1182, 1194, 1195, 1210, 1213, 1215, 1226, 1235, 1240, 1241, 1243, 1254, 1267, 1277, 1284.

Geometrical Problems relating to Maxima and Minima.

Prize 1712, 36, 47, 56, 62, 74, 92, 119, 134, prize 1729, 150, 156, prize 1732, 172, 174, 177, 189, 190, 197, prize 1737, 217, 226, 230, 232, 237, 250, 269, 282, 293, 299, 316, 318, 327, 330, prize 1750, 348, 354, 372, 374, 381, 382, 396, 409, 410, prize 1756, 430, 439, 446, 452, 453, 456, 465, 472, 479, 480, 493, 495, 548, 558, 595, 607, 615, 620, 645, 649, 656, 659, 660, 673, 676, 685, 689, 693, 701, 702, 707, 721, 729, 730, 732, 737, 743, 751, 766, 768, 785, 800, 805, 827, 830, 836, 846, 868, 872, 884, 905, 916, 922, 937, 945, 947, 950, 975, 1012, 1025, prize 1798, 1095, 1096, 1100, 1102, 1104, 1105, 1115, 1114, 1116, 1129, 1134, 1144, 1154, 1161, 1174, 1186, 1190, 1196, 1201, 1207, prize 1809, 1212, 1214, 1219, 1225, 1242, 1252, 1260, 1261, 1262, 1271, 1286, prize 1815.

Geometrical Problems resolved by the Arithmetic of Sines.

100, 167, 171, 182, 202, 235, 385, 395, 1015.

Geometrical Theorems; 153, 170, prize 1735, 380, 427, 451, 496, 601, 605, 623, 629, 639, 733, 734, 776, 811, 826, 828, 843, 987, 1006, 1037, 1061, 1138, 1142, 1150, 1183, 1187, 1197, 1211, 1230, 1244, 1258, 1275, 1294.

GNOMONICS.

78, 87, 125, 164, 169, prize 1743, 337, 445, 510, 542, 637, 928, 980, 1063, 1084, 1146, 1192, 1250.

HARMONICS.

319, 1086.

HYDROSTATICS.

89, 127, 131, prize 1752, 518, 577, 859, 991, 1056, 1080, 1093, 1263, 1269.

HYDRODYNAMICS.

Prize 1721, 142, 331, 356, 361, prize 1754, 459, 547, 562, 563, 608, 618, prize 1770, 632, 652, 704, 810, 891, 903, 929, 934, 952, 965, 976, prize 1797, 1052, 1083, 1114, 1127, prize 1801.

INDETERMINATE ANALYSIS.

Equations of the first degree.

8, 12, 67, 78, 173, 207, 245, 267, 281, 295, 296, 638, 723, 829, 1176. Magic Squares (61, 126); Perfect Numbers, 166; Amicable Numbers, 305.

Diophantine, or Indeterminate Problems of the second and higher orders.

51, 133, 192, 254, 275, 288, 304, 311, 341, 344, 443, 476, 491, 536, 583, 638, 716, 746, 853, 925, 932, prize 1792, 974, 992, 1062, 1075, 1099, 1111, prize 1808, 1131, 1132, 1155, 1203, 1217, 1218, 1227, 1231, 1270, 1291.

INTEREST, ANNUITIES, &c.

64, 75, prize 1731, 213, 277, 345, 365, 379, 400, 401, 415, 471, 485, 530, 555, 569, 599, 611, 624, 669, 886, 893, 955, 970, 983, 1108, 1169, 1209.

ISOPERIMETRY.

185, prize 1741, 264, prize 1754, prize 1757, prize 1801.

NAVIGATION.

22, 384, 395, prize 1755, 440, 441, 544, 571, 717, 725, 769, 780, 1021, 1145, 1153, 1252.

OPTICS.

66, 366, 411, 592, 912, 1186, 1195, 1277.

PERSPECTIVE.

587, 814.

PNEUMATICS.

328, 357, 833, 857, 875, 946, prize 1795, 1027, 1041, 1079, 1188.

PROJECTIONS.

Prize 1764, 640, 724, prize 1780, 839.

SERIES.

258, 420, 461, 594, 754, 763, 806, 815, 861, 993, 1060, 1204.

SPHERICS.

200, 211, prize 1739, 399, 429, 871, prize 1799.

STATICS.

44, 46, 128, 148, 244, 251, 317, 329, 350, prize 1753, 402, 414, 417, 549, 755, 849, 866, 867, 885, 901, 914, 944, 972, 1010, 1078, 1085, 1087, 1107, 1115, 1124, 1156, 1173, 1185, 1206, 1221, 1234, 1278, 1282, 1287.

THE END.

ERRATA.

VOL. I.

Page.	Line.	For:	Read.
13	10	$56^{\circ}57'45''$,	$\tan 56^{\circ}57'45''$.
24	11	442,	442 (one more than the square of 21)
30	4	$x = \frac{cb' -}{ab' - a'b}$,	$x = \frac{cb' - cb}{ab' - a'b}$. (In some copies.)
34	14	of the castle,	required.
	18	of the castle,	required.
43	51	Lydai,	Lydia.
82		163,	{ 103 in the 5th horizontal rank of the magic square.
84	13	$\frac{(135 + 111)}{3} 4\frac{1}{2}$,	$\frac{(135 + 111)}{3} \times 4\frac{1}{2}$.
110		3 fr. bot. 210,	2ro.
115	25	foot,	feet.
143	14	$18du^6$,	$20du^6$.
146	20	power,	{ power; which, wanting the odd powers, can be solved as a biquadratic, It is the same as Turner's above.
167	21	+	$+\frac{rx}{d}$.
168	14	biquadratic,	biquadratic.
183	21	$aa + aa$,	$aa + aa$.
199	8	cannot,	can.
239	16	$xx = xx$,	$xx = yy$.
248	30	the intricate,	this intricate.
268	14		dele the third).
272	20	Cooper,	Cowper.
276	11	$b^{\frac{1}{2}}$,	$b^{\frac{1}{2}}$.
286	1	cd,	cs.
292	27	this data,	these data.
294	15	863-052,	863-0532.
313	25	notwith-	notwith-
314	25	$(a - x)^2$,	$(a - x)^4$.
316	15	$-w$),	$-w$.
323	18	by 2 is,	by 2 is.
330	22	AST,	AST'.
347	22	affugiat,	affugiat.
349	29	triangle,	plane triangle.
365	11	$-\frac{v x}{y}$,	$-\frac{v x}{y}$.
		$-\frac{v x}{v x}$,	$-\frac{v x}{v x}$.
372	18	$\Lambda(x+1)^{n-1}$,	$\Lambda(x+1)^{n+1}$.

VOL. II.

Page 50. QUESTION 340. (Vol. II.)

The following remark was omitted in its proper place, at the end of the solution.
 "Mr. Charles Mason, jun. of Sapperton, Gloucestershire, sends us, that the above solution was done by him, though sent us in Mr. Davis's name."

ERRATA.

Page.	Line.	For.	Read.
68	23	s^2 ,	a^2 .
192	30	$\frac{b}{a+b}$,	$\frac{a}{a+b}$.
193	21	$+n$,	$n+$.
196	34	hypothense,	hypothesis.
279	5	From,	Form.
294	38	affected,	effected.
307	17	$\frac{1}{2}r \cdot s^2 - s$,	$\frac{1}{2}r \cdot (s^2 - s)$.

VOL. III.

12	3 from bott.	$(2r \div (r+s))^2$,	$(2r \div (r+s))^2$.
88	24	761,	785.
103	18	max,	min.
109	20	equation,	equations.
176	27	future,	subsequent.
205	7	of Ware,	of Ware. (Suppt.)
250	5 from bott.	946,	947.
270	13	hypothense,	hypothesse.
286	11	sz^2	sz^2 .

VOL. IV.

106	18	Tatlow,	Titlow.
137	11	Minia,	Minima.
207	4 from bot.	invariably,	inversely.
230	2	$21^\circ 38' 15\frac{1}{2}''$	$20^\circ 38' 15\frac{1}{2}''$.
233	4 from bot.	$<$,	$>$.
238	19 fr. bot.	magle,	magic.
255	32	quatities,	quantities.
	33	{ and the rest are rational func- tions of it,	{ and some power of each of the rest is a rational function of it. N. B. The equation must be independent and consistent.
258	22	$(14.)e^5 5le^6$,	$(14.)e^5 - 5le^6$.
267	In the first	figure the upper	should be 1.
328	8	side,	sides.
405	12	$\frac{sfss}{r}$,	$\frac{sfss}{r}$.

